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TWO-NUMBER OF SYMMETRIC R-SPACES

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Dedicated to Professor Shingo Murakami on his sixtieth birthday

Introduction

Chen-Nagano [2] introduced a Riemannian geometric invariant $\nu(M)$, called the 2-*number*, for a compact (connected) symmetric space M: Points $p, q \in M$ are said to be *antipodal* to each other, if $p = q$ or there is a closed geodesic of M on which p and q are antipodal to each other. A subset A of M is called an *antipodal subset* if every pair of points of A are antipodal to each other. Now the 2-number $\nu(M)$ is defined as the maximum possible cardinality $|A|$ of an antipodal subset A of M. The 2-number is finite.

In this note we will prove the following

THEOREM. *If M is a symmetric R-space* (*See* § 1 *for the definition*), *we have*

$$\nu(M) = \dim H(M, Z_2),$$

where $H(M, Z_2)$ denotes the homology group of M with coefficients $\boldsymbol{Z}_2$.

§ 1. Symmetric R-spaces

A compact symmetric space M is said to have a *cubic lattice* if a maximal torus of M is isometric to the quotient of $\boldsymbol{R}^r$ by a lattice of $\boldsymbol{R}^r$ generated by an orthogonal basis of the same length. A Riemannian product of several compact symmetric spaces with cubic lattices is called a *symmetric R-space*. We here recall some properties of symmetric R-spaces (cf. Takeuchi [4], [6], Loos [2]).

A symmetric R-space M has the complexification $\overline{M}$: There exists uniquely a connected complex projective algebraic manifold $\overline{M}$ defined over $\boldsymbol{R}$ such that the set $\overline{M}(\boldsymbol{R})$ of $\boldsymbol{R}$-rational points of $\overline{M}$ is identified

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with M. The group Aut $\overline{M}$ of holomorphic automorphisms of $\overline{M}$ is a complex linear algebraic group defined over $\boldsymbol{R}$. The identity component G of $(\mathrm{Aut}\,\overline{M})(\boldsymbol{R})$ is a semi-simple Lie group without compact factors and acts on M effectively and transitively. The identity component K of the group $\mathrm{I}(M)$ of isometries of M is a maximal compact subgroup of G. Thus there is an involutive automorphism τ of G such that

$$K = \{g \in G;\ \tau(g) = g\}\,.$$

We fix a point $o \in M$ once and for all, and set

$$U = \{g \in G;\ g \cdot o = o\}\,, \qquad K_0 = K \cap U\,.$$

Thus we have identifications: $M = G/U = K/K_0$. Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}\,, \qquad \mathfrak{k} = \mathrm{Lie}\,K$$

be the eigenspace decomposition of $\mathfrak{g} = \mathrm{Lie}\,G$ with respect to the differential of τ. Then there exists uniquely an element $E \in \mathfrak{p}$ such that $\mathfrak{u} = \mathrm{Lie}\,U$ is given by

$$\mathfrak{u} = \mathfrak{g}_0 + \mathfrak{g}_1\,,$$

where $\mathfrak{g}_p$ denotes the p-eigenspace of ad E. Furthermore the subgroup K_0 coincides with the centralizer of E in K, and so M is imbedded into $\mathfrak{p}$ as the K-orbit through E. We choose a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ with $E \in \mathfrak{a}$ and set

$$W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})\,, \qquad W_0 = N_{K_0}(\mathfrak{a})/Z_{K_0}(\mathfrak{a})\,,$$

where $N_K(\mathfrak{a})$ (resp. $N_{K_0}(\mathfrak{a})$) and $Z_K(\mathfrak{a})$ (resp. $Z_{K_0}(\mathfrak{a})$) denote the normalizer and the centralizer in K (resp. in K_0) of $\mathfrak{a}$. We may regard W_0 as a subgroup of W. These groups W and W_0 are finite groups called Weyl groups of $\mathfrak{g}$ and $\mathfrak{g}_0$. We define a subset A of M by

$$A = N_K(\mathfrak{a}) \cdot o\,.$$

Since $A \cong N_K(\mathfrak{a})/N_{K_0}(\mathfrak{a}) \cong W/W_0$, we have

$$|A| = |W/W_0|\,.$$

THEOREM 1. (Bott-Samelson [1]).

$$\dim H(M, Z_2) = |W/W_0|\,.$$

THEOREM 2 (Takeuchi [4]). *Let $M_1, \cdots, M_s$ be the connected compo-*

nents of fixed point set of the symmetry of M at o. Thus each M_i is a compact symmetric space with respect to the Riemannian metric induced from that of M. Then

(i) *Each M_i is also a symmetric R-space; and*

(ii) $\dim H(M, Z_2) = \sum_{i=1}^{s} \dim H(M_i, Z_2)$.

THEOREM 3 (Takeuchi [4], [5]). *There exists a maximal torus of M through o which includes antipodal points a_i $(1 \leq i \leq s)$ to o such that*

$$A = N_{K_0}(\mathfrak{a})\cdot\{a_1, \cdots, a_s\}.$$

§ 2. Proof of Theorem

We first show that the subset A is an antipodal subset. We remark that since $K_0 \subset \mathrm{I}(M)$ and $N_{K_0}(\mathfrak{a})\cdot o = \{o\}$, by Theorem 3 each point of A is antipodal to o. Let $p, q \in A$ be arbitrary. Since $A = N_K(\mathfrak{a})\cdot o$, there is $k \in N_K(\mathfrak{a})$ such that $k\cdot p = o$. By the above remark the point $k\cdot q \in A$ is antipodal to $o = k\cdot p$. It follows by $k \in \mathrm{I}(M)$ that q is antipodal to p. This proves the claim. Now, together with Theorem 1 this implies the inequality:

$$\dim H(M, Z_2) \leq \nu(M).$$

We will prove Theorem by induction on $\dim M$. We make use of the inequality in Chen-Nagano [2]:

$$\nu(M) \leq \sum_{i=1}^{s} \nu(M_i),$$

which holds for a general compact symmetric space. By Theorem 2 (i) and the assumption of the induction we have that $\nu(M_i) = \dim H(M_i, Z_2)$ for each i, and hence

$$\nu(M) \leq \sum_{i=1}^{s} \dim H(M_i, Z_2).$$

But the right hand side is equal to $\dim H(M, Z_2)$ by Theorem 2 (ii), and so we obtain the inequality:

$$\nu(M) \leq \dim H(M, Z_2).$$

Together with the previous opposite inequality we get

$$\nu(M) = \dim H(M, Z_2).$$

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