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THE MONOTONICITY OF ABSOLUTE NORMALIZED NORMS ON $\mathbb{C}^n$

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Abstract. In this paper, we characterize some monotonicity property of absolute normalized norms on $\mathbb{C}^n$ (resp. $\mathbb{R}^n$) by means of their corresponding continuous convex functions.

1. Introduction

Recently, some properties of absolute normalized norms on $\mathbb{C}^n$ have been studied by several authors (cf. [3, 5, 8, 9, 12, 13, 14, 15, 16], etc). A norm $\|\cdot\|$ on $\mathbb{C}^n$ (resp. $\mathbb{R}^n$) is called *absolute* if

$$\|(x_1, x_2, \ldots, x_n)\| = \|(|x_1|, |x_2|, \ldots, |x_n|)\|$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ (resp. $\mathbb{R}^n$), and *normalized* if

$$\|(1, 0, \ldots, 0)\| = \|(0, 1, 0, \ldots, 0)\| = \cdots = \|(0, \ldots, 0, 1)\| = 1.$$

The ℓ_p-norms $\|\cdot\|_p$ are such examples:

$$\|(x_1, \ldots, x_n)\|_p = \begin{cases} (|x_1|^p + \cdots + |x_n|^p)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{|x_1|, \ldots, |x_n|\} & \text{if } p = 1. \end{cases}$$

Let AN_n be the family of all absolute normalized norms on $\mathbb{C}^n$ (resp. $\mathbb{R}^n$). Bonsall and Duncan in [3] showed the following characterization of absolute normalized norms on $\mathbb{C}^2$ (resp. $\mathbb{R}^2$). Namely, the set AN_2 is in a one-to-one correspondence with the set Ψ_2 of all continuous convex functions ψ on the unit interval $[0, 1]$ satisfying $\max\{1-t, t\} \le \psi(t) \le 1$ for any t with $0 \le t \le 1$. The correspondence is given by the equation $\psi(t) = \|(1-t, t)\|$. Indeed, for all ψ in Ψ_2 we define the norm $\|\cdot\|_\psi$ as

$$\|(x, y)\|_\psi = \begin{cases} (|x| + |y|)\psi(\frac{|y|}{|x|+|y|}) & \text{if} \quad (x, y) \ne (0, 0), \\ 0 & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

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Then $\|\cdot\|_\psi$ belongs to AN_2 and satisfies $\psi(t) = \|(1-t,t)\|_\psi$.

Saito, Kato and Takahashi in [14] extended this fact to $\mathbb{C}^n$ (resp. $\mathbb{R}^n$). Namely, they showed that for any absolute normalized norm on $\mathbb{C}^n$ (resp. $\mathbb{R}^n$) there corresponds a continuous convex function on

$$\Delta_n = \{(t_1,t_2,\ldots,t_{n-1}) \in \mathbb{R}^{n-1} : t_j \geq 0 \ (\forall j),\ \sum_{j=1}^{n-1} t_j \leq 1\}$$

with some appropriate conditions. Indeed, for any $\|\cdot\| \in AN_n$ we define

$$\psi(s_1,\ldots,s_{n-1}) = ||(1-\sum_{i=1}^{n-1} s_i, s_1,\ldots,s_{n-1})|| \quad ((s_1,\ldots,s_{n-1}) \in \Delta_n). \tag{1.1}$$

Then ψ is a continuous convex function on Δ_n, and satisfies the following conditions:

(A_0) $\quad \psi(0,0,\ldots,0) = \psi(1,0,0,\ldots,0) = \psi(0,1,0,\ldots,0)$
$\qquad = \cdots = \psi(0,\ldots,0,1) = 1,$

(A_1) $\quad \psi(s_1,\ldots,s_{n-1}) \geq$
$$(s_1+\cdots+s_{n-1})\psi\left(\frac{s_1}{s_1+\cdots+s_{n-1}},\ldots,\frac{s_{n-1}}{s_1+\cdots+s_{n-1}}\right),$$
$\qquad$ if $s_1+\cdots+s_{n-1} \neq 0,$

(A_2) $\quad \psi(s_1,\ldots,s_{n-1}) \geq (1-s_1)\psi\left(0,\frac{s_2}{1-s_1},\ldots,\frac{s_{n-1}}{1-s_1}\right), \quad$ if $s_1 \neq 1,$

$\vdots \qquad\qquad \vdots$

(A_n) $\quad \psi(s_1,\ldots,s_{n-1}) \geq (1-s_{n-1})\psi\left(\frac{s_1}{1-s_{n-1}},\ldots,\frac{s_{n-2}}{1-s_{n-1}},0\right), \quad$ if $s_{n-1} \neq 1.$

Let Ψ_n be the set of all continuous convex functions ψ on Δ_n satisfying (A_0), (A_1), $\cdots$, (A_n). Conversely, for every $\psi \in \Psi_n$, we define

$$\|(x_1,x_2,\ldots,x_n)\|_\psi = \begin{cases} (|x_1|+\cdots+|x_n|)\psi\left(\frac{|x_2|}{|x_1|+\cdots+|x_n|},\cdots,\frac{|x_n|}{|x_1|+\cdots+|x_n|}\right) & \text{if } (x_1,\ldots,x_n) \neq (0,\ldots,0), \\ 0 & \text{if } (x_1,\ldots,x_n) = (0,\ldots,0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_n$ and satisfies (1.1). Hence AN_n and Ψ_n are in a one-to-one correspondence under (1.1). From this result, we have a plenty of concrete absolute normalized norms of $\mathbb{C}^n$ which is not ℓ_p-type. As applications, Saito, Kato and Takahashi in [14] characterized the strict convexity of absolute norms on $\mathbb{C}^n$ (resp. $\mathbb{R}^n$) in terms of their continuous convex function in Ψ_n (see also [15, 16]). We remark that a norm $\|\cdot\|$ on $\mathbb{C}^n$ is absolute if and only if it has the following monotonicity:

$$|z_i| \leq |w_i| \ (\forall i = 1,\ldots,n) \Rightarrow \|(z_1,\ldots,z_n)\| \leq \|(w_1,\ldots,w_n)\|$$

(see [2]).

In this paper, we characterize some monotonicity properties of absolute normalized norms on $\mathbb{C}^n$ in terms of continuous convex functions in Ψ_n. Moreover, as applications, we calculate the modulus of convexity for absolute normalized norms on $\mathbb{R}^2$.

2. Preliminaries

We recall some well-known results about monotonicity of norms on $\mathbb{C}^n$. At first, we have the following

Proposition 2.1. (cf. [2, Proposition IV.1.1]) *Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. Then the following assertions are equivalent:*
(i) *for any $(z_1,\dots,z_n),\ (w_1,\dots,w_n)\in\mathbb{C}^n$ with $|z_i|\le|w_i|$ for all i,*

$$\|(z_1,\dots,z_n)\|\le\|(w_1,\dots,w_n)\|.$$

(ii) *for any $(z_1,\dots,z_n),\ (w_1,\dots,w_n)\in\mathbb{C}^n$ with $|z_i|<|w_i|$ for all i,*

$$\|(z_1,\dots,z_n)\|<\|(w_1,\dots,w_n)\|.$$

(iii) *$\|\cdot\|$ is absolute.*

For the reader, we repeat the proof.

Proof. We only show (i) $\Leftrightarrow$ (ii). (ii) $\Rightarrow$ (i). Let $z_i,w_i\in\mathbb{C}$ with $|z_i|\le|w_i|$. Fix $\varepsilon>0$. Then, since $|z_i|<(1+\varepsilon)|w_i|$, it follows by (ii) that

$$\|(z_1,\dots,z_n)\|<\|((1+\varepsilon)w_1,\dots,(1+\varepsilon)w_n)\|=(1+\varepsilon)\|(w_1,\dots,w_n)\|.$$

Since ε is arbitrary, we obtain (i).

(i) $\Rightarrow$ (ii). Let $z_i,w_i\in\mathbb{C}$ with $|z_i|<|w_i|$. Take λ with $0<\lambda<1$ such that $|z_i|\le\lambda|w_i|$ for all i. By (i),

$$\|(z_1,\dots,z_n)\|\le\|(\lambda w_1,\dots,\lambda w_n)\|=\lambda\|(w_1,\dots,w_n)\|<\|(w_1,\dots,w_n)\|.$$

Thus we obtain (ii). □

For all $\psi\in\Psi_n$, $(\mathbb{R}^n,\|\cdot\|_\psi)$ is called strictly monotone if $(x_1,\dots,x_n),(y_1,\dots,y_n)\in\mathbb{R}^n$ with $|x_i|\le|y_i|$ for all i and $|x_{i_0}|<|y_{i_0}|$ for some i_0, then we have

$$\|(x_1,\dots,x_n)\|_\psi<\|(y_1,\dots,y_n)\|_\psi.$$

In [16], Takahashi, Kato and Saito showed the following.

Proposition 2.2 ([16]). *Let $\psi\in\Psi_2$. Then the following are equivalent:*

(i) *$\psi(t)>t$ for all t with $0<t<1$.*

(ii) $\psi(t)/t$ *is strictly decreasing on* $(0,1]$.
(iii) *If* $|z| < |u|$ *and* $|w| \le |v|$, *then* $\|(z,w)\|_\psi < \|(u,v)\|_\psi$.

Proposition 2.3 ([16]). *Let* $\psi \in \Psi_2$. *Then the following are equivalent:*

(i) $\psi(t) > 1-t$ *for all* t *with* $0 < t < 1$.
(ii) $\psi(t)/(1-t)$ *is strictly increasing on* $[0,1)$.
(iii) *If* $|z| \le |u|$ *and* $|w| < |v|$, *then* $\|(z,w)\|_\psi < \|(u,v)\|_\psi$.

Thus we have the following.

Proposition 2.4 ([16]). *Let* $\psi \in \Psi_2$. *Then* $(\mathbb{R}^2, \|\cdot\|_\psi)$ *is strictly monotone if and only if* $\psi(t) > \psi_\infty(t)(=\max\{1-t,t\})$ *for all* t *with* $0 < t < 1$.

Dowling and Turett in [6] extended this result to $(\mathbb{R}^n, \|\cdot\|_\psi)$ in order to characterize the complex strict convexity of $(\mathbb{C}^n, \|\cdot\|_\psi)$.

Proposition 2.5 ([6]). *Let* $\psi \in \Psi_n$. *Then* $(\mathbb{R}^n, \|\cdot\|_\psi)$ *is strictly monotone if and only if* ψ *satisfies the following conditions* $(sA_1), (sA_2), \ldots, (sA_n)$*:*

(sA_1)

$$\psi(s_1,\ldots,s_{n-1}) > (s_1+\cdots+s_{n-1})\psi\left(\frac{s_1}{s_1+\cdots+s_{n-1}},\ldots,\frac{s_{n-1}}{s_1+\cdots+s_{n-1}}\right), \quad \text{if } 0 < s_1+\cdots+s_{n-1} < 1,$$

(sA_2)

$$\psi(s_1,\ldots,s_{n-1}) > (1-s_1)\psi\left(0,\frac{s_2}{1-s_1},\ldots,\frac{s_{n-1}}{1-s_1}\right), \quad \text{if } 0 < s_1 < 1,$$

$\vdots \qquad \vdots$

(sA_n)

$$\psi(s_1,\ldots,s_{n-1}) > (1-s_{n-1})\psi\left(\frac{s_1}{1-s_{n-1}},\ldots,\frac{s_{n-2}}{1-s_{n-1}},0\right), \quad \text{if } 0 < s_{n-1} < 1.$$

3. Monotonicity of absolute normalized norms on $\mathbb{C}^2$

In this section, we first consider the monotonicity of absolute normalized norms on $\mathbb{C}^2$.

Theorem 3.1 ([12]). *Let* $\psi \in \Psi_2$ *and* $0 \le s_0 \le 1/2$. *Then the following are equivalent:*

(i) $s_0 = \max\{t \in [0,1/2] : \psi(t) = 1-t\}$.
(ii) $\psi(t)/(1-t)$ *is strictly increasing on* $[s_0,1)$, *and* $\psi(t) = 1-t$ *for all* $t \in [0,s_0]$.

(iii) *Let* $z,\ w,\ u \in \mathbb{C}$ *such that* $|w| < |u|$.

(a) *If* $\frac{|u|}{|z|+|u|} > s_0$, *then* $\|(z,w)\|_\psi < \|(z,u)\|_\psi$.

(b) *If* $\frac{|u|}{|z|+|u|} \le s_0$, *then* $\|(z,w)\|_\psi = \|(z,u)\|_\psi$.

For the readers we repeat the proof.

Proof. (i)⇒(ii). Assume (i). Let $s_0 < s < t < 1$. Then by $\psi(t) > 1-t$, we have

$$\begin{aligned}\frac{\psi(s)}{1-s} - \frac{\psi(t)}{1-t} &\le \frac{1}{1-s}\Big\{\frac{s}{t}\psi(t) + \frac{t-s}{t}\psi(0)\Big\} - \frac{\psi(t)}{1-t} \\ &= \frac{s-t}{t(1-s)(1-t)}\{\psi(t) - (1-t)\} < 0.\end{aligned}$$

Therefore $\frac{\psi(t)}{1-t}$ is strictly increasing on $(s_0, 1)$. From the convexity of ψ we have $\psi(t) = 1-t$ for all $t \in [0, s_0]$. Thus we obtain (ii).

(ii)⇒(iii). Assume (ii). Let $|w| < |u|$. Suppose that $\frac{|u|}{|z|+|u|} > s_0$. In the case of $\frac{|u|}{|z|+|u|} > \frac{|w|}{|z|+|w|} > s_0$ we have

$$\begin{aligned}\|(z,w)\|_\psi &= (|z|+|w|)\psi\Big(\frac{|w|}{|z|+|w|}\Big) \\ &= |z| \times \frac{\psi\Big(\frac{|w|}{|z|+|w|}\Big)}{1-\frac{|w|}{|z|+|w|}} \\ &< |z| \times \frac{\psi\Big(\frac{|u|}{|z|+|u|}\Big)}{1-\frac{|u|}{|z|+|u|}} = \|(z,u)\|_\psi.\end{aligned}$$

In the case of $\frac{|u|}{|z|+|u|} > s_0 \ge \frac{|w|}{|z|+|w|}$ we have

$$\begin{aligned}\|(z,w)\|_\psi &= |z| \times \frac{\psi\Big(\frac{|w|}{|z|+|w|}\Big)}{1-\frac{|w|}{|z|+|w|}} \\ &= |z| \times \frac{\psi(s_0)}{1-s_0} \\ &< |z| \times \frac{\psi\Big(\frac{|u|}{|z|+|u|}\Big)}{1-\frac{|u|}{|z|+|u|}} = \|(z,u)\|_\psi.\end{aligned}$$

If $\frac{|u|}{|z|+|u|} \le s_0$, then by $\frac{|w|}{|z|+|w|} < \frac{|u|}{|z|+|u|} \le s_0$ and $\psi(t) = 1-t$ on $[0, s_0]$ we have $\|(z,w)\|_\psi = |z| = \|(z,u)\|_\psi$. Thus we obtain (iii).

(iii)⇒(i). The proof is easy and so is omitted. □

In the same way we have the following

Theorem 3.2 ([12]). *Let $\psi \in \Psi_2$ and $1/2 \le t_0 \le 1$. Then the following are equivalent:*

(i) $t_0 = \min\{t \in [1/2, 1] : \psi(t) = t\}$.
(ii) $\psi(t)/t$ *is strictly decreasing on* $(0, t_0)$, *and* $\psi(t) = t$ *for all* $t \in [t_0, 1]$.
(iii) *Let* $z, u, w \in \mathbb{C}$ *such that* $|z| < |u|$.
 (a) *If* $\frac{|w|}{|u|+|w|} < t_0$, *then* $\|(z, w)\|_\psi < \|(u, w)\|_\psi$.
 (b) *If* $\frac{|w|}{|u|+|w|} \ge t_0$, *then* $\|(z, w)\|_\psi = \|(u, w)\|_\psi$.

4. Monotonicity of absolute normalized norms on $\mathbb{C}^n$

In this section we consider the monotonicity of absolute normalized norms on $\mathbb{C}^n$(resp. $\mathbb{R}^n$).

Theorem 4.1. *Let $\psi \in \Psi_n$. Fix $(s_1, \dots, s_{n-1}) \in \Delta_n$ with $s_1 + \cdots + s_{n-1} = 1$. Then the following assertions are equivalent:*
(i) *There exists λ_0 with $0 \le \lambda_0 < 1$ satisfying*

$$0 \le \lambda < \lambda_0 \Rightarrow \psi(\lambda s_1, \dots, \lambda s_{n-1}) > \lambda \psi(s_1, \dots, s_{n-1}), \tag{4.1}$$

$$\lambda_0 \le \lambda \le 1 \Rightarrow \psi(\lambda s_1, \dots, \lambda s_{n-1}) = \lambda \psi(s_1, \dots, s_{n-1}). \tag{4.2}$$

(ii) *There exists λ_0 with $0 \le \lambda_0 < 1$ such that*

$$f(\lambda) := \frac{\psi(\lambda s_1, \dots, \lambda s_{n-1})}{\lambda}$$

is strictly decreasing on $(0, \lambda_0]$, and $f(\lambda)$ is constant $(= \psi(s_1, \dots, s_{n-1}))$ on $(\lambda_0, 1]$.
(iii) *There exists λ_0 with $0 \le \lambda_0 < 1$ such that, for $a_1 \in \mathbb{C}$ and $(p_1, \dots, p_n) \in \mathbb{C}^n$ satisfying that*

$$s_1 = \frac{|p_2|}{|p_2| + \cdots + |p_n|}, s_2 = \frac{|p_3|}{|p_2| + \cdots + |p_n|}, \cdots, s_{n-1} = \frac{|p_n|}{|p_2| + \cdots + |p_n|},$$

if $0 \le |p_1| < |a_1|$ and $\frac{|p_2|+\cdots+|p_n|}{|a_1|+|p_2|+\cdots+|p_n|} < \lambda_0$, then

$$\|(p_1, p_2, \dots, p_n)\|_\psi < \|(a_1, p_2, \dots, p_n)\|_\psi, \tag{4.3}$$

and, if $0 \le |p_1| < |a_1|$ and $\frac{|p_2|+\cdots+|p_n|}{|a_1|+|p_2|+\cdots+|p_n|} \ge \lambda_0$, then

$$\|(p_1, p_2, \dots, p_n)\|_\psi = \|(a_1, p_2, \dots, p_n)\|_\psi. \tag{4.4}$$

Proof. (i)$\Rightarrow$(ii). Let $0 < \lambda_1 < \lambda_2 \le \lambda_0$. Then

$$\psi(\lambda_2 s_1, \dots, \lambda_2 s_{n-1}) \le \frac{1 - \lambda_2}{1 - \lambda_1} \psi(\lambda_1 s_1, \dots, \lambda_1 s_{n-1}) + \frac{\lambda_2 - \lambda_1}{1 - \lambda_1} \psi(s_1, \dots, s_{n-1}).$$

Therefore

$f(\lambda_1) - f(\lambda_2)$

$$\geq \frac{\psi(\lambda_1 s_1,\dots,\lambda_1 s_{n-1})}{\lambda_1} - \frac{1}{\lambda_2}\left\{\frac{1-\lambda_2}{1-\lambda_1}\psi(\lambda_1 s_1,\dots,\lambda_1 s_{n-1}) + \frac{\lambda_2-\lambda_1}{1-\lambda_1}\psi(s_1,\dots,s_{n-1})\right\}$$

$$= \frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2(1-\lambda_1)}\big\{\psi(\lambda_1 s_1\cdots,\lambda_1 s_{n-1}) - \lambda_1\psi(s_1,\dots,s_{n-1})\big\}.$$

By (4.1) we have $f(\lambda_1) > f(\lambda_2)$.

(ii)$\Rightarrow$(iii). Let $0 \leq |p_1| < |a_1|$ and $(|p_2|+\cdots+|p_n|)/(|a_1|+|p_2|+\cdots+|p_n|) \leq \lambda_0$. Put

$$\lambda_1 = \frac{|p_2|+\cdots+|p_n|}{|p_1|+|p_2|+\cdots+|p_n|} \text{ and } \lambda_2 = \frac{|p_2|+\cdots+|p_n|}{|a_1|+|p_2|+\cdots+|p_n|},$$

respectively. Suppose that $f(\lambda)$ is strictly decreasing on $(0,\lambda_0]$. Since $s_1+\cdots+s_{n-1}=1, \lambda_1>\lambda_2$ and $\lambda_2\leq\lambda_0$, we have

$$\begin{aligned}
\|(p_1,p_2,\dots,p_n)\|_\psi &= (|p_1|+\cdots+|p_n|)\times\psi\left(\frac{|p_2|}{|p_1|+\cdots+|p_n|},\cdots,\frac{|p_n|}{|p_1|+\cdots+|p_n|}\right)\\
&= (|p_2|+\cdots+|p_n|)\times\frac{\psi(\lambda_1 s_1,\cdots,\lambda_1 s_{n-1})}{\lambda_1}\\
&< (|p_2|+\cdots+|p_n|)\times\frac{\psi(\lambda_2 s_1,\cdots,\lambda_2 s_{n-1})}{\lambda_2}\\
&= \|(a_1,p_2,\dots,p_n)\|_\psi.
\end{aligned}$$

Suppose that $f(\lambda)=\psi(s_1,\dots,s_{n-1})$ for all λ with $\lambda_0\leq\lambda\leq 1$. Then $\lambda_1>\lambda_2>\lambda_0$. Hence we have

$$\begin{aligned}
\|(p_1,p_2,\dots,p_n)\|_\psi &= (|p_2|+\cdots+|p_n|)\times\frac{\psi(\lambda_1 s_1,\cdots,\lambda_1 s_{n-1})}{\lambda_1}\\
&= (|p_2|+\cdots+|p_n|)\times\psi(s_1,\dots,s_{n-1})\\
&= \|(0,p_2,\dots,p_n)\|_\psi.
\end{aligned}$$

We similarly have $\|(a_1,p_2,\dots,p_n)\|_\psi = \|(0,p_2,\dots,p_n)\|_\psi$.
(iii)$\Rightarrow$(i). Let $0\leq\lambda\leq\lambda_0$ and let $(s_1,\dots,s_{n-1})\in\Delta_n$ with $s_1+\cdots+s_{n-1}=1$. Then $1-\lambda>0$ and $(\lambda s_1+\cdots+\lambda s_{n-1})/(1-\lambda+\lambda s_1+\cdots+\lambda s_{n-1})\leq\lambda_0$. Hence, by (4.3) we have

$$\begin{aligned}
\psi(\lambda s_1,\dots,\lambda s_{n-1}) &= \|(1-\lambda,\lambda s_1,\dots,\lambda s_{n-1})\|_\psi\\
&> \|(0,\lambda s_1,\dots,\lambda s_{n-1})\|_\psi\\
&= \lambda\psi(s_1,\dots,s_{n-1}).
\end{aligned}$$

If $\lambda_0<\lambda\leq 1$, then we have $\psi(\lambda s_1,\dots,\lambda s_{n-1}) = \lambda\psi(s_1,\dots,s_{n-1})$. This completes the proof. □

For $\psi \in \Psi_n$ let

$$\widetilde{\psi}(s_1,\dots,s_{n-1}) = \psi(1-s_1-\cdots-s_{n-1}, s_2,\dots,s_{n-1}) \quad ((s_1,\dots,s_{n-1}) \in \Delta_n).$$

Then $\widetilde{\psi} \in \Psi_n$ and for $s_1+\cdots+s_{n-1}=1$,

$$\widetilde{\psi}(\lambda s_1,\dots,\lambda s_{n-1}) = \psi(1-\lambda, \lambda s_2,\dots,\lambda s_{n-1}),$$
$$\widetilde{\psi}(s_1,\dots,s_{n-1}) = \psi(0, s_2,\dots,s_{n-1}).$$

Therefore, by Theorem 4.1 we have

Theorem 4.2. *Let $\psi \in \Psi_n$. Fix $(0, s_2,\dots,s_{n-1}) \in \Delta_n$. Then the following are equivalent:*
(i) *There exists λ_0 with $0 \le \lambda_0 < 1$ satisfying*

$$0 \le \lambda \le \lambda_0 \Rightarrow \psi(1-\lambda, \lambda s_2, \cdots, \lambda s_{n-1}) > \lambda\psi(0, s_2,\dots,s_{n-1}),$$
$$\lambda_0 < \lambda \le 1 \Rightarrow \psi(1-\lambda, \lambda s_2, \cdots, \lambda s_{n-1}) = \lambda\psi(0, s_2,\dots,s_{n-1}).$$

(ii) *There exists λ_0 with $0 \le \lambda_0 < 1$ such that*

$$f(\lambda) := \frac{\psi(1-\lambda, \lambda s_2,\dots,\lambda s_{n-1})}{\lambda}$$

is strictly decreasing on $(0, \lambda_0]$, and $f(\lambda)$ is constant $(= \psi(0, s_2,\dots,s_{n-1}))$ on $(\lambda_0, 1]$.
(iii) *There exists λ_0 with $0 \le \lambda_0 < 1$ such that, for $a_2 \in \mathbb{C}$ and $(p_1,\dots,p_n) \in \mathbb{C}^n$ satisfying that*

$$s_2 = \frac{|p_3|}{|p_1|+|p_3|+\cdots+|p_n|}, \cdots, s_{n-1} = \frac{|p_n|}{|p_1|+|p_3|+\cdots+|p_n|},$$

if $0 \le |p_2| < |a_2|$ and $\frac{|p_1|+|p_3|+\cdots+|p_n|}{|p_1|+|a_2|+|p_3|+\cdots+|p_n|} \le \lambda_0$, then

$$\|(p_1,p_2,\dots,p_n)\|_\psi < \|(p_1,a_2,p_3,\dots,p_n)\|_\psi,$$

and, if $0 \le |p_2| < |a_2|$ and $\frac{|p_1|+|p_3|+\cdots+|p_n|}{|p_1|+|a_2|+|p_3|\cdots+|p_n|} > \lambda_0$, then

$$\|(p_1,p_2,\dots,p_n)\|_\psi = \|(p_1,a_2,p_3,\dots,p_n)\|_\psi.$$

In general, we have the following theorem.

Theorem 4.3. *Let $\psi \in \Psi_n$ and $i \in \{1,2,\dots,n-1\}$. Fix $(s_1,\dots,s_{i-1},0,s_{i+1},\dots,s_{n-1}) \in \Delta_n$. Then the following are equivalent:*
(i) *There exists λ_0 with $0 \le \lambda_0 < 1$ satisfying*

$$0 \le \lambda \le \lambda_0$$
$$\Rightarrow \psi(\lambda s_1,\dots,\lambda s_{i-1}, 1-\lambda, \lambda s_{i+1},\dots,\lambda s_{n-1}) > \lambda\psi(s_1,\dots,s_{i-1},0,s_{i+1},\dots,s_{n-1}),$$
$$\lambda_0 < \lambda \le 1$$
$$\Rightarrow \psi(\lambda s_1,\dots,\lambda s_{i-1}, 1-\lambda, \lambda s_{i+1},\dots,\lambda s_{n-1}) = \lambda\psi(s_1,\dots,s_{i-1},0,s_{i+1},\dots,s_{n-1}).$$

(ii) *There exists* λ_0 *with* $0 \leq \lambda_0 < 1$ *such that*

$$f(\lambda) := \frac{\psi(\lambda s_1, \ldots, \lambda s_{i-1}, 1-\lambda, \lambda s_{i+1}, \ldots, \lambda s_{n-1})}{\lambda}$$

is strictly decreasing on $(0, \lambda_0]$, *and* $f(\lambda)$ *is constant* $(= \psi(s_1, \ldots, s_{i-1}, 0, s_{i+1}, \ldots, s_{n-1}))$ *on* $(\lambda_0, 1]$.

(iii) *There exists* λ_0 *with* $0 \leq \lambda_0 < 1$ *such that, for* $a_i \in \mathbb{C}$ *and* $(p_1, \ldots, p_n) \in \mathbb{C}^n$ *satisfying that*

$$s_j = \frac{|p_{j+1}|}{|p_1| + |p_2| + \cdots + |p_{i-1}| + |p_{i+1}| + \cdots + |p_n|},$$

where $j = 1, \ldots, n-1$ *with* $j \neq i$, *if* $0 \leq |p_i| < |a_i|$ *and* $\frac{|p_1|+\cdots+|p_{i-1}|+|p_{i+1}|+\cdots+|p_n|}{|p_1|+\cdots+|p_{i-1}|+|a_i|+|p_{i+1}|+\cdots+|p_n|} \leq \lambda_0$, *then*

$$\|(p_1, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_n)\|_\psi < \|(p_1, \ldots, p_{i-1}, a_i, p_{i+1}, \ldots, p_n)\|_\psi,$$

and, if $0 \leq |p_i| < |a_i|$ *and* $\frac{|p_1|+\cdots+|p_{i-1}|+|p_{i+1}|+\cdots+|p_n|}{|p_1|+\cdots+|p_{i-1}|+|a_i|+|p_{i+1}|+\cdots+|p_n|} > \lambda_0$, *then*

$$\|(p_1, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_n)\|_\psi = \|(p_1, \ldots, p_{i-1}, a_i, p_{i+1}, \ldots, p_n)\|_\psi.$$

5. Applications

In this section, as applications of Section 2, we study some geometrical constants of Banach spaces. Let X be a Banach space. In 2006, Yang and Wang [17] introduced the geometrical constant $\gamma_X(t)$ of Banach spaces in order to compute the von Neumann-Jordan constant for Day-James spaces $\ell_2 - \ell_1$ and $\ell_\infty - \ell_1$:

$$\gamma_X(t) = \sup\left\{\frac{\|x+ty\|^2 + \|x-ty\|^2}{2} : x, y \in S_X\right\}.$$

They also presented the following characterization for uniformly non-squareness.

Proposition 5.1 ([17]). *Let* X *be a Banach space. Then the following are equivalent:*

(i) X *is uniformly non-square.*

(ii) $\gamma_X(t) < (1+t)^2$ *for all (resp. some)* t *with* $0 < t \leq 1$.

Mitani and Saito in [12] generalized the notion of the geometrical constant $\gamma_X(t)$ by using the ψ-direct sum $X \oplus_\psi X$, as follows:

Definition 5.2. For a Banach space $X = (X, \|\cdot\|_X)$ and $\psi \in \Psi_2$, let ψ-direct sum $X \oplus_\psi X$ be the direct sum $X \oplus X$ equipped with the norm

$$\|(x, y)\|_\psi = \|(\|x\|_X, \|y\|_X)\|_\psi.$$

Then we define

$$\gamma_{X,\psi}(t) = \sup\{\|(x+ty, x-ty)\|_\psi : x, y \in S_X\},$$

where $S_X = \{x \in X : \|x\|_X = 1\}$.

See also [16] for the notion of ψ-direct sum. Note that in the case of $\psi = \psi_2$ (ψ_2 is the corresponding function with ℓ_2-norm $\|\cdot\|_2$) it holds that

$$\gamma_{X,\psi_2}(t) = \sqrt{2\gamma_X(t)}.$$

By using Theorems 3.1 and 3.2, Mitani and Saito presented the following characterization for uniform non-squareness:

Theorem 5.3 ([12])**.** *For a Banach space X and $\psi \in \Psi_2$ with $\psi \neq \psi_\infty$, X is uniformly non-square if and only if*

$$\gamma_{X,\psi}(t) < 2(1+t)\psi(1/2)$$

for any (resp. some) t with $0 < t \leq 1$.

As an application of Section 1, we discuss the characteristic of convexity of $(\mathbb{R}^2, \|\cdot\|_\psi)$. The modulus of convexity of a Banach space X is the function $\delta : [0,2] \to [0,1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X, \|x-y\| = \varepsilon\right\}.$$

The characteristic of convexity of X is the number $\varepsilon_0(X)$ defined by

$$\varepsilon_0(X) = \sup\{\varepsilon : \delta_X(\varepsilon) = 0\}.$$

It is well-known that X is uniformly convex if and only if $\varepsilon_0(X) = 0$, and that X is uniformly non-square if and only if $\varepsilon_0(X) < 2$ (see [7, 10]). We consider the case where $s_0 = \max\{t \in [0,1] : \psi(t) = 1-t\}$ and $t_0 = \min\{t \in [0,1] : \psi(t) = t\}$ in the theorems above. We shall compute the characteristic of convexity for $(\mathbb{R}^2, \|\cdot\|_\psi)$ in terms of s_0, t_0.

Theorem 5.4. *Let $\psi \in \Psi_2$. Let $s_0 = \max\{t \in [0,1] : \psi(t) = 1-t\}$ and $t_0 = \min\{t \in [0,1] : \psi(t) = t\}$. If ψ is strictly convex on $[s_0, t_0]$, that is, $\psi((s+t)/2) < (\psi(s)+\psi(t))/2$ for all s, t with $s_0 \leq s < t \leq t_0$. then*

$$\varepsilon_0((\mathbb{R}^2, \|\cdot\|_\psi)) = 2\max\left\{\frac{s_0}{1-s_0}, \frac{1-t_0}{t_0}\right\}. \tag{5.1}$$

Proof. Note that

$$\begin{aligned}\varepsilon_0((\mathbb{R}^2, \|\cdot\|_\psi)) &= \sup\{\|x-y\|_\psi : x, y, (x+y)/2 \in S_{(\mathbb{R}^2,\|\cdot\|_\psi)}\} \\ &= \sup\{\|x-y\|_\psi : x, y \in S_{(\mathbb{R}^2,\|\cdot\|_\psi)}, \overline{xy} \text{ in } S_{(\mathbb{R}^2,\|\cdot\|_\psi)}\}\end{aligned}$$

because $\dim(\mathbb{R}^2, \|\cdot\|_\psi) = 2$ where $\overline{xy}$ is a straight line between the point x and the point y, that is, $\overline{xy} = \{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\}$. Put the points $P_i (i = 0, 1, \ldots, 7)$ in the unit sphere $S_{(\mathbb{R}^2,\|\cdot\|_\psi)}$ as

$$P_0 = (1, s_0'),\ P_1 = (t_0', 1),\ P_2 = (-t_0', 1),\ P_3 = (-1, s_0'),$$

$$P_4 = -P_0,\ P_5 = -P_1,\ P_6 = -P_2,\ P_7 = -P_3$$

where $s_0' = \frac{s_0}{1-s_0}$ and $t_0' = \frac{1-t_0}{t_0}$. Since ψ is strictly convex on $[s_0, t_0]$, it is easy to see that the unit sphere $S_{(\mathbb{R}^2, \|\cdot\|_\psi)}$ has only straight lines $\overline{P_1P_2}$, $\overline{P_3P_4}$, $\overline{P_5P_6}$, $\overline{P_7P_0}$ (see Figure 1). If $x, y \in \overline{P_1P_2}$, then

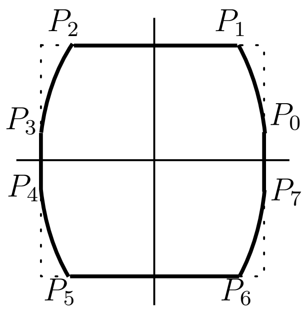

Figure 1. the unit sphere $S_{(\mathbb{R}^2, \|\cdot\|_\psi)}$

$$\|x - y\|_\psi \le \|P_1 - P_2\|_\psi = \|(t_0', 1) - (-t_0', 1)\|_\psi = 2t_0'.$$

If $x, y \in \overline{P_0P_7}$, then

$$\|x - y\|_\psi \le \|P_1 - P_2\|_\psi = \|(1, s_0') - (1, -s_0')\|_\psi = 2s_0'.$$

The rest cases are similar. Thus we have (5.1). $\square$

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