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ON THE EXISTENCE OF OPTIMAL CONTROL FOR CONTROLLED STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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§ 1. Introduction

In this paper we are concerned with stochastic control problems of the following kind. Let $Y(t)$ be a d'-dimensional Brownian motion defined on a probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ and $u(t)$ an admissible control. We consider the Cauchy problem of stochastic partial differential equations (SPDE in short)

$$(1.1) \qquad \begin{cases} dp(t, x) = L(Y(t), u(t))p(t, x)dt + M(Y(t))p(t, x)dY(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^d, \ t > 0 \\ p(0, x) = \phi(x) \end{cases}$$

where $L(y, u)$ is the 2nd order elliptic differential operator and $M(y)$ the 1st order differential operator.

By a solution $p(t) = p^u(t)$, we mean $H^{1(\dagger)}$-valued $\mathscr{F}_t$-adapted process which satisfies

$$(p(t), \eta) = (\phi, \eta) + \int_0^t \langle L(Y(s), u(s))p(s), \eta\rangle ds$$
$$+ \int_0^t (M(Y(s))p(s), \eta)dY(s), \quad t \geq 0$$

for any smooth η where $\langle \cdot, \cdot \rangle$ is the pairing between H^{-1} and H^1 and $(\cdot, \cdot)$ is $L^2(\mathbb{R}^d)$ inner product (see [4], [7]).

The SPDE (1.1) is related to the filtering, stochastic control with partial observation, population genetics etc. and investigated by Pardoux, Krylov & Rozovskii and Rozovskii & Shimizu, etc.

The purpose of this paper is to prove the existence of optimal controls for the following problem. Define a criterion $J(u)$ by

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(†) $H^l = H^l(\mathbb{R}^d)$ denotes the Sobolev space $W_2^l(\mathbb{R}^d)$, $l = 0, \pm 1, \cdots$.

$$J(u) = E[F(p^u) + G(p^u(T))] \tag{1.2}$$

where F and G are real valued functions on $L^2(0, T; L^2(\mathbb{R}^d))$ and $L^2(\mathbb{R}^d)$ respectively. Now we want to minimize $J(u)$ by a suitable choice of an admissible process u.

In Section 2 we will recall some known results in our convenient way and formulate our problem precisely. In Section 3 we will prove that the solution p^u depends on u continuously which derives the existence of optimal control [Theorem 3.2]. In Section 4 we apply our results to stochastic control with partial observation, where an observation noise may depend on a state noise.

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§ 2. Notation and preliminaries

We assume the following conditions (A.1)~(A.3).

(A.1)
$$\begin{aligned} b&: \mathbb{R}^d \times \mathbb{R}^{d'} \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^L \\ \sigma&: \mathbb{R}^d \times \mathbb{R}^{d'} \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^{d'} \\ a&: \mathbb{R}^d \times \mathbb{R}^{d'} \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d \\ h&: \mathbb{R}^d \times \mathbb{R}^{d'} \longrightarrow \mathbb{R}^{d'} \end{aligned}$$

are bounded and continuous and a is symmetric.

(A.2) There exists $\delta > 0$ such that

$$2a(x, y) - 3\sigma(x, y)\sigma^*(x, y) \geq \delta I \qquad \text{for any } (x, y) \in \mathbb{R}^d \times \mathbb{R}^{d'}$$

where σ^* is the transposed matrix of σ.

(A.3)
$$\begin{aligned} &a(\cdot, y),\ \sigma(\cdot, y) \text{ are } C^{\hat{m}+1}\text{-class in } x \in \mathbb{R}^d, \\ &h(\cdot, y),\ b(\cdot, y) \text{ are } C^{\hat{m}}\text{-class in } x \in \mathbb{R}^d, \end{aligned}$$

and their derivatives are bounded and continuous in $(x, y) \in \mathbb{R} \times \mathbb{R}^{d'}$, where $\hat{m} = \max\{2, m\}$ and m is a given nonnegative integer.

Let Γ be a convex and compact subset of $\mathbb{R}^L$.

DEFINITION 2.1. $\mathscr{A} = (\Omega, \mathscr{F}, P, Y, u)$ is called an admissible system, if $(\Omega, \mathscr{F}, P)$ is a probability space and u is a Γ-valued measurable process and Y is a d'-dimensional $(\mathscr{F}_t)$-Brownian motion on $(\Omega, \mathscr{F}, P)$, where $\mathscr{F}_t = \sigma\left\{Y(s), \int_0^s u(\tau)d\tau; s \leq t\right\}$.

$\mathfrak{A}$ denotes the totality of admissible systems.

For $\mathscr{A} \in \mathfrak{A}$, $\pi^{\mathscr{A}}$ denotes the image measure of (Y, u) on $C(0, T; \mathbb{R}^{d'}) \times L^2(0, T; \Gamma)$.

Endowing the uniform topology on $C(0, T; \mathbb{R}^{d'})$ and the weak topology on $L^2(0, T; \Gamma)$, we have

LEMMA 2.1. *$\{\pi^{\mathscr{A}}; \mathscr{A} \in \mathfrak{A}\}$ is compact under the Prokhorov metric. (See Fleming & Pardoux* [2] *Lemma* 2.3.)

Define $L(y, u) \in \mathscr{L}(H^1, H^{-1})$, $M^k(y) \in \mathscr{L}(H^1, L^2(\mathbb{R}^d))$ $(k = 1, \cdots, d', y \in \mathbb{R}^{d'}, u \in \Gamma)$ by

$$\langle L(y, u)p, q\rangle = -\sum_{i,j=1}^{d}\left(a_{ij}(\cdot, y)\frac{\partial p}{\partial x_i}, \frac{\partial q}{\partial x_j}\right) + \sum_{j=1}^{d}\left(\tilde{b}_j(\cdot, y, u)p, \frac{\partial q}{\partial x_j}\right) \tag{2.1}$$

$$(M^k(y)p, \eta) = -\sum_{i=1}^{d}\left(\sigma_{ik}(\cdot, y)\frac{\partial p}{\partial x_i}, \eta\right) + (\tilde{h}_k(\cdot, y)p, \eta) \tag{2.2}$$

for $p, q \in H^1$ and $\eta \in L^2(\mathbb{R}^d)$, where $(\cdot, \cdot) =$ the inner product in $L^2(\mathbb{R}^d)$, $\langle\cdot, \cdot\rangle =$ the duality pairing between H^{-1} and H^1 and

$$\tilde{b}_j(x, y, u) = \sum_{l=1}^{L} b_{jl}(x, y)u_l - \sum_{i=1}^{d}\frac{\partial a_{ij}}{\partial x_i}(x, y)$$

$$\tilde{h}_k(x, y) = h(x, y) - \sum_{i=1}^{d}\frac{\partial \sigma_{ik}}{\partial x_i}(x, y)\,.$$

By (A.1)~(A.3), there exists $\alpha > 0$ and $\lambda \in \mathbb{R}$ such that

$$-2\langle L(y, u)p, p\rangle + \lambda\|p\|_0^2 \geq \alpha\|p\|_1^2 + 3\sum_{k=1}^{d'}\|M^k(y)p\|_0^2 \tag{2.3}$$

$$\text{for any } p \in H^1,\ y \in \mathbb{R}^d,\ u \in \Gamma$$

where $\|\cdot\|_l =$ the H^l-norm $(l = 0, \pm 1, \cdots)$ (for the proof, see § 2 of Krylov & Rozovskii [4]).

(2.3) is called the coercivity condition.

For an admissible system $\mathscr{A} = (\Omega, \mathscr{F}, P, Y, u)$, putting $L^{\mathscr{A}}(t) = L(Y(t), u(t))$ and $M^{\mathscr{A}k}(t) = M^k(Y(t))$, we consider the Cauchy problem of SPDE on $(\Omega, \mathscr{F}, P)$,

$$\begin{cases} dp(t) = L^{\mathscr{A}}(t)p(t)dt + M^{\mathscr{A}}(t)p(t)dY(t) \\ \qquad\qquad\qquad\qquad\qquad t > 0 \\ p(0) = \phi \in H^{\hat{m}} \end{cases} \tag{2.4}$$

where $M^{\mathscr{A}}(t) = (M^{\mathscr{A}1}(t), \cdots, M^{\mathscr{A}d'}(t))$.

DEFINITION 2.2. By a solution of SPDE (2.4), we mean an H^1-valued $\mathscr{F}_t$-adapted process $p(t)$ defined on $(\Omega, \mathscr{F}, P)$ such that

(1) $E\left[\int_0^T \|p(t)\|_1^2\, dt\right] < \infty$

(2) for any $\eta \in H^1$ and $t \in [0, T]$

$$(p(t), \eta) = (\phi, \eta) + \int_0^t \langle L^{\mathscr{A}}(s)p(s), \eta\rangle ds + \int_0^t (M^{\mathscr{A}}(s)p(s), \eta)\, dY(s) \tag{2.5}$$

holds.

By the coercivity condition (2.3), we have the following proposition. (See [5], [7].)

PROPOSITION 2.1. *For each* $\mathscr{A} \in \mathfrak{A}$, *the equation* (2.4) *has a unique solution* $p = p^{\mathscr{A}}$ *which satisfies*

$$p \in L^2((0, T) \times \Omega; H^{\hat{m}+1}) \cap L^2(\Omega; C(0, T; H^{\hat{m}})) \tag{2.7}$$

and

$$\begin{aligned} \|p(t)\|_0^2 = \|\phi\|_0^2 &+ 2\int_0^t \langle L^{\mathscr{A}}(s)p(s), p(s)\rangle ds \\ &+ 2\int_0^t (M^{\mathscr{A}}(s)p(s), p(s))\, dY(s) + \int_0^t \|M^{\mathscr{A}}p(s)\|_0^2 ds\,. \end{aligned} \tag{2.8}$$

The solution $p = p^{\mathscr{A}}$ *of the SPDE* (2.4) *is called the response for* $\mathscr{A}$.

Remark 2.1. We can apply the results of Pardoux [7] also to the triplet (V, H, V^*), where $V = H^{l+1}$, $H = H^l$ and $V^* = H^{l-1}$ $(l = 0, 1, \cdots, \hat{m})$. Define $\tilde{L}(y, u) \in \mathscr{L}(H^{l+1}, H^{l-1})$, $\tilde{M}(y) \in \mathscr{L}(H^{l+1}, H^l)$ similarly to $L(y, u)$, $M(y)$, where we replace $\langle\cdot, \cdot\rangle$ and $(\cdot, \cdot)$ by "$\langle\cdot, \cdot\rangle_l$ = the duality pairing between H^{l-1} and H^{l+1}" and "$(\cdot, \cdot)_l$ = the inner product in H^l" repsectively in (2.1), (2.2). Then the coercivity condition holds. (In (2.3), $\|\cdot\|_0$ and $\|\cdot\|_1$ are replaced by $\|\cdot\|_l$ and $\|\cdot\|_{l+1}$ respectively.) Appealing to Krylov & Rozovskii [4], the solution p of (2.4) turns out a unique solution of SPDE (2.9)

$$\begin{cases} dp(t) = \tilde{L}(Y(t), u(t))p(t)dt + \tilde{M}(Y(t))p(t)dY(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad t > 0 \\ p(0) = \phi\,. \end{cases} \tag{2.9}$$

Moreover $p(t)$ satisfies similar equality to (2.8). (i.e. "0" is replaced by "l".)

Let $F : L^2(0, T; H^{m+1}) \to \mathbb{R}$ and $G : H^m \to \mathbb{R}$ be weakly continuous functions.

For $\mathscr{A} \in \mathfrak{A}$, we define the pay-off function $J(\mathscr{A})$ by

$$J(\mathscr{A}) = E[F(p^{\mathscr{A}}) + G(p^{\mathscr{A}}(T))]\,. \tag{2.10}$$

We want to minimize its value by a suitable choice of $\mathscr{A} \in \mathfrak{A}$.

§ 3. Existence of optimal control

First of all we will prove that the solution $p^{\mathscr{A}}$ of (2.4) depends on $\mathscr{A}$ continuously.

THEOREM 3.1. *If $\pi^{\mathscr{A}(n)} \to \pi^{\mathscr{A}}$ in law, then*

(3.1) $p^{\mathscr{A}(n)} \longrightarrow p^{\mathscr{A}}$ *in law as $L^2(0, T; H^{m+1})$-random variable*

and

(3.2) $p^{\mathscr{A}(n)}(T) \longrightarrow p^{\mathscr{A}}(T)$ *in law as H^m-random variable,*

where we endow the weak topologies on $L^2(0, T; H^{m+1})$ and H^m.

For the proof we need the following two lemmas.

LEMMA 3.1. There exists a constant $K > 0$ such that

$$E\left\{\int_0^T \|p^{\mathscr{A}}(t)\|_{l+1}^2\, dt\right\} \leq K\|\phi\|_l^2 \tag{3.3}$$

$$E\{\sup_{0\leq t\leq T} \|p^{\mathscr{A}}(t)\|_l^2\} \leq K\|\phi\|_l^2 \tag{3.4}$$

$$E\left\{\int_0^T \|p^{\mathscr{A}}(t)\|_l^4\, dt\right\} \leq K\|\phi\|_l^4 \tag{3.5}$$

for any $\mathscr{A} \in \mathfrak{A}$. $(l = 0, 1, \cdots, \hat{m})\,.$

According to [6] we introduce the spaces $\mathscr{H}_\gamma(D)$ and $\mathscr{H}_\gamma(T, D)$ as follows. Set $\hat{\psi}(\cdot, x) =$ the Fourier transformation in t of $\psi(\cdot, x)$, $\|\cdot\|_{2,D} =$ the $H^2(D)$-norm and $\|\cdot\|_* =$ the norm of the dual space $(H^2(D))^*$, where we identify $H^1(D)$ with its dual space.

$$\mathscr{H}_\gamma(D) = \left\{\psi \in L^2(-\infty, \infty; H^2(D));\ \int_{-\infty}^{\infty} |\tau|^{2\gamma} \|\hat{\psi}(\tau)\|_*^2\, d\tau < \infty\right\}$$

where

$$\|\psi\|_{\mathscr{H}_\gamma(D)} = \left\{\int_{-\infty}^{\infty} \|\psi(t)\|_{2,D}^2\, dt + \int_{-\infty}^{\infty} |\tau|^{2\gamma} \|\hat{\psi}(\tau)\|_*^2\, d\tau\right\}^{1/2}$$

$$\mathscr{H}_\gamma(T, D) = \{\psi|_{[0,T]};\ \psi \in \mathscr{H}_\gamma(D)\}$$

where

$$\|\psi\|_{\mathscr{H}_\gamma(T,D)} = \inf\{\|\varphi\|_{\mathscr{H}_\gamma(D)}; \varphi(t) = \psi(t) \text{ a.e. on } [0, T]\}.$$

Remark 3.1. If D is a bounded and open subset of $\mathbb{R}^d$ with a smooth boundary, then, by the compactness lemma ([6] p. 60) the imbedding: $\mathscr{H}_\gamma(T, D) \to L^2(0, T; H^1(D))$ is compact.

LEMMA 3.2. *Let* $0 < \gamma < 1/4$, *then for each* $\mathscr{A} \in \mathfrak{A}$,

$$p^{\mathscr{A}} \in \mathscr{H}_\gamma(T, D) \text{ a.s.}$$

and there exists $K > 0$ *such that*

$$E[\|p^{\mathscr{A}}\|^2_{\mathscr{H}_\gamma(T,D)}] \leq K\|\phi\|^2_2 \qquad \forall \mathscr{A} \in \mathfrak{A}. \tag{3.6}$$

Proof of Lemma 3.1. (3.3) and (3.4) are easy variants of Corollary 2.2 of Krylov & Rozovskii [4]. Now we will show (3.5). Since the response p is the solution of (2.9), using Itô's formula, we get

$$\begin{aligned} \|p(t)\|^4_l &= \|\phi\|^4_l + 4\int_0^t \|p(s)\|^2_l \langle \tilde{L}(s)p(s), p(s)\rangle_l \, ds \\ &\quad + 2\int_0^t \|p(s)\|^2_l \|\tilde{M}(s)p(s)\|^2_l \, ds + 4\sum_{k=1}^{d'} \int_0^t (\tilde{M}^k(s)p(s), p(s))^2_l \, ds \\ &\quad + 4\int_0^t \|p(t)\|^2_l (\tilde{M}(s)p(s), p(s))_l \, dY(s) \end{aligned} \tag{3.7}$$

where $\tilde{L}(t) = \tilde{L}(Y(t), u(t))$ and $\tilde{M}(t) = \tilde{M}(Y(t))$.

Hence, using the coercivity condition, we have

$$\begin{aligned} E[\|p(t)\|^4_l] - \|\phi\|^4_l &= 2E\left[\int_0^t \|p(s)\|^2_l \{2\langle \tilde{L}(s)p, p\rangle_l + \|\tilde{M}(s)p\|^2_l\} ds\right] \\ &\quad + 4E\left[\int_0^t \sum_{k=0}^{d'} (\tilde{M}(s)p, p)^2_l \, ds\right] \\ &\leq 2E\left[\int_0^t \|p(s)\|^2_l \{\lambda'\|p(s)\|^2_l - \alpha'\|p(s)\|^2_{l+1}\} \, ds\right] \\ &\leq 2\lambda' E\left[\int_0^t \|p(s)\|^4_l \, ds\right]. \end{aligned} \tag{3.8}$$

So the Gronwall's inequality derives (3.5).

Proof of Lemma 3.2. For the convenience, we extend $p(t)$ on $(-\infty, \infty)$ in the following way

$$\begin{aligned} p(t) &= p(t), \ t \in [0, T] \\ &= 0, \quad t \in (-\infty, \infty)\backslash[0, T]. \end{aligned}$$

Since $p(t)$ is a solution of (2.9), applying Itô's formula, we obtain

$$
\begin{aligned}
(3.9)\qquad 2\pi i\tau(\hat{p}(\tau),\eta)_2 &= (\phi,\eta)_2 - (p(T),\eta)_2\exp\{-2\pi i\tau T\} \\
&\quad + \langle\widehat{\tilde{L}p}(\tau),\eta\rangle_2 + \int_0^T \exp\{-2\pi i\tau t\}(\tilde{M}(t)p,\eta)_2\,dY(t)
\end{aligned}
$$

for any $\eta \in H^3$.

Let $\{\eta_k\}_{k\geq 1}$ be an orthonormal basis in H^3. Using (3.3), (3.4) and (3.9), we have

$$
(3.10)\quad 4\pi^2\tau^2 E[\|\hat{p}(\tau)\|_1^2] = 4\pi^2\tau^2\sum_{k=1}^{\infty} E\{|(\hat{p}(\tau),\eta_k)_2|^2\} \leq K_1\|\phi\|_2^2 + K_2 E[\|\widehat{\tilde{L}p}(\tau)\|_1^2]\,.
$$

Let $0<\gamma<1/4$ and $0<\kappa<3/2$, then

$$
\begin{aligned}
\int_{-\infty}^{\infty} E\{|\tau|^{2\gamma}\|\hat{p}(\tau)\|_*^2\}d\tau &\leq \int_{|\tau|\leq 1} E[\|\hat{p}(\tau)\|_1^2]d\tau + \int_{|\tau|\geq 1} E\left[\frac{2|\tau|^2}{1+|\tau|^\kappa}\|\hat{p}(\tau)\|_1^2\right]d\tau \\
(3.11)\qquad &\leq K_3\left\{E\left[\int_{-\infty}^{\infty}\|p(t)\|_1^2\,dt\right] + \int_{-\infty}^{\infty}\frac{d\tau}{1+|\tau|^\kappa}\|\phi\|_1^2 + E\left[\int_{-\infty}^{\infty}\|\tilde{L}(t)p\|_1^2\,dt\right]\right\} \\
&\leq K_4\|\phi\|_2^2\,.
\end{aligned}
$$

This concludes the lemma.

Remark 3.2. (3.5) implies the uniform integrability of

$$
\int_0^T \|p^{\mathscr{A}}(t)\|_1^2\,dt, \quad \mathscr{A}\in\mathfrak{A}\,.
$$

Remark 3.3. We define the metric d on $H = L^2(0,T;H^{m+1}(\mathbb{R}^d))$ by

$$
d(p,q) = \sum_{k=1}^{\infty}\frac{1}{2^k}\min(|(e_k, p-q)|, 1\} \quad p,q\in H
$$

where $(\cdot,\cdot)$ is the inner product on H and $\{e_k\}_{k=1}^{\infty}$ is the orthonormal basis on H. Then Lemma 3.1 and Prokhorov's theorem imply that the totality of image measure $p^{\mathscr{A}}$ $(\mathscr{A}\in\mathfrak{A})$ is relatively compact as a set of measures on the metric space (H,d).

On the other hand, on each bounded set of H the weak topology is metrizable by the metric d. Therefore, for any weakly closed set F of H, $F\cap\{q\in H;\|q\|\leq r\}$ $(r>0)$ is closed with respect to the metric d.

Under this observation, $\{p^{\mathscr{A}};\mathscr{A}\in\mathfrak{A}\}$ is relatively compact as a set of measures on H associated with the weak topology.

Proof of Theorem 3.1. Let D_k $(k=1,2,\cdots)$ be bounded and open subsets of $\mathbb{R}^d$ with smooth boundary, $\overline{D_k}\subset D_{k+1}$ and $\bigcup_{k=1}^{\infty}D_k = \mathbb{R}^d$. For an admissible system $\mathscr{A} = (\Omega,\mathscr{F},P,Y,u)$,

$$\mu^{\mathscr{A}} = \text{the image measure of } (Y, u, p^{\mathscr{A}}) \text{ on } S\,,$$
$$\mu_k^{\mathscr{A}} = \text{the image measure of } (Y, u, p^{\mathscr{A}}) \text{ on } S_k$$

where

$$S = C(0, T; \mathbb{R}^{d'}) \times L^2(0, T; \Gamma) \times L^2(0, T; H^{m+1}(\mathbb{R}^d))\,,$$

and

$$S_k = C(0, T; \mathbb{R}^{d'}) \times L^2(0, T; \Gamma) \times L^2(0, T; H^1(D_k))$$

endowing the weak topology on $L^2(0, T; H^{m+1}(\mathbb{R}^d))$ and the strong topology on $L^2(0, T; H^1(D_k))$. By the compactness of $\{\pi^{\mathscr{A}}; \mathscr{A} \in \mathfrak{A}\}$ and Remark 3.3, $\mathfrak{P} = \{\mu^{\mathscr{A}}; \mathscr{A} \in \mathfrak{A}\}$ is relatively compact. Moreover, by Lemma 3.2 and Remark 3.1, $\mathfrak{P}_k = \{\mu_k^{\mathscr{A}}; \mathscr{A} \in \mathfrak{A}\}$ is relatively compact.

Hence there exist a subsequence $\{\mathscr{A}(n')\}_{n'}$, a probability μ on S and a probability μ_k on S_k $(k = 1, 2, \cdots)$ such that

$$\mu^{\mathscr{A}(n')} \longrightarrow \mu \quad \text{in law as} \quad n' \longrightarrow \infty \tag{3.12}$$

and

$$\mu_k^{\mathscr{A}(n')} \longrightarrow \mu_k \quad \text{in law as} \quad n' \longrightarrow \infty\,. \tag{3.13}$$

By Skorohod's theorem, we can construct the S_k-valued random variables $(Y_{n'}, u_{n'}, p_{n'})$, (Y, u, p), $n' = 1, 2, \cdots$, on a probability space $(\Omega, \mathscr{F}, P)$ such that

(3.14) the law of $(Y_{n'}, u_{n'}, p_{n'}) = \mu_k^{\mathscr{A}(n')}$, $n' = 1, 2, \cdots$,

(3.15) the law of $(Y, u, p) = \mu_k$

and

(3.16) $(Y_{n'}, u_{n'}, p_{n'}) \longrightarrow (Y, u, p)$ almost surely $(n' \longrightarrow \infty)$

as S_k-valued random variables.

Now we will prove the following lemma.

LEMMA 3.3. *Let ψ: $[0, T] \to \mathbb{R}$ be an absolutely continuous function with $\psi' \in L^2(0, T)$ and $\psi(T) = 0$ and $\eta \in C_0^\infty(\mathbb{R}^d)$ with* $\mathrm{supp}(\eta) \subset D_k$, *then (Y, u, p) of* (3.16) *satisfies*

$$\begin{aligned}(\phi, \eta)\psi(0) &+ \int_0^T \psi'(t)(p(t), \eta)\,dt + \int_0^T \psi(t)\langle L(Y(t), u(t))p, \eta\rangle\,dt \\ &+ \int_0^T \psi(t)(M(Y(t))p, \eta)\,dY(t) = 0\,.\end{aligned} \tag{3.17}$$

Proof. Since $p_{n'}$ is the solution of the SPDE (2.4) for $(Y_{n'}, u_{n'})$, using Itô's formula to (2.5), we get

$$(3.17)_{n'} \quad \begin{aligned}(\phi, \eta)\psi(0) &+ \int_0^T \psi'(t)(p_{n'}(t), \eta)dt + \int_0^T \psi(t)\langle L(Y_{n'}(t), u_{n'}(t))p_{n'}, \eta\rangle dt \\ &+ \int_0^T \psi(t)(M(Y_{n'}(t))p_{n'}, \eta)dY_{n'}(t) = 0 .\end{aligned}$$

By Remark 3.2 and (3.16), we get

$$(3.18) \qquad E\left[\int_0^T \|p_{n'}(t) - p(t)\|^2_{1, D_k} dt\right] \longrightarrow 0 \quad (n' \to \infty)$$

Recalling "$\mathrm{supp}(\eta) \subset D_k$", we obtain

$$(3.19) \quad \begin{aligned}&\int_0^T \psi(t)\langle L(Y_{n'}(t), u_{n'}(t))p_{n'}, \eta\rangle dt \\ &\longrightarrow \int_0^T \psi(t)\langle L(Y(t), u(t))p, \eta\rangle dt \qquad \text{in } L^2(\Omega) .\end{aligned}$$

$$(3.20) \qquad \psi(t)(p_{n'}(t), \eta) \longrightarrow \psi(t)(p(t), \eta) \qquad \text{in } L^2([0, T] \times \Omega)$$

and

$$(3.21) \quad \psi(t)(M(Y_{n'}(t))p_{n'}, \eta) \longrightarrow \psi(t)(M(Y(t))p, \eta) \qquad \text{in } L^2([0, T] \times \Omega) .$$

For the proof of (3.19), putting

$$q_{n'}(t) = \psi(t)(b_{il}(\cdot, Y_{n'}(t))p_{n'}(t), \eta)$$
$$q(t) = \psi(t)(b_{il}(\cdot, Y(t))p(t), \eta)$$

and $u(t) = (u^1(t), \cdots, u^L(t))$, we have

$$(3.22) \quad \begin{aligned}&\int_0^T \psi(t)(b_{il}(\cdot, Y_{n'}(t))p_{n'}(t), \eta)u^l_{n'}(t)dt - \int_0^T \psi(t)(b_{il}(\cdot, Y(t))p(t), \eta)u^l(t)dt \\ &= \int_0^T u^l_{n'}(t)(q_{n'}(t) - q(t))dt + \int_0^T (u^l_{n'}(t) - u^l(t))q(t)dt .\end{aligned}$$

By (3.18), the 1st term of the right hand side of (3.22) converges to 0 in $L^2(\Omega)$. By Remark 3.2 and (3.16), we get

$$(3.23) \qquad E\left[\left\{\int_0^T (u^l_{n'}(t) - u^l(t))q(t)dt\right\}^2\right] \longrightarrow 0 .$$

This implies (3.19). (3.20) and (3.21) can be proved similarly. Moreover, combining (3.21) with (3.16), we get

$$(3.24) \quad \begin{aligned}&\int_0^T \psi(t)(M(Y_{n'}(t))p_{n'}, \eta)dY_{n'}(t) \\ &\longrightarrow \int_0^T \psi(t)(M(Y(t))p, \eta)dY(t) \qquad \text{in } L^2(\Omega) .\end{aligned}$$

Hence, by taking limit of $(3.17)_{n'}$, we obtain (3.17).

Let $i_k: S \to S_k$ be the canonical injection. Then by the definition

$$i_k(\mu^{\mathscr{A}(n')}) = \mu_k^{\mathscr{A}(n')} \quad \text{and} \quad i_k(\mu) = \mu_k \,. \tag{3.25}$$

Let $(\tilde{Y}, \tilde{u}, \tilde{p})$ be S-valued random variable whose law $= \mu$. Then (3.25) implies that the law of $(\tilde{Y}, \tilde{u}, \tilde{p}|_{D_k}) = \mu_k$.

Hence, by Lemma 3.3, $(\tilde{Y}, \tilde{u}, \tilde{p}|_{D_k})$ satisfies the equation (3.17). Noting that $\mathrm{supp}(\eta) \subset D_k$, we obtain

$$\begin{aligned}(\phi, \eta)\psi(0) &+ \int_0^T \psi'(t)(\tilde{p}(t), \eta)\,dt + \int_0^T \psi(t)\langle L(\tilde{Y}(t), \tilde{u}(t))\tilde{p}, \eta\rangle dt \\ &+ \int_0^T \psi(t)(M(\tilde{Y}(t))\tilde{p}, \eta)\,d\tilde{Y}(t) = 0\,.\end{aligned} \tag{3.26}$$

Since k is arbitrary, (3.26) holds for any $\eta \in C_0^\infty(\mathbb{R}^d)$.

By the same argument as Theorem 1.3 in [7], $\tilde{p}$ becomes a solution of SPDE (2.4) for $(\tilde{Y}, \tilde{u})$. Since the law of $(\tilde{Y}, \tilde{u}) = \pi^{\mathscr{A}}$, we get

$$\mu = \text{the law of } (\tilde{Y}, \tilde{u}, \tilde{p}) = \mu^{\mathscr{A}}\,. \tag{3.27}$$

This means that any convergent subsequence of $\{\mu^{\mathscr{A}(n')}\}$ converges to $\mu^{\mathscr{A}}$. Hence the original sequence $\{\mu^{\mathscr{A}(n)}\}$ converges to $\mu^{\mathscr{A}}$. So we get (3.1). Next we consider the law of $(Y, u, p^{\mathscr{A}}, p^{\mathscr{A}}(T))$ then by the similar argument we can prove (3.2).

THEOREM 3.2. *If F and G are bounded from below, then there exists an optimal admissible system $\tilde{\mathscr{A}} \in \mathfrak{A}$ that is*

$$\inf\{J(\mathscr{A});\, \mathscr{A} \in \mathfrak{A}\} = J(\tilde{\mathscr{A}})\,. \tag{3.28}$$

Proof. By theorem 3.1,

$$J_n(\mathscr{A}) = E[\min\{F(p^{\mathscr{A}}), n\} + \min\{G(p^{\mathscr{A}}(T)), n\}]$$

is continuous on $\mathfrak{A}$. Since $J(\mathscr{A})$ is the limit function of non-decreasing sequence $\{J_n(\mathscr{A})\}_{n=1}^\infty$, it is lower-semicontinuous on $\mathfrak{A}$. This concludes the theorem.

§ 4. Optimal control for partially observed diffusions

In this section we will apply Theorem 3.2 to the stochastic control problems for partially observed diffusions where an observation noise may depend on a state noise.

We assume the following conditions (A.4)$\sim$(A.6).

(A.4) $\hat{\sigma}: \mathbb{R}^d \times \mathbb{R}^{d'} \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ is bounded and continuous

(A.5) There exists $\delta > 0$ such that

$$\hat{\sigma}(x, y)\hat{\sigma}^*(x, y) - 2\sigma(x, y)\sigma^*(x, y) \geq \delta \mathrm{I} \qquad \text{for } \forall\ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d$$

(A.6) $\hat{\sigma}(\cdot, y)$ is C^3-class in $x \in \mathbb{R}^d$ and all derivatives are bounded and continuous in $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$.

Put $a(x, y) = (\hat{\sigma}(x, y)\hat{\sigma}^*(x, y) + \sigma(x, y)\sigma^*(x, y))/2$, then $a(x, y)$ and $\sigma(x, y)$ satisfy (A.2).

Now we will consider the optimal control problems of the following kind. Let $X(t)$ denote the state process being controlled, $Y(t)$ the observation process and $u(t)$ the control process. The state and observation processes are governed by the stochastic differential equations

$$\text{(4.1)} \quad \begin{cases} dX(t) = b(X(t), Y(t))u(t)dt + \hat{\sigma}(X(t), Y(t))d\hat{W}(t) + \sigma(X(t), Y(t))dW(t) \\ X(0) = \xi \end{cases}$$

and

$$\text{(4.2)} \qquad \begin{cases} dY(t) = h(X(t))dt + dW(t) \\ Y(0) = 0 \end{cases}$$

where $\hat{W}$ and W are independent Brownian motions with values in $\mathbb{R}^d$ and $\mathbb{R}^{d'}$ respectively on a probability space $(\Omega, \mathscr{F}, \hat{P})$.

The problem is to minimize a criterion of the form

$$\text{(4.3)} \qquad J(u) = \hat{E}\left[\int_0^T f(X(t))dt + g(X(T))\right].$$

In the customary version of stochastic control under partial observation, $u(t)$ is a function of the observation process $Y(s)$, $s \leq t$. Instead of discussing the problem of this type, we treat some wider class of admissible controls inspired by Fleming & Pardoux [2].

Let

$$\text{(4.4)} \qquad \rho(t) = \exp\left\{\int_0^t h(X(s))dY(s) - \frac{1}{2}\int_0^t |h(X(s))|^2 ds\right\}.$$

Then $\hat{W}$ and Y become independent Brownian motions under a new probability P defined by

$$\text{(4.5)} \qquad dP = \rho(T)^{-1}d\hat{P}$$

and $X(t)$ becomes a solution of the following SDE

$$
(4.6) \qquad \begin{cases} dX(t) = \{b(X(t), Y(t))u(t) - \sigma(X(t), Y(t))h(X(t))\}dt \\ \qquad\qquad + \hat{\sigma}(X(t), Y(t))d\hat{W}(t) + \sigma(X(t), Y(t))dY(t) \\ X(0) = \xi . \end{cases}
$$

Suppose ξ has a probability density $\phi \in H^2(\mathbb{R}^d)$.

DEFINITION 4.1. $\mathscr{A} = (\Omega, \mathscr{F}, P, \hat{W}, Y, u, \xi)$ is called an admissible system, if

(1) $(\Omega, \mathscr{F}, P)$ is a probability space

(2) u is Γ-valued measurable process

(3) Y is a d'-dimensional $(\mathscr{F}_t)$-Brownian motion where

$$
\mathscr{F}_t = \sigma\left\{Y(s), \int_0^s u(\tau)d\tau; s \le t\right\}
$$

(4) $\hat{W}$ is a d-dimensional Brownian motion

(5) ξ is a d-dimensional random variable and its distribution has the density ϕ

(6) ξ, $\hat{W}$ and (Y, u) are independent with respect to P.

For an admissible system $\mathscr{A}$, the solution $X(t) = X^{\mathscr{A}}(t)$ of the SDE (4.6) is called the response for $\mathscr{A}$. Putting $d\hat{P} = \rho(T)dP$, we define the pay-off function by

$$
(4.7) \qquad J(\mathscr{A}) = \hat{E}\left[\int_0^T f(X^{\mathscr{A}}(t))dt + g(X^{\mathscr{A}}(T))\right]
$$

where f, $g \in L^2(\mathbb{R}^d)$ and non-negative.

By the similar argument as Rozovskii [8], we obtain the following.

PROPOSITION 4.1. *Let $p^{\mathscr{A}}$ be a solution of the SPDE* (2.4) *for an admissible system $\mathscr{A}$, then $p^{\mathscr{A}}(t)$ is the unnormalized conditional density of $X^{\mathscr{A}}(t)$ with respect to $\mathscr{F}_t$. Namely, for every $\varphi \in L^\infty(\mathbb{R}^d)$, $t \in [0, T]$*

$$
(4.8) \qquad E[\varphi(X^{\mathscr{A}}(t))\rho(t)|\mathscr{F}_t] = (\varphi, p^{\mathscr{A}}(t)) \ P\text{-a.s.}
$$

holds, where $(\cdot, \cdot)$ is the inner product in $L^2(\mathbb{R}^d)$.

Using (4.8), *we get*

$$
(4.9) \qquad J(\mathscr{A}) = E\left[\int_0^T (f, p^{\mathscr{A}}(t))dt + (g, p^{\mathscr{A}}(T))\right].
$$

Since $(f, p^{\mathscr{A}}(t))$ and $(g, p^{\mathscr{A}}(T))$ are non-negative, Theorem 3.2 assures the existence of an optimal admissible system. Namely, THEOREM 4.1. *There exists an optimal admissible system $\tilde{\mathscr{A}}$, that is*

$$\inf_{\mathscr{A}:\text{ad. sys.}} J(\mathscr{A}) = J(\tilde{\mathscr{A}}). \tag{4.10}$$

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