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TOPOLOGICAL STABILITY OF SOLENOIDAL AUTOMORPHISMS

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§ 0. Introduction

In [10] A. Morimoto proved that every topologically stable homeomorphism of a compact manifold M has the pseudo-orbit tracing property in the case $\dim(M) \geqslant 2$. Further, in studying relation between the topological stability and other stability of diffeomorphisms, he showed the following

THEOREM A. *Let $\boldsymbol{R}^r$ be the r-dimensional vector group and φ be a group automorphism of $\boldsymbol{R}^r$. Then the following conditions are mutually equivalent;*

(i) *φ is hyperbolic,*
(ii) *φ is expansive,*
(iii) *φ is structually stable,*
(iv) *φ has the pseudo-orbit tracing property,*
(v) *φ is topologically stable.*

The statement further is true for toral automorphisms.

We know (cf. see § 1) that every toral automorphism is contained in the class of solenoidal automorphisms. Thus it will be natural to ask what kind of solenoidal automorphisms have the pseudo-orbit tracing property. Our aim is to investigate this problem by using results in [2] and A. Morimoto [9, 10, 11].

§ 1. A main result and preparatory lemmas

Let $f\colon X \longleftrightarrow$ be a homeomorphism of a compact metric space (X, d). We denote by $\mathscr{H}(X)$ the group of all homeomorphisms of X. Then $\mathscr{H}(X)$ becomes a complete topological group with the topology given by the metric $d(f, g) = \max\{d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x))\colon x \in X\}$ $(f, g \in \mathscr{H}(X))$. We

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call f to be *topologically stable* iff for every $\varepsilon > 0$ there is $\delta > 0$ with the property that for every $g \in \mathscr{H}(X)$ with $d(f, g) < \delta$ there is a continuous map $h: X \longleftrightarrow$ such that

$$\text{i)}\quad h \circ g = f \circ h, \qquad \text{ii)}\quad d(h(x), x) < \varepsilon \ (x \in X).$$

A sequence of points $\{x_i\}_{i\in(a,b)}$ $(-\infty \leqslant a < b \leqslant +\infty)$ is called a *δ-pseudo-orbit* of f iff $d(f(x_i), x_{i+1}) < \delta$. Given $\varepsilon > 0$, a δ-pseudo-orbit $\{x_i\}$ is called to be *ε-traced* by a point $y \in X$ iff $d(f^i(y), x_i) < \varepsilon$ for every $i \in (a, b)$. We call f to have the *pseudo-orbit tracing property* (abbrev. P.O.T.P.) iff for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit of f can be ε-traced by some point $y \in X$. We denote by $\mathrm{Orb}^\delta(f)$ the set of all (finite or infinite) δ-pseudo-orbits of f and by $\mathrm{Tr}^\varepsilon(\{x_i\}, f) = \mathrm{Tr}^\varepsilon(\{x_i\})$ the set of all $y \in X$ such that $\{x_i\}$ is ε-traced by y. We call (X, f) to have *weak specification* iff for every $\varepsilon > 0$ there is $M(\varepsilon) > 0$ such that for every $k \geqslant 1$ and k points $x_1, \cdots, x_k \in X$ and for every set of integers $a_1 \leqslant b_1 < a_2 \leqslant b_2 \cdots < a_k \leqslant b_k$ with $a_{i+1} - b_i \geqslant M(\varepsilon)$ $(1 \leqslant i \leqslant k-1)$ there is $x \in X$ with $d(f^n(x), f^n(x_i)) < \varepsilon$ $(a_i \leqslant n \leqslant b_i,\ 1 \leqslant i \leqslant k)$.

We say that X is *solenoidal* iff X is a compact connected finite-dimensional abelian group. Every finite-dimensional torus is clearly solenoidal.

Hereafter X will be an r-dimensional solenoidal group and σ will be an automorphism of X. Our main result is the following

THEOREM 1. *The following* (A) *and* (B) *are equivalent*;

(A) *(X, σ) is topologically stable,*

(B) *(X, σ) has the P.O.T.P..*

Further there exist solenoidal automorphisms with P.O.T.P. such that one of the following conditions holds:

(C) *(X, σ) is not expansive,*

(D) *(X, α) is not densely periodic.*

The second statement of Theorem 1 will follow from Remark 1 mentioned below.

Denote by (G, γ) the dual of (X, σ) $((\gamma g)(x) = g(\sigma x),\ g \in G$ and $x \in X)$. Let $\bar{G}$ be a minimal divisible extension of G (p. 168, [7]). Since G is torsion free, $\bar{G}$ is so and $\mathrm{rank}(\bar{G}) = \mathrm{rank}(G) = r < \infty$ (p. 34, [7]). It is well known that γ induces an automorphism $\bar{\gamma}$ of $\bar{G}$. We shall write $\gamma = \bar{\gamma}$ for the sake of simplicity.

Let $\boldsymbol{Q}[x, x^{-1}]$ be the ring of polynomials in x and x^{-1} with coefficients in $\boldsymbol{Q}$ (the notation $\boldsymbol{Q}$ means the rational field). Since $\bar{G}$ is divisible and torsion free, for every $\bar{f}\in\bar{G}$ and every natural number n there is a unique $\bar{g}\in\bar{G}$ such that $n\bar{g}=\bar{f}$. So we consider $\bar{g}$ to be $(1/n)\bar{f}$ and $\boldsymbol{Q}[x, x^{-1}]$ to act on $\bar{G}$ by $(\sum_{j=-m}^{n} b_j x^j)\bar{g}=\sum_{j=-m}^{n} b_j \gamma^j \bar{g}$ ($b_j\in\boldsymbol{Q}$ and $\bar{g}\in\bar{G}$). Then $\bar{G}$ becomes a $\boldsymbol{Q}[x, x^{-1}]$-module. Since $\boldsymbol{Q}[x, x^{-1}]$ is a principal ideal domain, it follows (cf. p. 397, [6]) that there is in G a finite sequence $\{g_1, \cdots, g_s\}$ such that $\bar{G}$ splits into a direct sum $\bar{G}=\bar{G}_{g_1}\oplus\cdots\oplus\bar{G}_{g_s}$ of γ-invariant subgroups $\bar{G}_{g_i}$ where $\bar{G}_{g_i}=\{f\in\bar{G}: mf\in \mathrm{gp}\{\gamma^j g_i: j\in\boldsymbol{Z}\}$ for some $m\neq 0\}$ for $1\leqslant i\leqslant s$ (the notation gp E means the subgroup generated by a set E). Since $\gamma\bar{G}_{g_i}=\bar{G}_{g_i}$ for $1\leqslant i\leqslant s$, we can find a polynomial $q_i(x)\in\boldsymbol{Z}[x]$ with minimal degree r_i such that $q_i(\gamma)g_i=0$ holds, so that $\Theta=\{g_1, \cdots, \gamma^{r_1-1}g_1, \cdots, g_s, \cdots, \gamma^{r_s-1}g_s\}$ is linearly independent (the notation $\boldsymbol{Z}[x]$ means the ring of polynomials with integer coefficients). Hence the factor group $G/\mathrm{gp}\,\Theta$ is a torsion group; i.e. $\mathrm{gp}\,\Theta$ is full in G. Numbering the elements of Θ as $\Theta=\{e_1, \cdots, e_r\}$, every $0\neq g\in G$ is expressed as $ag=a_1e_1+\cdots+a_re_r$ for some $a\neq 0$ and some $a_1, \cdots, a_r$ with $(a_1. \cdots, a_r)\neq(0, \cdots, 0)$. Since the existence of $(a_1/a, \cdots, a_r/a)$ is unique, we can define an into isomorphism $\varphi: G\to\boldsymbol{Q}^r$ by $\varphi(g)=(a_1/a, \cdots, a_r/a)$. To simplify the notations, we identify g with $(a_1/a, \cdots, a_r/a)$ under φ. Then Θ is the canonical basis of $\boldsymbol{Z}^r$ (i.e. $e_1=(1, 0, \cdots, 0), \cdots, e_r=(0, \cdots, 0, 1)$), so that $\mathrm{gp}\,\Theta=\boldsymbol{Z}^r\subset G\subset\bar{G}=\boldsymbol{Q}^r\subset\boldsymbol{R}^r$. We extend γ on $\boldsymbol{R}^r$ by the natural way, and denote it by the same symbol. For $t=(t_1, \cdots, t_r)\in\boldsymbol{R}^r$, define $\psi(t)g=t_1a_1/a+\cdots+t_ra_r/a$ (addition mod 1) for all $g=(a_1/a, \cdots, a_r/a)\in G$. Then we get $\psi(t)\in X$ (p. 251, [12]). In fact, $\psi: \boldsymbol{R}^r\to X$ is an into homomorphism. The adjoint map $\hat{\gamma}$ of γ is defined by $\psi(\hat{\gamma}t)g=\psi(t)\gamma g=(\sigma\psi(t))g$ ($t\in\boldsymbol{R}^r$, $g\in G$). Since $\hat{\gamma}$ and γ are isomorphic, we denote $\hat{\gamma}$ by γ again, and say $(\boldsymbol{R}^r, \gamma)$ to be the *lifting system* of (X, σ).

LEMMA B ((P.2(i)), [2]). *Under the above notations, $\psi(\boldsymbol{R}^r)$ is dense in X. If X is a torus then $\psi(\boldsymbol{R}^r)=X$.*

LEMMA C ((P.2), [2]). *Let F be the annihilator of* gp Θ *in X. Then* (i) *F is totally disconnected and $\psi^{-1}\{\psi(\boldsymbol{R}^r)\cap F\}=\boldsymbol{Z}^r$,* (ii) *$X=\psi(\boldsymbol{R}^r)+F$ and* (iii) *there is a small closed neighborhood U of* 0 *in $\boldsymbol{R}^r$ such that $\psi(U)\cap F=\{0\}$ and the direct product $U\times F$ is homeomorphic to $\psi(U)+F$. And $\psi(U)+F$ is a closed neighborhood of* 0 *in X (We write $\psi(U)\oplus F$ such a neighborhood $\psi(U)+F$).*

LEMMA D. *Under the above notations, the followings hold;*

(i) *there exist a torus* V_1 *and a vector group* V_2 *such that* ψ *induces a* 1–1 *homomorphism* ψ^* *from the direct product group* $V_1 \oplus V_2$ *onto* $\psi(\boldsymbol{R}^r)$,

(ii) $\psi(\boldsymbol{Z}^r)$ *is a closed subgroup in* $\psi(\boldsymbol{R}^r)$,

(iii) $\psi(\boldsymbol{Z}^r)$ *is dense in* F.

Proof. Let K be the kernel of ψ. Then $K \subset \boldsymbol{Z}^r$ by Lemma C(i), so that there are a torus V_1 and a vector subgroup V_2 such that $\boldsymbol{R}^r/K = V_1 \oplus V_2$. Therefore ψ induces a 1–1, onto homomorphism ψ^*: $V_1 \oplus V_2 \to \psi(\boldsymbol{R}^r)$ in the natural way. (i) was proved. It is clear that $\overline{\psi(\boldsymbol{Z}^r)} \subset F$. By Lemma C (ii), $\overline{\psi(\boldsymbol{Z}^r)} \cap \psi(\boldsymbol{R}^r) \subset F \cap \psi(\boldsymbol{R}^r) = \psi(\boldsymbol{Z}^r)$ and so $\overline{\psi(\boldsymbol{Z}^r)} \cap \psi(\boldsymbol{R}^r) = \psi(\boldsymbol{Z}^r)$. This shows (ii). Put $B = \overline{\psi(\boldsymbol{Z}^r)}$. Then $X/B = \{(\psi(\boldsymbol{R}^r) + B)/B\} + \{F/B\}$. Since $\psi(\boldsymbol{R}^r)/\psi(\boldsymbol{Z}^r)$ is a factor group of $\boldsymbol{R}^r/\boldsymbol{Z}^r$, it is a torus and so $(\psi(\boldsymbol{R}^r) + B)/B$ is also a torus since $(\psi(\boldsymbol{R}^r) + B)/B \cong \psi(\boldsymbol{R}^r)/\psi(\boldsymbol{Z}^r)$. On the other hand, X/B is connected, from which we have $F = B$.

LEMMA E. *Let* V_1, V_2 *and* ψ^* *be as in Lemma* D. *For* $\alpha_2 > 0$ *small enough, let* $B(\alpha_2)$ *be a closed neighborhood with the radius* α_2 *of* 0 *in* $V_1 \oplus V_2$. *Then* $B(\alpha_2)$ *is a closed neighborhood of* $\boldsymbol{R}^r$ *and* $\psi^*(v) = \psi(v)$ *for every* $v \in B(\alpha_2)$.

Proof. This is clear by the proof of Lemma D (i).

LEMMA F((P. 8), [2]). *If* $H = \operatorname{ann}(X, \operatorname{gp} \bigcup_{-\infty}^{\infty} \gamma^j \Theta)$, *then there exist subgroups* F^- *and* F^+ *of* F *satisfying the conditions*;

(i) $\sigma H = H$ *and the topological entropy of* σ_H *equals zero*,

(ii) H *contains a sequence* $H = H_0 \supset H_1 \supset \cdots \supset \bigcap H_n = \{0\}$ *of subgroups such that for* $n \geqslant 0$, $\sigma H_n = H_n$ *and* H/H_n *is finite*,

(iii) $F^- \supset \sigma^{-1}F^- \supset \cdots \supset \bigcap_0^\infty \sigma^{-n}F^- = \{0\}$,

(iv) $F^+ \supset \sigma F^+ \supset \cdots \supset \bigcap_0^\infty \sigma^n F^+ = \{0\}$,

(v) $\sigma F^-/F^-$ *and* $F^+/\sigma F^+$ *are finite*,

(vi) $F = F^- \oplus F^+ \oplus H$.

By Lemmas C (iv) and F (vi), we have $X = \psi(\boldsymbol{R}^r) + \{F^- \oplus F^+ \oplus H\}$. Since X is connected, it follows that $X = \psi(\boldsymbol{R}^r) + \{F^- \oplus F^+\}$ when H is finite.

LEMMA G ((P. 4), [2]). (i) $H = \{0\}$ *iff* $G = \operatorname{gp} \bigcup_{-\infty}^{\infty} \gamma^j \Theta$, (ii) H *is finite iff* (G, γ) *is finitely generated under* γ; i.e. *there is a finite set* Λ *in* G *such that* $G = \operatorname{gp} \bigcup_{-\infty}^{\infty} \gamma^j \Lambda$.

The space $\boldsymbol{R}^r$ splits into a direct sum $\boldsymbol{R}^r = E^u \oplus E^s \oplus E^c$ of γ-invariant subspaces E^u, E^s and E^c such that the eigenvalues of γ_{E^u}, γ_{E^s} and γ_{E^c} have modulus > 1, < 1 and 1 respectively. We call that $(\boldsymbol{R}^r, \gamma)$ is *hyperbolic* iff $E^c = \{0\}$. It is easily proved that there are norms $\|\cdot\|_u$ and $\|\cdot\|_s$ of E^u and E^s, respectively and $\lambda_0 \in (0, 1)$ such that $\|\gamma^n x\|_u \leqslant \lambda_0^{-n}\|x\|_u$ ($x \in E^u$ and $n \leqslant 0$) and $\|\gamma^n x\|_s \leqslant \lambda_0^n\|x\|_s$($x \in E^s$ and $n \geqslant 0$). If $E^c \neq \{0\}$, by using Jordan normal form in the real field for (E^c, γ) we get a finite direct sum $E^c = E^{c_0} \oplus \cdots \oplus E^{c_k}$ of the subspaces E^{c_i} satisfying the following conditions; for $0 \leqslant i \leqslant k$, the dimension of E^{c_i} is 1 or 2, and

$$\gamma_{E^c} = \begin{bmatrix} \gamma_0 & I_1 & & & 0 \\ & \gamma_1 & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & \ddots & I_k \\ & & & & \gamma_k \end{bmatrix}$$

where $\gamma_i \colon E^{c_i} \longleftrightarrow$ is an isometry under some norm $\|\cdot\|_{c_i}$ of E^{c_i} and each $I_i \colon E^{c_i} \to E^{c_{i-1}}$ is either a zero map or a map corresponding to the identity matrix. We call that $(\boldsymbol{R}^r, \gamma)$ is *central spin* iff $E^c \neq \{0\}$ and each I_i is a zero map. If the characteristic polynomial $p(x)$ of γ is irreducible over $\boldsymbol{Q}$ and $p(x)$ has roots of modulus one, then $(\boldsymbol{R}^r, \gamma)$ is central spin. Define a norm $\|\cdot\|_c$ of E^c by $\|x\|_c = \max\{\|x^i\|_{c_i} \colon 0 \leqslant i \leqslant k\}$ $(x = x^0 + \cdots + x^k \in \oplus_0^k E^{c_i})$. Then we get easily that $d_0(x, y) = \max\{\|x^u - y^u\|_u, \|x^s - y^s\|_s, \|x^c - y^c\|_c\}$ is a metric of $\boldsymbol{R}^r$ satisfying the following conditions; (i) d_0 is translation invariant, (ii) there is $\lambda_0 \in (0, 1)$ with

$$d_0(\gamma^n x, 0) \leqslant \begin{cases} \lambda_0^{-n} d_0(x, 0) & (x \in E^u,\ n \leqslant 0)\ , \\ \lambda_0^n d_0(x, 0) & (x \in E^s,\ n \geqslant 0)\ , \end{cases}$$

and (iii) if $E^c \neq \{0\}$ then each of γ_i (under the above notations) is d_0-isometry. We see that there are $\alpha_1 > 0$ such that for every $\alpha \in (0, \alpha_1]$, if $B(\alpha) = \{x \in \boldsymbol{R}^r \colon d_0(x, 0) \leqslant \alpha\}$ then $\psi B(\alpha) \oplus F^- \oplus F^+ \oplus H$ is a closed neighborhood of X.

Let λ_0 be as above. Then the functions

$$d_-(x, y) = \begin{cases} \lambda_0^n & \text{if } x - y \in \sigma^{-n}F^- \text{ and } x - y \notin \sigma^{-(n+1)}F^- \\ 0 & \text{if } x = y\ , \end{cases}$$

$$d_+(x, y) = \begin{cases} \lambda_0^n & \text{if } x - y \in \sigma^{n}F^+ \text{ and } x - y \notin \sigma^{n+1}F^+ \\ 0 & \text{if } x = y \end{cases}$$

are metrics generating the original topology of F^- and F^+ respectively. For $x = x_0 + x_1 + x_2 + x_3 \in \psi B(\alpha_1) \oplus F^- \oplus F^+ \oplus H$, put

$$\rho(x) = \min\{\alpha_1, \max\{d_0(\psi^{-1}x_0, 0), d_-(x_1, 0), d_+(x_2, 0), d(x_3, 0)\}\} .$$

Then the metric of X defined by

$$d(x, y) = \begin{cases} \rho(x-y) & \text{if } x - y \in \psi B(\alpha_1) \oplus F^- \oplus F^+ \oplus H \\ \alpha_1 & \text{otherwise} \end{cases}$$

is compatible with the original topology. It follows that for $\varepsilon \in (0, \alpha_1)$, $B(\varepsilon) = B^u(\varepsilon) \oplus B^s(\varepsilon) \oplus B^c(\varepsilon)$ where $B^u(\varepsilon) = B(\varepsilon) \cap E^u$, $B^s(\varepsilon) = B(\varepsilon) \cap E^s$ and $B^c(\varepsilon) = B(\varepsilon) \cap E^c$.

In proving our results, it is important that closed neighborhoods are chosen to be proper subsets of X, so that we take and fix a number α_0 such that

$$(*) \quad 0 < \alpha_0 < \min\{\alpha_1, \alpha_2\} .$$

Here α_2 is the number chosen in Lemma E. For $\varepsilon \in (0, \alpha_0]$, a closed neighborhood $W(\varepsilon) = \{x \in X: d(x, 0) \leqslant \varepsilon\}$ is expressed as

$$(**) \quad W(\varepsilon) = W^u(\varepsilon) \oplus W^s(\varepsilon) \oplus W^c(\varepsilon)$$

where $W^u(\varepsilon) = W(\varepsilon) \cap \{\psi B^u(\varepsilon) \oplus F^-\}$, $W^s(\varepsilon) = W(\varepsilon) \cap \{\psi B^s(\varepsilon) \oplus F^+\}$ and $W^c(\varepsilon) = W(\varepsilon) \cap \{\psi B^c(\varepsilon) \oplus H\}$. Let d be a metric of X defined as above. For $x = x^u + x^s + x^c \in W^u(\alpha_0) \oplus W^s(\alpha_0) \oplus W^c(\alpha_0)$, we have

$$(***) \quad \begin{aligned} & d(x, 0) = \max\{d(x^u, 0), d(x^s, 0), d(x^c, 0)\} \quad \text{and} \\ & d(\sigma^n x, 0) \leqslant \begin{cases} \lambda_0^{-n} d(x, 0) & (x \in W^u(\alpha_0), n \leqslant 0) \\ \lambda_0^n d(x, 0) & (x \in W^s(\alpha_0), n \geqslant 0) . \end{cases} \end{aligned}$$

LEMMA H. *If (X, σ) is ergodic under the Haar measure, then $\psi(E^c)$ is dense in X.*

Proof. There is in E^c a γ-invariant subspace E^{c_1} such that $\gamma_{E^{c_1}}$ is d_0-isometry. Hence E^c is expressed as $E^c = E^{c_1} \oplus E^{c_2}$ where E^{c_2} is a subspace. Assume that $\psi(E^c)$ is not dense in X, and put $A = \overline{\psi(E^{c_1})}$. Then A is a σ-invariant connected subgroup of X. Obviously, σ_A is d-isometry; i.e. the topological entropy of σ_A is zero ($\text{ent}(\sigma_A) = 0$). As before let (G, γ) be the dual of (X, σ) and G_A be the annihilator of A in G. Since $G_A \subset G \subset \boldsymbol{R}^r$, we denote by V_A the smallest vector subgroup of $\boldsymbol{R}^r$ containing G_A. Then $\boldsymbol{R}^r$ is expressed as $\boldsymbol{R}^r = V_A \oplus V'$ where V' is a subspace. We see that $\boldsymbol{R}^r/V_A \cong V'$ is the smallest subspace containing $\dot{G}_A = G/G_A$. Since G is finitely generated under γ, so is $\dot{G}_A$. Let $p(x)$ be the characteristic polynomial of $\gamma_{\hat{G}_A}$. Then the Kolmogorov entropy of σ_A equals $h(\sigma_A) = \sum_{|\lambda|>1} \log|\lambda| + \log \Delta$ where λ's are the eigenvalues of $\gamma_{\hat{G}_A}$ and Δ is the smallest positive integer such that $\Delta p(x)$ has the integer coefficients (see [13]). Since $\mathrm{ent}(\sigma_A) = h(\sigma_A) = 0$, we have $\Delta = 1$ (hence $p(x) \in \boldsymbol{Z}[x]$) and all the roots of $p(x)$ are modulus one. It follows from a result in the number theory that they are the roots of unity since $p(x) \in \boldsymbol{Z}[x]$. On the other hand, since (X, σ) is ergodic, all the roots of the characteristic polynomial of γ are not the roots of unity. This is a contradiction.

Remark 1. Let M be a compact manifold and φ be a diffeomorphism of M. It is proved in [10] that the set of all periodic points is dense in the non-wandering set when (M, φ) is topologically stable. In general this is not true for homeomorphisms on compact metric spaces. For example, let γ be an automorphism of $\boldsymbol{Q}^r$. Consider to $\boldsymbol{Q}^r$ be an abelian group imposed with the discrete topology. If $(\boldsymbol{R}^r, \gamma)$ is hyperbolic, then $(\gamma^j - I)\boldsymbol{Q}^r = \boldsymbol{Q}^r$ for every $j>0$. From this we get that the dual (X, σ) of $(\boldsymbol{Q}^r, \gamma)$ has no periodic points except 0; i.e. (X, σ) is not densely periodic. By Theorem 2 in the next section, (X, σ) has the P.O.T.P., and (X, σ) is topologically stable by the first statement of Theorem 1.

Since $\boldsymbol{Q}^r$ is not finitely generated under γ, (X, σ) is not expansive (by Theorem 1, [2]). Therefore it will follow that there is a solenoidal automorphism which has the P.O.T.P., but is not expansive.

Remark 2. The set $\mathscr{C}(X)$ of all non-empty closed subsets of X is a compact metric space by the Hausdorff metric $\bar{d}$. Denote by $\widetilde{\mathrm{Orb}}^{\varepsilon}(\sigma)$ the set of all $A \in \mathscr{C}(X)$ for which there is $\{x_i\} \in \mathrm{Orb}^{\varepsilon}(\sigma)$ such that $A = \overline{\{x_i : i \in \boldsymbol{Z}\}}$. Let $E(\sigma)$ denote the set of all $A \in \mathscr{C}(X)$ such that for every $\varepsilon > 0$ there is $A_\varepsilon \in \widetilde{\mathrm{Orb}}^{\varepsilon}(\sigma)$ with $\bar{d}(A, A_\varepsilon) < \varepsilon$. Then $E(\sigma)$ is closed in $\mathscr{C}(X)$. We define $O(\sigma) = \overline{\{O_\sigma(x) : x \in X\}} \subset \mathscr{C}(X)$ where $O_\sigma(x) = \overline{\{\sigma^i(x) : i \in \boldsymbol{Z}\}}$ for $x \in X$. Obviously $O(\sigma) \subset E(\sigma)$. We call σ to have the OE-*property* iff $O(\sigma) = E(\sigma)$. A. Morimoto asks in [11] whether the lifting system $(\boldsymbol{R}^r, \gamma)$ of (X, σ) is hyperbolic if σ has the OE-property. In [3] it is proved that for every automorphism β of a compact metric group, β has the OE-property iff β has the P.O.T.P.. From this result together with Theorem 2 in the next section, we shall see that Morimoto's problem is completely solved.

From Theorem 1 and Remark 2 we get the following

COROLLARY. *The following* (A)′, (B)′ *and* (C)′ *are equivalent*;

(A)′ (X, σ) *is topologically stable,*
(B)′ (X, σ) *has the* P.O.T.P.,
(C)′ (X, σ) *satisfies the OE-property.*

Hereafter, the restriction and the factor of σ will be denoted by the same symbol if there is no confusion.

§ 2. An auxiliary result

In this section we shall prove the following

THEOREM 2. *The lifting system* $(\boldsymbol{R}^r, \gamma)$ *of* (X, σ) *is hyperbolic iff* (X, σ) *has the P.O.T.P..*

For the proof we need the following lemmas.

LEMMA 2.1. (X, σ) *has the P.O.T.P., then* (X, σ) *is topologically mixing.*

Proof. By (Theorem 2, [1]), X contains σ-invariant subgroups X_1 and X_2 such that (X_1, σ) has zero entropy, (X_2, σ) is ergodic and X splits into a sum $X = X_1 + X_2$. Since X/X_2 is a factor group of X_1, $(X/X_2, \sigma)$ has zero entropy. As we saw in the proof of Lemma H, X/X_2 is a torus and $(X/X_2, \sigma)$ is not hyperbolic. It is easy to see that $(X/X_2, \sigma)$ has the P.O.T.P., so that we must have $X_1 = \{0\}$ by Theorem A. Therefore (X, σ) is ergodic and hence (X, σ) is topologically mixing.

LEMMA 2.2. *If* (X, σ) *has the P.O.T.P., then* (X, σ) *satisfies weak specification.*

Proof. By Lemma 2.1, (X, σ) is topologically mixing. Let $\varepsilon > 0$ be given. Choose $\delta = \delta(\varepsilon) > 0$ as in the definition of the P.O.T.P. Cover X by a finite family $\mathscr{U}$ of δ-balls. For any two U_i, $U_j \in \mathscr{U}$ there is $M_{ij} > 0$ such that $\sigma^n U_i \cap U_j \neq \phi$ for $n \geqslant M_{ij}$. Put $M = \max\{M_{ij}: i, j\} < \infty$. Let $x_1, \cdots, x_k$ be points in X and $a_1 \leqslant b_1 < \cdots < a_k \leqslant b_k$ be integers with $a_j - b_{j-1} \geqslant M$ for $2 \leqslant j \leqslant k$. For $z \in X$ we denote by $U(z)$ some $U \in \mathscr{U}$ with $z \in U$. For $1 \leqslant j \leqslant k$ there is a point $y_j \in U(\sigma^{b_j} x_j)$ such that $\sigma^{a_{j+1} - b_j} y_j \in U(\sigma^{a_{j+1}} x_{j+1})$. Consider the δ-pseudo-orbit $\{z_i: a_1 \leqslant i \leqslant a_k\}$ defined by $z_i = \sigma^i x_j$ for $a_j \leqslant i \leqslant b_j$ and $z_i = \sigma^{i - b_j}(y_j)$ for $b_j < i < a_{j+1}$. Then there is a point $x \in X$ which ε-traces the orbit. From this we get $d(\sigma^i(x), \sigma^i(x_j)) < \varepsilon$ for $a_j \leqslant i \leqslant b_j$ $(1 \leqslant j \leqslant k)$.

LEMMA 2.3. *Let* σ *be an automorphism of a compact metric group* Y

and K be a completely σ-invariant normal subgroup of $Y(\sigma(K) = K)$. If both $(Y/K, \sigma)$ and (K, σ) have the P.O.T.P., then so is (Y, σ).

Proof. By assumption, for every $\varepsilon > 0$ there is a $\delta > 0$ with $\delta < \varepsilon$ such that for every δ-pseudo-orbit in K, a point in K $\varepsilon/2$-traces the orbit. Choose η with $0 < \eta < \delta/3$ such that the following conditions hold;

(a) $d(\sigma(x), \sigma(y)) < \delta/3$ when $d(x, y) < \eta$ and

(b) for an arbitrary η-pseudo-orbit $\{x_i: a < i < b\}$ of Y, Y/K contains a point $x + K \in Y/K$ with $d(\sigma^i(x + K), x_i + K) < \delta/3 (a < i < b)$ (here d is a metric on X/K defined by $d(x + K, y + K) = \inf\{d(x + k, y + k'): k, k' \in K\}$).

By (b), for $a < i < b$ there is $k_i \in K$ such that $d(\sigma^i(x) + k_i, x_i) < \eta$.
By (a), $d(\sigma^{i+1}(x) + \sigma(k_i), \sigma(x_1)) < \delta/3$. We calculate that for $a < i < b - 1$

$$\begin{aligned} d(\sigma(k_i), k_{i+1}) &= d(\sigma^{i+1}(x) + \sigma(k_i), \sigma^{i+1}(x) + k_{i+1}) \\ &\leqslant d(\sigma^{i+1}(x) + \sigma(k_i), \sigma(x_i)) + d(\sigma(x_i), x_{i+1}) + d(x_{i+1}, \sigma^{i+1}(x) + k_{i+1}) \\ &< \delta , \end{aligned}$$

from which there is a point $k \in K$ $\varepsilon/2$-tracing the orbit $\{k_i: a < i < b\}$. Since

$$d(\sigma^i(x + k), x_i) \leqslant d(\sigma^i(x + k), \sigma^i(x) + k_i) + d(\sigma^i(x) + k_i, x_i) < \varepsilon ,$$

the point $x + k$ ε-traces the orbit $\{x_i: a < i < b\}$ in Y and the proof is completed.

LEMMA 2.4. *Let Y and σ be as in Lemma 2.3. If Y contains a sequence $Y = K_0 \supset K_1 \supset \cdots \supset \cap K_n = \{e\}$ of normal subgroups such that for $n \geqslant 0$, $\sigma K_n = K_n$ and Y/K_n is finite, then (Y, σ) has the P.O.T.P.*

Proof. For every $\varepsilon > 0$, there are $n > 0$ and δ with $0 < \delta < \varepsilon$ such that

$$\{x \in Y: d(x, 0) < \delta\} \subset K_n \subset \{x \in Y: d(x, 0) < \varepsilon\}.$$

Let $\{x_i: a < i < b\}$ be an arbitrary δ-pseudo-orbit in Y; i.e. $d(\sigma x_i, x_{i+1}) < \delta$, $a < i < b - 1$ (without loss of generality we may assume $a + 1 \leqslant 0$). Then $\sigma x_i - x_{i+1} \in K_n$ $(a < i < b - 1)$. Hence $\sigma^i x_0 - x_i \in K_n$ since $\sigma K_n = K_n$; i.e. $d(\sigma^i x_0, x_i)$ $(a < i < b)$. This shows that $(Y, \sigma$ has the P.O.T.P.)

Proof of Theorem 2.

As we saw in Section 1, $\boldsymbol{R}^r$ splits into a direct sum $\boldsymbol{R}^r = E^u \oplus E^s \oplus E^c$ of the γ-invariant subspaces.

Proof of $\Leftarrow$): By Lemma 2.2, (X, σ) obeys weak specification. Hence $(\boldsymbol{R}^r, \gamma)$ is central spin by (Theorem 2, [2]); i.e. if $E^c \neq \{0\}$ then γ on E^c is d_0-isometry (the metric d_0 is defined as in Section 1). We shall now prove that if $E^c \neq \{0\}$ then (X, σ) has not the P.O.T.P.. To do this, assume that (X, σ) has the P.O.T.P.; i.e. for every $\varepsilon \in (0, \alpha_0)$ there is $\delta > 0$ with $\delta < \varepsilon$ such that every δ-pseudo-orbit is $\varepsilon/2$-traced by some point of X. Fix $0 \neq v_0 \in E^c$ with $d(v_0, 0) < \delta$ and set $z_j = j\gamma^j(v_0)$ for $j \in \boldsymbol{Z}$. Then it follows that

$$d(\gamma(z_j), z_{j+1}) = d_0(j\gamma^{j+1}(v_0),\ (j+1)\gamma^{j+1}(v_0)) = d_0(0, v_0) < \delta\ ,$$

and so $\{z_j\} \in \mathrm{Orb}^\delta(\gamma)$. Put $V_\varepsilon(v) = \{u \in \boldsymbol{R}^r : d_0(v, u) < \varepsilon\}$ for $v \in \boldsymbol{R}^r$. Then there is $k > 0$ such that $V_\varepsilon(kv_0) \cap V_\varepsilon(0) = \phi$. From the relation between the metrics d_0 and d, we have

$$\delta > d_0(\gamma(z_j), z_{j+1}) = d(\sigma\psi(z_j), \psi(z_{j+1}))$$

for $j \in \boldsymbol{Z}$, and so $\{\psi(z_j)\} \in \mathrm{Orb}^\delta(\sigma)$. By the assumption there is $x \in X$ such that $d(\sigma^j(x), \psi(z_j)) < \varepsilon/2$ for $j \in \boldsymbol{Z}$. Since $\psi(E^c)$ is dense in X (by Lemma H), we can find $0 \neq y \in \psi(E^c)$ with $\max\{d(\sigma^i(y), \sigma^i(x)) : 0 \leqslant i \leqslant k\} \leqslant \varepsilon/2$. Hence $d(\sigma^j(y), \psi(z_j)) < \varepsilon$ for $0 \leqslant j \leqslant k$, from which we have

$$\begin{aligned} d(y, \psi(z_0)) &= d(y, 0) = d_0(\psi^{-1}(y), 0) < \varepsilon \quad \text{and} \\ d(\sigma^k(y), \psi(z_k)) &= d_0(\gamma^k\psi^{-1}(y), z_k) = d_0(\gamma^k\psi^{-1}(y), \gamma^k(kv_0)) \\ &= d_0(\psi^{-1}(y), kv_0) < \varepsilon\ . \end{aligned}$$

Therefore $\psi^{-1}(y) \in V_\varepsilon(0) \cap V_\varepsilon(kv_0)$, which is a contradiction.

Proof of $\Rightarrow$). Since H is the annihilator of $\mathrm{gp}\bigcup_{-\infty}^{\infty}\gamma^n\Theta$ in X (see Lemma F), $(\mathrm{gp}\bigcup_{-\infty}^{\infty}\gamma^n\Theta, \gamma)$ is the dual of $(X/H, \sigma)$ and $G/\mathrm{gp}\bigcup_{-\infty}^{\infty}\gamma^n\Theta$ is a torsion group. Hence the dimension of $\dot{X} = X/H$ is equal to that of X since H is zero-dimensional. Since $\mathrm{gp}\bigcup_{-\infty}^{\infty}\gamma^n\Theta$ is clearly finitely generated under γ and since the lifting system $(\boldsymbol{R}^r, \gamma)$ of $(\dot{X}, \sigma)$ is hyperbolic, it follows that $(\dot{X}, \sigma)$ is expansive (see Theorem 1, [2]). We see that $\dot{X}$ is expressed as $\dot{X} = \psi(\boldsymbol{R}^r) + \{F^- \oplus F^+\}$ by Lemma G (ii). Let $\varepsilon_0 > 0$ be an expansive constant for $(\dot{X}, \sigma)$. Then, for $0 < \varepsilon < \varepsilon_0$ we have a coordinate neighborhood $W(\varepsilon) = W^u(\varepsilon) \oplus W^s(\varepsilon)$ of 0 in $\dot{X}$ where $W^u(\varepsilon) \neq \{0\}$ and $W^s(\varepsilon) \neq \{0\}$. It is easily seen that there is $\delta = \delta(\varepsilon) > 0$ such that if $d(x, y) < \delta$ $(x, y \in \dot{X})$ then $\{W^u(\varepsilon) + x\} \cap \{W^s(\varepsilon) + y\}$ consists only of one point. Therefore $(\dot{X}, \sigma)$ has the P.O.T.P. using (***) (for the proof, see p. 74, R. Bowen [4]). From this fact together with Lemmas 2.3 and 2.4 and Lemma F (ii), we get the conclusion.

§ 3. Proof of Theorem 1

To see the statement (A) $\Rightarrow$ (B), we shall prepare the following three lemmas.

LEMMA 3.1. *If* (X, σ) *is topologically stable, then* $X \backslash \mathrm{Fix}(\sigma)$ *is dense in* X *where* $\mathrm{Fix}(\sigma) = \{x \in X : \sigma(x) = x\}$.

Proof. Notice that $\mathrm{Fix}(\sigma)$ is a subgroup of X. Assume that $X \backslash \mathrm{Fix}(\sigma)$ is not dense in X. Then $\mathrm{Fix}(\sigma)$ is open in X. Since X is connected, we get $X = \mathrm{Fix}(\sigma)$; i.e. $\sigma = \mathrm{id}$. Take $\varepsilon > 0$ with $2\varepsilon < \mathrm{diameter}(X)$ and let $\delta > 0$ be as in the definition of topological stability. Now we can find $a \in X$ with $d(a, 0) < \delta$ such that $\{na : n \in Z\}$ is dense in X (see [5]). Let $f_a : X \hookleftarrow$ be a homeomorphism defined by $f_a(x) = x + a (x \in X)$, then $d(f_a(x), x) = d(f_a(x), \sigma(x)) < \delta$. Hence there is a continuous map $h : X \hookleftarrow$ with $h \circ f_a = \sigma \circ h$ and $d(h(x), x) < \varepsilon$ $(x \in X)$. Since $h(an) = h(f_a^n(0)) = \sigma^n h(0) = h(0)$ for all n, we get $h(x) = h(0)$ for all $x \in X$, and so $\varepsilon > d(h(0), x)$ for $x \in X$. On the other hand, since $\varepsilon < \mathrm{diameter}(X)/2$, there is $y \in X$ with $d(h(0), y) > \varepsilon$, which is a contradiction.

LEMMA 3.2 ([10]). *Let* φ *be a homeomorphism of a connected metric space* Y. *Assume that* φ *is uniformly continuous and* $Y \backslash \mathrm{Fix}(\varphi)$ *is dense in* Y. *Take and fix a constant* $\delta_1 > 0$ *and an integer* $k > 0$. *Then for every* $\{x_i\} \in \mathrm{Orb}^{\delta_1}(\varphi)$ *and* $\varepsilon_1 > 0$, *there is* $\{x'_i\} \in \mathrm{Orb}^{3\delta_1}(\varphi)$ *such that* i) $d(x_i, x'_i) < \varepsilon_1$ *for* $0 \leqslant i \leqslant k$ *and* ii) $Y_i = \{\varphi(x'_i), x'_{i+1}\}$ $(0 \leqslant i \leqslant k-1)$ *are disjoint.*

We shall describe here a proof given in [10] for completeness. We can assume $\varepsilon_1 < \delta_1$. For this ε_1, there is $\varepsilon'_1 > 0$ with $\varepsilon_1 > \varepsilon'_1$ such that $d(x, y) < \varepsilon'_1$ implies $d(\varphi(x), \varphi(y)) < \varepsilon_1$. First we can find $x'_i \in Y$ $(0 \leqslant i \leqslant k)$ such that $x'_i \neq x'_j$ $(i \neq j)$ and $d(x_i, x'_i) < \varepsilon'_1$ $(0 \leqslant i \leqslant k)$. Next we shall show by induction that $Y_0, \cdots, Y_{k-1}$ are disjoint by taking x'_i suitably. Assume that $Y_i = \{\varphi(x'_i), x'_{i+1}\}$ $(0 \leqslant i \leqslant k-2)$ are disjoint. We shall show that, by changing x'_{k-1} and x'_k if necessary, $Y_i \cap Y_j = \phi$ $(0 \leqslant i \neq j \leqslant k-1)$. Consider the point $\varphi(x'_{k-1})$ and assume $\varphi(x'_{k-1}) \in \bigcup_{i=0}^{k-2} Y_i$. Then there is a unique $i \leqslant k-1$ such that $\varphi(x'_{k-1}) = x'_i$ since $x'_{k-1} \neq x'_j$ $(j \leqslant k-2)$ implies $\varphi(x'_{k-1}) \neq \varphi(x'_j)$. If $i \leqslant k-2$, we can find x''_{k-1} near x'_{k-1} such that $\varphi(x''_{k-1}) \neq x'_i$. If $i = k-1$ $(\varphi(x'_{k-1}) = x'_{k-1})$, then we can find x''_{k-1} near x'_{k-1} such that $\varphi(x''_{k-1}) \neq x''_{k-1}$, since $Y \backslash \mathrm{Fix}(\varphi)$ is dense and open in Y. We denote x''_{k-1} by x'_{k-1} again. Then we can assume that $x'_k \notin \bigcup_{i=0}^{k-1} Y_i$, since $\bigcup_{i=0}^{k-2} Y_i$ is a finite set. Thus we have proved that $Y_0, Y_1, \cdots, Y_{k-1}$ are disjoint.

For $i < 0$ (resp. $i > k$) we define $x'_i = \varphi^{-i}(x'_0)$ (resp. $x'_i = \varphi^{i-k}(x'_k)$). Then we see that $\{x'_i\} \in \mathrm{Orb}^{3\delta_1}(\varphi)$ since

$$d(\varphi(x'_i), x'_{i+1}) \leqslant d(\varphi(x'_i), \varphi(x_i)) + d(\varphi(x_i), x_{i+1}) + d(x_{i+1}, x'_{i+1})$$
$$< \varepsilon_1 + \delta_1 + \varepsilon'_1 < 3\delta_1$$

for $0 \leqslant i \leqslant k-1$. This completes the proof of Lemma 3.2.

LEMMA 3.3 ([8]). *Let M be a differentiable manifold of* $\dim(M) \geqslant 2$ *with a metric d. Let $M_i = \{p_i, q_i\}$ $(1 \leqslant i \leqslant k)$ be a subset of M consisting of at most two points p_i and q_i with $d(p_i, q_i) < \delta$. Assume that $M_i \cap M_j = \phi$ $(i \neq j)$. Then there is an onto homeomorphism $\eta\colon M \longleftrightarrow$ such that $d(\eta(x), x) < 2\pi\delta$ for $x \in M$, and that $\eta(p_i) = q_i$ $(1 \leqslant i \leqslant k)$.*

Proof of the statement (A) $\Rightarrow$ (B). The proof will be done along the following two cases.

Case (1). For the case $\dim(X) = 1$, we get that $(\boldsymbol{R}^1, \gamma)$ is hyperbolic by applying Lemma 3.1. Therefore (X, σ) has the P.O.T.P. by Theorem 2.

Case (2). We can use Lemma 3.3 when $\dim(X) \geqslant 2$. For every $\varepsilon > 0$, choose $\delta > 0$ $(\delta < \varepsilon)$ as in the definition of topological stability. Since $\overline{\psi(\boldsymbol{R}^r)} = X$ by Lemma B, for every $\{x_i\} \in \mathrm{Orb}^{\delta/12\pi}(\sigma)$ there is $\{x'_i\} \subset \psi(\boldsymbol{R}^r)$ such that $d(x_i, x'_i) < \delta/24\pi$, $d(\sigma(x_i), \sigma(x'_i)) \langle \delta/24\pi$ and $\sigma(x'_i) - x'_{i+1} \in W(\delta/6\pi)$ $(i \in \boldsymbol{Z})$ where $W(\delta/6\pi)$ is a closed neighborhood with the radius $\delta/6\pi$ of 0 in X. Since (X, σ) is topologically stable and $\psi(\boldsymbol{R}^r)$ is connected, we have that $\psi(\boldsymbol{R}^r)\backslash(\psi(\boldsymbol{R}^r) \cap \mathrm{Fix}(\sigma))$ is dense in $\psi(\boldsymbol{R}^r)$ (by using Lemma 3.1).

Take and fix an integer $k > 0$. Notice that σ and σ^{-1} are uniformly continuous on $\psi(\boldsymbol{R}^r)$. By Lemma 3.2 there is a sequence $\{x''_i\} \in \mathrm{Orb}^{\delta/2\pi}(\sigma)$ such that $d(x'_i, x''_i) < \delta/2\pi$ $(0 \leqslant i \leqslant k)$ and $\{\sigma(x''_i), x''_{i+1}\}$ $(0 \leqslant i \leqslant k-1)$ are mutually disjoint. Choose two closed balls B' and B of 0 in $V_1 \oplus V_2$ such that $\{x''_i\}_1^k \cup \{\sigma x''_i\}_1^{k-1} \subset \psi^*(B') \subsetneqq \psi^*(B)$. Since $\psi^*(B)$ is a differentiable manifold, as in Lemma 3.3 there is an onto homeomorphism $\eta_1\colon \psi^*(B) \longleftrightarrow$ such that $d(\eta_1(x), x) < \delta$ $(x \in \psi^*(B))$, $\eta_1(x) = x$ for $x \in \psi^*(B)\backslash\psi^*(B')$ and $\eta_1\sigma(x''_i) = x''_{i+1}$ $(1 \leqslant i \leqslant k-1)$. Choose a small open subgroup F' of F such that $\psi^*(B + B) \cap F' = \{0\}$ and $\psi^*(B) + F'$ is a closed neighborhood of X. Define a map $\eta'_1\colon \psi^*(B) + F' \longleftrightarrow$ by $\eta'_1(x + y) = \eta_1(x) + y$ $(x \in \psi^*(B)$ and $y \in F')$. Then it is clear that η'_1 is 1–1 and onto. Let η be a map from X onto itself defined by

$$\eta(x) = \begin{cases} \eta'_1(x) & \text{if } x \in \psi^*(B) + F' \\ x & \text{if } x \notin \psi^*(B) + F' . \end{cases}$$

It is easily checked that $\eta: X \hookleftarrow$ is an onto homeomorphism having the properties; $d(\eta(x), x) < \delta$ for $x \in X$, $\eta(x) = x$ on $\psi^*(B') + F$ and $\eta\sigma(x_i'') = x_{i+1}''$ $(1 \leqslant i \leqslant k-1)$. Put $\varphi = \eta \circ \sigma$. Since $d(\varphi(x), \sigma(x)) < \delta$ for $x \in X$, we have $\{\varphi^i(x)\} \in \mathrm{Orb}^\delta(\sigma)$. By the property for $\delta > 0$, there is a continuous map $h: X \hookleftarrow$ such that $h \circ \varphi = \sigma \circ h$ and $d(h(x), x) < \varepsilon$ for $x \in X$. If $x = h(x_0'')$, then for $0 \leqslant i \leqslant k$

$$\begin{aligned} d(\sigma^i(x), x_i) &= d(\sigma^i(h(x_0'')), x_i) = d(h(\varphi^i(x_0''), x_i) \\ &\leqslant d(h(x_i''), x_i'') + d(x_i'', x_i') + d(x_i', x_i) < 2\varepsilon , \end{aligned}$$

which shows $x \in \mathrm{Tr}^{2\varepsilon}(\{x_i\}_0^k, \sigma)$. Since k is arbitrary, (X, σ) has the P.O.T.P.

It remains to prove the statement (B) $\Rightarrow$ (A). First we shall prepare the following two lemmas.

LEMMA 3.4. *Let α_0 be as in* $(*)$. *For every $f \in \mathscr{H}(X)$ with* $\max\{d(\sigma, f), d(\sigma^{-1}, f^{-1})\} < \alpha_0/2$, *there exist $y_0 \in F$ and $f_0 \in \mathscr{H}(X)$ with $f_0(\psi(\boldsymbol{R}^r)) = \psi(\boldsymbol{R}^r)$ such that $f(x) = y_0 + f_0(x)$ for $x \in X$.*

Proof. Put $g(x) = f(x) - y_0'$ $(x \in X)$ where $y_0' = f(0)$. Since $d(f(0), \sigma(0)) = d(y_0', 0) < \alpha_0/2$, we get $d(g(x), f(x)) < \alpha_0/2$ $(x \in X)$. Notice that $\{x \in X: d(x, 0) < \alpha_0\} \subset \psi B(\alpha_0) \oplus F$ (see $(**)$). Put $\kappa(x) = g(x) - \sigma(x)$ $(x \in X)$. Since $d(\kappa(x), 0) \leqslant d(g(x), f(x)) + d(f(x), \sigma(x)) < \alpha_0$ for all $x \in X$, we have $\kappa(x) \in \psi B(\alpha_0) \oplus F$ for $x \in X$ and so $\kappa(\psi \boldsymbol{R}^r) \subset \psi B(\alpha_0) \oplus F$. Since $\psi(\boldsymbol{R}^r)$ contains the identity 0 and $\kappa(\psi \boldsymbol{R}^r)$ is connected, we have $\kappa(\psi \boldsymbol{R}^r) \subset \psi B(\alpha_0)$, from which $g(\psi \boldsymbol{R}^r) \subset \psi(\boldsymbol{R}^r)$. In the same way, it follows that $g^{-1}(\psi \boldsymbol{R}^r) \subset \psi(\boldsymbol{R}^r)$, so that $g(\psi \boldsymbol{R}^r) = \psi(\boldsymbol{R}^r)$. Since $y_0' \in X = \psi(\boldsymbol{R}^r) + F$, y_0' splits into the sum $y_0' = \psi(v) + y_0$ with $\psi(v) \in \psi(\boldsymbol{R}^r)$ and $y_0 \in F$. Put $f_0(x) = \psi(v) + g(x)$ $(x \in X)$. Then $f_0(x)$ satisfies all the conditions of the lemma.

LEMMA 3.5 ([10]). *Let Y be a metric space such that every bounded set is relatively compact. Let f: $Y \hookleftarrow$ be a homeomorphism with the* P.O.T.P.. *If (Y, f) is expansive, then (Y, f) is topologically stable.*

We shall give here a proof due to A. Morimoto [10]. The proof will be used in proving the statement (B) $\Rightarrow$ (A). For every $\varepsilon > 0$, there is $\delta > 0$ such that $\{x_i\} \in \mathrm{Orb}^\delta(f)$ implies $\mathrm{Tr}^\varepsilon(\{x_i\}, f) = \phi$. We can assume $\varepsilon < \varepsilon_0/4$, where ε_0 is an expansive constant of f. Take a $g \in \mathscr{H}(Y)$ with $d(g, f) < \delta$. We shall prove that there exists a continuous map h: $Y \hookleftarrow$ having the property in the definition of topological stability. Take a point $x \in Y$. It is easy to see that $\{g^i(x)\} \in \mathrm{Orb}^\delta(f)$. Hence there is $y \in$

$\mathrm{Tr}^{\varepsilon}(\{g^i(x)\}, f)$; i.e. $d(f^i(y), g^i(x)) \leqslant \varepsilon$ for $i \in Z$. If $y' \in \mathrm{Tr}^{\varepsilon}(\{g^i(x)\}, f)$, then we have

$$d(f^i(y), f^i(y')) \leqslant d(f^i(y), g^i(x)) + d(g^i(x), f^i(y')) \leqslant 2\varepsilon < \varepsilon_0 ,$$

which implies $y = y'$. Thus by putting $h(x) = y$, we get a well-defined map $h\colon Y \hookleftarrow$ with the property

$$d(f^i(h(x)), g^i(x)) \leqslant \varepsilon \qquad \text{for } i \in Z . \tag{1}$$

Putting $i = 0$ in (1) we get

$$d(h(x), x) \leqslant \varepsilon \qquad \text{for } x \in Y . \tag{2}$$

Next we have, again by (1) for x and $g(x)$,

$$\begin{aligned} d(f^i(f(h(x))), f^i(h(g(x)))) \leqslant\; & d(f^i(f(h(x))), g^i(g(x))) \\ & + d(f^i(h(g(x))), g^i(g(x))) \leqslant 2\varepsilon < \varepsilon_0 \end{aligned}$$

for every $i \in Z$, which implies $f(h(x)) = h(g(x))$. Finally we shall prove the continuity of h. Assume that h is not continuous at $x_0 \in Y$. Then, there is a sequence $x_\nu \to x_0$ $(\nu \to \infty)$ such that $y_\nu = h(x_\nu)$ does not tend $y_0 = h(x_0)$ as $\nu \to \infty$. Since $\{x_\nu\}$ is bounded and $d(h(x_\nu), x_\nu) \leqslant \varepsilon$ for $\nu > 0$, the set $\{h(x_\nu)\}$ is also bounded. Hence we can assume, by taking a subsequence if necessary, that $y_\nu \to y_0' \neq y_0$ $(\nu \to \infty)$. Since f is expansive, there is $k \in Z$ such that $d(f^k(y_0'), f^k(y_0)) \geqslant \varepsilon_0$. Fixing k, we can find $\nu_0 > 0$ such that for $\nu > \nu_0$

$$d(f^k(y_\nu), f^k(y_0')) < \varepsilon_0/4 , \tag{3}$$

since f^k is continuous and $y_\nu \to y_0'$ $(\nu \to \infty)$. We can assume

$$d(g^k(x_\nu), g^k(x_0)) < \varepsilon \quad (\nu \geqslant \nu_0) , \tag{4}$$

since g^k is continuous and $x_\nu \to x_0$ $(\nu \to \infty)$. Now we have, using (2) and (4),

$$\begin{aligned} d(f^k(y_\nu), f^k(y_0)) &= d(f^k(h(x_\nu)), h(g^k(x_0))) \\ &= d(h(g^k(x_\nu)), h(g^k(x_0))) \\ &\leqslant d(h(g^k(x_\nu)), g^k(x_\nu)) + d(g^k(x_\nu), g^k(x_0)) \\ &\quad + d(g^k(x_0), h(g^k(x_0))) < 3\varepsilon \end{aligned}$$

and hence by (3) we obtain

$$\begin{aligned} \varepsilon_0 \leqslant d(f^k(y_0'), f^k(y_0)) &\leqslant d(f^k(y_0'), f^k(y_\nu)) + d(f^k(y_\nu), f^k(y_0)) \\ &\leqslant \varepsilon_0/4 + 3\varepsilon < \varepsilon_0 , \end{aligned}$$

which is a contradiction. This completes the proof of Lemma 3.5.

Proof of the statement (B) $\Rightarrow$ (A). Remark that $\psi^*(V_1 \oplus V_2) = \psi(\boldsymbol{R}^r)$ (by Lemma D (i)). By Theorem 2, $(\boldsymbol{R}^r, \gamma)$ is hyperbolic. Hence $(\boldsymbol{R}^r, \gamma)$, and so $(V_1 \oplus V_2, \gamma)$, is expansive and has the P.O.T.P. (see Theorem A). Using Lemma 3.5, we get that $(V_1 \oplus V_2, \gamma)$ is topologically stable; i.e. take $\varepsilon \in (0, \alpha_0/2)$ such that $\gamma B(3\varepsilon) \subset B(\alpha_0)$ and $\gamma^{-1}B(3\varepsilon) \subset B(\alpha_0)$ and let $\delta > 0$ ($\delta < \varepsilon$) be the number with the property of topological stability.

Take $f \in \mathscr{H}(X)$ with $d(f, \sigma) < \delta$ (Then we may assume that the number δ is chosen such that $d(f^{-1}, \sigma^{-1}) < \alpha_0/2$). By Lemma 3.4 there are $y_0 \in F$ and $f_0 \in \mathscr{H}(X)$ such that $f_0(\psi(\boldsymbol{R}^r)) = \psi(\boldsymbol{R}^r)$ and $f(x) = f_0(x) + y_0$ for $x \in X$. Since $\overline{\psi(\boldsymbol{Z}^r)} = F$ by Lemma D (iii), we can choose in $\psi(\boldsymbol{Z}^r)$ a sequence $\{y_n\}_{n \geqslant 1}$ such that $y_n \to y_0$ as $n \to \infty$. Put $f_n(x) = y_n + f_0(x)$. Then $d(f_n, \sigma) < \delta$ for n large enough. Fix such an integer n and define $\hat{f}_n(v) = \psi^{*-1} f_n \psi^*(v)$ $(v \in V_1 \oplus V_2)$. Then we claim that $\hat{f}_n \colon V_1 \oplus V_2 \longleftrightarrow$ is uniformly continuous. Indeed, we denote by $F(\varepsilon)$ a closed neighborhood with the radius ε of 0 in F. Since $f_n \colon X \longleftrightarrow$ is uniformly continuous, for every $\lambda \in (0, \alpha_0]$ there is $\alpha > 0$ with $\alpha < \lambda$ such that for every $v \in V_1 \oplus V_2$

$$f_n(\psi^* B(\alpha) \oplus F(\alpha) + \psi^*(v)) \subset \psi^* B(\lambda) \oplus F(\lambda) + f_n \psi^*(v),$$

from which

$$f_n(\psi^* B(\alpha) + \psi^*(v)) \subset \psi^* B(\lambda) + f_n \psi^*(v).$$

Hence, $\hat{f}_n(B(\alpha) + v) \subset B(\lambda) + \hat{f}_n(v)$; i.e. our requirement was obtained.

Let d_0 be a metric of $V_1 \oplus V_2$ defined as in Section 1. Since $\gamma = \psi^{-1} \circ \sigma \circ \psi$, we have

$$\delta > d(\sigma\psi^*(v), f_n\psi^*(v)) = d_0(\psi^{*-1}\sigma\psi^*(v), \psi^{*-1}f_n\psi^*(v)) = d_0(\gamma(v), \hat{f}_n(v)),$$

so that $\{\hat{f}_n(v)\} \in \mathrm{Orb}^\delta(\gamma)$ for $v \in V_1 \oplus V_2$. Hence there is $w \in V_1 \oplus V_2$ with $d_0(\hat{f}_n^j(v), \gamma^j(w)) < \varepsilon$ $(j \in \boldsymbol{Z})$ since $(V_1 \oplus V_2, \gamma)$ has the P.O.T.P.. Put $w = \hat{h}_n(v)$. Since $(V_1 \oplus V_2, \gamma)$ is expansive, from the proof of Lemma 3.5 it follows that $\hat{h}_n \colon V_1 \oplus V_2 \longleftrightarrow$ is a continuous map such that $\hat{h}_n \circ \hat{f}_n = \alpha \circ \hat{h}_n$ and $d_0(\hat{h}_n, \mathrm{id}) < \varepsilon$. Put $h_n = \psi^* \circ \hat{h}_n \circ \psi^{*-1}$. Obviously, $h_n \circ f_n = \sigma \circ h_n$ on $\psi(\boldsymbol{R}^r)$ and $d(h_n(x), x) < \varepsilon$ for $x \in \psi(\boldsymbol{R}^r)$.

We now prove that h_n is uniformly continuous. Since $d(\hat{f}_n^j(v), \gamma^j \hat{h}_n(v)) < \varepsilon$ for $j \in \boldsymbol{Z}$ and $v \in V_1 \oplus V_2$, we have

$$\psi^{*-1} f_n^j \psi^*(v) - \gamma^j \hat{h}_n(v) = \hat{f}_n^j(v) - \gamma^j \hat{h}_n(v) \in B(\varepsilon),$$

so that for all $v \in V_1 \oplus V_2$ and all $j \in \boldsymbol{Z}$

$$f_n^j \psi^*(v) - \sigma^j h_n \psi^*(v) \in \psi^* B(\varepsilon)\ . \tag{5}$$

Since $(\boldsymbol{R}^r, \gamma)$ is expansive, it is easily checked that for every $\lambda > 0$ ($\lambda < \varepsilon$) there is $N \geqslant 1$ such that $d(h_n(x), h_n(y)) < \lambda$ when $\sigma^j h_n(x) - \sigma^j h_n(y) \in \psi^* B(3\varepsilon)$ for j with $|j| \leqslant N$. Take $\alpha > 0$ such that if $d(x, y) < \alpha$ for $x, y \in \psi(\boldsymbol{R}^r)$ then $\max\{d(f_n^j(x), f_n^j(y)): -N \leqslant j \leqslant N\} < \lambda$. Then for j with $|j| \leqslant N$,

$$\begin{aligned} d(\sigma^j h_n(x), \sigma^j h_n(y)) \leqslant\ & d(\sigma^j h_n(x), f_n^j(x)) + d(f_n^j(x), f_n^j(y)) \\ & + d(f_n^j(y), \sigma^j h_n(y)) < 2\varepsilon + \lambda < 3\varepsilon\ , \end{aligned}$$

which shows that $d(h_n(x), h_n(y)) < \lambda$. Indeed, fix $y \in \psi(\boldsymbol{R}^r)$ and put $\kappa_y(x) = f_n^j(x) - f_n^j(y)$. Then $\kappa_y(W(\alpha) + y) \subset \psi^* B(\lambda) \oplus F(\lambda)$ where $W(\alpha) = \psi^* B(\alpha) \oplus F(\alpha)$. Since $\kappa_y(\psi(\boldsymbol{R}^r)) = \psi(\boldsymbol{R}^r)$ and $\psi(\boldsymbol{R}^r) = \cup\, \kappa_y(W(\alpha) + y)$, we have $\kappa_y(x) \in \kappa_y(W(\alpha) + y) \subset \psi^* B(\lambda)$ and hence $\sigma^j h_n(x) - \sigma^j h_n(y) \in \psi^* B(3\varepsilon)$.

Therefore h_n is uniformly extended to a continuous map from X into itself. We shall denote it by the same symbol. By using (5) we have for m and n large enough and for all $x \in X$

$$\sigma^j\{h_n(x) - h_m(x)\} + \{f_m^j(x) - f_n^j(x)\} \in \psi^* B(2\varepsilon)\ .$$

Since for fixed j

$$\lim_{n, m \to \infty} d(f_n^j, f_m^j) = 0\ ,$$

there is $N(j) > 0$ such that $f_n^j(x) - f_m^j(x) \in \psi^* B(\varepsilon)$ for n, $m \geqslant N(j)$ and $x \in X$. Hence for n, $m \geqslant N(j)$ and $x \in X$

$$h_n(x) - h_m(x) \in \sigma^{-j} \psi^* B(3\varepsilon)\ . \tag{6}$$

As before we have $B(3\varepsilon) = B(3\varepsilon)^u \oplus B(3\varepsilon)^s$ where $B(3\varepsilon)^u = B(3\varepsilon) \cap E^u$ and $B(3\varepsilon)^s = B(3\varepsilon) \cap E^s$. Hence $\psi^* B(3\varepsilon) = \psi^* B(3\varepsilon)^u \oplus \psi^* B(3\varepsilon)^s$. Since $\gamma B(3\varepsilon) \subset B(\alpha_0)$ and $\gamma^{-1} B(3\varepsilon) \subset B(\alpha_0)$, obviously $\sigma \psi^* B(3\varepsilon) \subset \psi^* B(\alpha_0)$ and $\sigma^{-1} \psi^* B(3\varepsilon) \subset \psi^* B(\alpha_0)$. It follows easily that $\sigma \psi^* B(3\varepsilon)^u \subset \psi^* B(\alpha_0)^u$ and $\sigma^{-1} \psi^* B(3\varepsilon)^s \subset \psi^* B(\alpha_0)^s$. Hence $\bigcap_{-\infty}^{\infty} \sigma^j \psi^* B(3\varepsilon) = \bigcap_{-\infty}^{\infty} \sigma^j \psi^* B(3\varepsilon)^u \oplus \bigcap_{-\infty}^{\infty} \sigma^j \psi^* B(3\varepsilon)^s = \{0\}$. From (6) we have for n, $m \geqslant \max\{N(j): -i \leqslant j \leqslant i\}$ and $x \in X$

$$h_n(x) - h_m(x) \in \bigcap_{j=-i}^{i} \sigma^j \psi^* B(3\varepsilon)\ .$$

For any open neighborhood U of 0 there is $i > 0$ such that $\bigcap_{j=-i}^{i} \sigma^j \psi^* B(3\varepsilon) \subset U$. This implies that $\lim_{n, m \to \infty} d(h_n, h_m) = 0$; i.e. $\{h_n\}$ converges uniformly to some continuous map h of X. Since $h_n \circ f_n = \sigma \circ h_n$ on X and $d(h_n, \mathrm{id}) < \varepsilon$ for an arbitrary large n, it follows that $h \circ f = \sigma \circ h$ on X and $d(h, \mathrm{id}) \leqslant \varepsilon$. Therefore (X, σ) is topologically stable. The proof of Theorem 1 is completed.

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