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KTS–spaces and natural reductivity

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Abstract

Simply connected KTS–spaces are naturally reductive homogeneous. The main purpose of this paper is to characterize the KTS–spaces in the class of naturally reductive spaces.

1. Introduction

φ–symmetric spaces have been introduced in Sasakian geometry by Takahashi [19] as generalizations of Sasakian space forms and also as analogs of Hermitian symmetric spaces. φ–symmetric spaces have been extensively studied by various authors (see for example, [1], [2], [3], [9], [13], [14], [25] and references therein). In particular, Blair and Vanhecke [1] proved that complete simply connected φ–symmetric spaces are necessarily naturally reductive homogeneous spaces. In [2] one studied the characterization of φ–symmetric spaces inside the class of naturally reductive spaces and the following result is proved:

Let M be a complete, connected, simply connected Sasakian manifold. Then M is a globally φ–symmetric space if and only if M is a naturally reductive homogeneous space with invariant Sasakian structure.

Further, in [4], [5], [6], [7] the notion of a *Killing–transversally symmetric space* (briefly KTS–*space*) is introduced as a generalization of Sasakian φ–symmetric spaces. This class is defined by using isometric reflections with respect to the flow lines generated by a unit Killing vector field (*isometric flow*) and these spaces form a subclass of the class of *transversally symmetric Riemannian foliations* studied in [21], [22]. Although the class of KTS–spaces is much broader than that of the φ–symmetric spaces (see examples in [5], [7]) their geometries are reasonably similar and this fact leads to a list of analogous characteristic properties. For example, it is proved in [7] that a simply connected KTS–space is a naturally reductive

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homogeneous space. The main purpose of this paper is to characterize the KTS–spaces in the class of naturally reductive spaces generalizing, in particular, the result stated above for φ–symmetric spaces.

The paper is organized as follows. In Section 2 we collect some basic material. In Section 3 we study the homogeneity of a KTS–space and we give a first characterization of KTS–spaces in the class of naturally reductive spaces. Other characterizations in terms of the canonical connection of the second kind are given in Section 4. The proofs are kept as short as possible. More details and extended proofs may be found in [8].

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2. Preliminaries

Let (M,g) be an n–dimensional smooth connected Riemannian manifold with $n \geq 2$. ∇ denotes the Levi Civita connection of (M,g) and R the corresponding Riemannian curvature tensor given by

$$R_{UV} = \nabla_{[U,V]} - [\nabla_U, \nabla_V] \quad , \quad U, V \in \mathfrak{X}(M)$$

where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M.

Let ξ denote a unit Killing vector field on (M,g) and $\mathfrak{F}_\xi$ the flow generated by it. Such a flow is necessarily Riemannian and it is called an *isometric flow* (see [20, p. 136]). Note that the flow lines are geodesics and that a geodesic which is orthogonal to ξ at one of its points is orthogonal to it at all of its points. A geodesic with this property is called a *transversal geodesic* with respect to the flow $\mathfrak{F}_\xi$.

Because the local submersions associated to a Riemannian foliation are Riemannian, the O'Neill tensors A and T may be used. We refer to [18] for more details (see also [20]). In our case $T = 0$ since the leaves are totally geodesic. For the integrability tensor A we have

$$A_U\xi = \nabla_U\xi \quad , \quad A_\xi U = 0 \tag{2.1}$$

for all tangent vector fields U. Further, we put

$$HU = -A_U\xi \tag{2.2}$$

and define the $(0,2)$–tensor h by

$$h(U,V) = g(HU,V). \tag{2.3}$$

Then, since ξ is a Killing vector field, we have

$$h(U,V)+h(V,U)=0.$$

Let η be the one–form on M defined by $\eta(U)=g(U,\xi)$, $U\in\mathfrak{X}(M)$. For all horizontal vector fields X,Y, that is, vector fields orthogonal to ξ, it is easy to prove that

$$A_XY=h(X,Y)\xi=\frac{1}{2}\eta([X,Y])\xi \tag{2.4}$$

and this gives

$$h=-d\eta. \tag{2.5}$$

Using the formulas above we get the following identities:

$$\begin{aligned}(\nabla_\xi h)(X,Y) &= 0,\\ R(X,Y,Z,\xi) &= (\nabla_Z h)(X,Y), &(2.6)\\ R(X,\xi,Y,\xi) &= g(HX,HY) &(2.7)\end{aligned}$$

for all horizontal vectors X,Y,Z.

We define the tensor field T of type $(1,2)$ (unrelated to the O'Neill tensor given above) by

$$T_UV=-d\eta(U,V)\xi+\eta(V)\nabla_U\xi-\eta(U)\nabla_V\xi \tag{2.8}$$

or equivalently,

$$T_UV=g(HU,V)\xi+\eta(U)HV-\eta(V)HU \tag{2.9}$$

for all $U,V\in\mathfrak{X}(M)$. Put

$$\overline{\nabla}=\nabla-T. \tag{2.10}$$

This connection $\overline{\nabla}$ will be called the *canonical connection* of the isometric flow $\mathfrak{F}_\xi$. It is a metric connection and the tensor fields ξ and η are parallel with respect to this connection. Moreover, we get $T_UU=0$. This means that $\overline{\nabla}$ and ∇ have the same geodesics or equivalently, they are projectively related. For the torsion $\overline{K}$ of $\overline{\nabla}$ we get easily

$$\overline{K}(U,V)=-2T_UV \tag{2.11}$$

for all vector fields U,V. In what follows we shall derive some properties for $\overline{\nabla}$ giving in particular the motivation for the notion "canonical". First, we state a useful result concerning a special type of isometric flow.

Definition 2.1. An isometric flow $\mathfrak{F}_\xi$ on a Riemannian manifold (M, g) is said to be a *normal flow* if its curvature tensor R satisfies

$$R(X, Y, X, \xi) = 0 \tag{2.12}$$

for all horizontal vector fields X, Y.

The condition (2.12) means that the horizontal subspace of the flow $\mathfrak{F}_\xi$ at each point of M is invariant by the curvature transformations R_{XY}, X, Y orthogonal to ξ, and it is equivalent to the condition

$$(\nabla_U H)V = g(HU, HV)\xi + \eta(V)H^2U \tag{2.13}$$

for all vector fields U, V in M, by virtue of (2.6) and (2.9).

Proposition 2.1. [5] *Let (M, g) be a Riemannian manifold and $\mathfrak{F}_\xi$ an isometric flow on it. Then the following statements are equivalent:*

(i) $\mathfrak{F}_\xi$ *is normal;*

(ii) $\overline{\nabla} H = 0$;

(iii) $\overline{\nabla} T = 0$.

On a Riemannian manifold with an isometric flow we have, using (2.5) or (2.7) that the following statements are equivalent:

(i) the sectional curvature $K(X, \xi)$ is positive for each horizontal vector X;

(ii) the endomorphism H is of maximal rank;

(iii) the one–form η on M is a contact form.

Moreover, if one of these conditions is satisfied, then n is necessarily odd. This leads to

Definition 2.2. An isometric flow on (M, g) is said to be a *contact flow* if η is a contact form.

Using [26, Proposition 6.10] we get

Proposition 2.2. *An n–dimensional Riemannian manifold (M, g) equipped with a normal contact flow is irreducible and its homogeneous holonomy group coincides with the group $SO(n)$ of all isometries.*

Next, let (M, g) be a Riemannian manifold equipped with an isometric flow $\mathfrak{F}_\xi$. For each point $m \in M$ we consider the flow line σ through it. A local diffeomorphism s_m of M defined in a neighborhood $\mathcal{U}$ of the point $m \in M$ is said to be a (*local*) *reflection with respect to* σ if for every transversal geodesic $\gamma(s)$, where $\gamma(0)$ lies in the intersection of $\mathcal{U}$ with σ, we have

$$(s_m \circ \gamma)(s) = \gamma(-s)$$

for all s with $\gamma(\pm s) \in \mathcal{U}$, s being the arc length. Note that the isomorphism $S_m = (s_m)_*(m)$ of $T_m M$ is given by

$$S_m = (-I + 2\eta \otimes \xi)(m)$$

and clearly, it is a linear isometry.

Definition 2.3. A *locally Killing–transversally symmetric space* (briefly a *locally* KTS–*space*) is a Riemannian manifold (M, g) equipped with an isometric flow $\mathfrak{F}_\xi$ such that the local reflection s_m with respect to the flow line through it is a (local) isometry for all $m \in M$.

In [5], we have characterized these spaces in terms of the curvature tensor of (M, g) and of the metric connection $\overline{\nabla}$ given in (2.10) (see also [21]). Namely

Theorem 2.1. *Let $\mathfrak{F}_\xi$ be an isometric flow on (M, g). Then the following statements are equivalent:*

(i) *(M, g, ξ) is a locally* KTS–*space;*

(ii) *$\mathfrak{F}_\xi$ is normal and $(\nabla_X R)(X, Y, X, Y) = 0$ for all horizontal X, Y;*

(iii) *$\overline{\nabla} R = \overline{\nabla} H = 0$ (or equivalently, $\overline{\nabla}\,\overline{R} = \overline{\nabla} H = 0$).*

A useful characterization for the class of locally contact KTS–spaces is obtained in [6]:

Theorem 2.2. *Let $\mathfrak{F}_\xi$ be a contact flow on (M, g). Then (M, g, ξ) is a locally* KTS–*space if and only if $\mathfrak{F}_\xi$ is normal and*

$$(\nabla_X R)(X, HX, X, HX) = 0$$

for all horizontal X.

We focus now on the class of locally KTS–spaces with a complete Killing vector field and such that the local reflections with respect to the flow lines of this field can be extended to global isometries.

Definition 2.4. Let (M,g) be a Riemannian manifold and ξ a non–vanishing complete Killing vector field on M. Then (M,g,ξ) is said to be a *(globally) Killing–transversally symmetric space* (briefly, a KTS–*space*) if and only if for each $m \in M$ there exists a (unique) global isometry $s_m : M \to M$ with derived map

$$(s_m)_*(m) = -I_{T_mM} + 2\eta_m \otimes \xi_m \tag{2.14}$$

on T_mM.

Note that (2.14) implies that ξ is a *unit* Killing vector field. The isometry s_m is called the *reflection* of M at m with respect to the flow line through m. Since it reverses the transversal geodesics through m, s_m is the unique global extension of the local reflection at m. Moreover, we have

Theorem 2.3. [4] *A complete, simply connected locally* KTS*–space is a* KTS*–space.*

3. Naturally reductive spaces and KTS–spaces

Let (M,g,ξ) be a KTS–space. In [7] it is proved that the group $A(M)$ of all ξ–preserving isometries of M is a transitive Lie transformation group of M and the $(1,2)$–tensor field T defined in (2.9) determines a naturally reductive homogeneous structure [23]. Hence, we have

Theorem 3.1. [7] *If (M,g,ξ) is a simply connected* KTS*–space, then (M,g) is a naturally reductive homogeneous space.*

Now, using the homogeneous structure T one can construct a connected Lie group acting transitively and effectively on (M,g,ξ) as a group of isometries [23]. In [5] it is proved that, for a fixed arbitrary point o of M, the corresponding curvature and torsion tensor vectors $\overline{R}_o$, $\overline{K}_o$ of the canonical connection $\overline{\nabla}$ of the flow $\mathfrak{F}_\xi$ determine an infinitesimal model on $(V = T_oM, g_o)$. We refer to [11], [15], [16] and [24] for more information and details concerning infinitesimal models. Following Nomizu [17] (see also [11]), one can then reconstruct our homogeneous manifold (M, g, ξ) in a standard way. Let $\mathfrak{h}$ be the Lie subalgebra of the Lie algebra $\mathfrak{so}(V)$ defined by

$$\mathfrak{h} = \{A \in \mathfrak{so}(V)/A \cdot \overline{R}_o = A \cdot \overline{K}_o = 0\}.$$

Here A acts as a derivation on the tensor algebra of V. Further, let $\mathfrak{g}$ be the direct sum of V and $\mathfrak{h}$ and put

$$\begin{cases} [X,Y] &= \overline{R}_{oXY} - \overline{K}_o(X,Y), \\ [A,X] &= A(X), \\ [A,B] &= A \circ B - B \circ A \end{cases}$$

for all $X, Y \in V$ and $A, B \in \mathfrak{h}$. Then $\mathfrak{g}$ becomes a Lie algebra. Now, let G be the unique connected and simply connected Lie group whose Lie algebra is $\mathfrak{g}$ and let H be the connected Lie subgroup of G corresponding to $\mathfrak{h}$, then $M = G/H$. Let $\mathfrak{k} \subset \mathfrak{h}$ be the Lie subalgebra generated by all projections $[X,Y]_{|\mathfrak{h}}$, $X, Y \in V$. Then $\mathfrak{k}$ can also be considered as the algebra generated by all curvature transformations $\overline{R}_{oXY}$ on the tangent space T_oM. The Lie subalgebra $\hat{\mathfrak{g}} \subset \mathfrak{g}$, $\hat{\mathfrak{g}} = V \oplus \mathfrak{k}$, is called the *transvection algebra* and the corresponding connected Lie subgroup $\hat{G} \subset G$ is the *transvection group* of the reductive homogeneous space G/H, or better, of the affine reductive space $(M, \overline{\nabla})$ (see [11]). We then have a new representation $(M, g) = \hat{G}/K$ by a new reductive homogeneous space with $\overline{\nabla}$ as the canonical connection of the second kind. Here, K is isomorphic to the restricted holonomy group of $(M, \overline{\nabla})$ at the origin. According to [11, Proposition I.38], all $\overline{\nabla}$–parallel tensor fields on M are also $\hat{G}$–invariant. Hence, as $\overline{\nabla} g = \overline{\nabla} \xi = 0$, $\hat{G} \subseteq A(M)$.

In terms of the Lie algebras, an arbitrary homogeneous Riemannian manifold (M, g) is said to be *naturally reductive* if there exists a reductive representation of the form $(M, g) = G/H$, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, satisfying the identity (see [10])

$$< [X,Y]_{\mathfrak{m}}, Z > + < [X,Z]_{\mathfrak{m}}, Y > = 0 \tag{3.1}$$

for any $X, Y, Z \in \mathfrak{m}$ where $<\,,\,>$ denotes the induced metric on $\mathfrak{m}$. In terms of the canonical connection $\tilde{\nabla}$ associated to the reductive decomposition, we can also write (3.1) in the form

$$g_o(\tilde{K}_o(X,Y), Z) + g_o(\tilde{K}_o(X,Z), Y) = 0, \tag{3.2}$$

where X, Y, Z are arbitrary vectors on T_oM and $\tilde{K}$ denotes the torsion tensor of $\tilde{\nabla}$.

Any reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ satisfying (3.1) (or any canonical connection $\tilde{\nabla}$ satisfying (3.2)) will be said to be *adapted.* Let us notice that the same homogeneous Riemannian manifold (M, g) may have more than one naturally reductive representation, and thus more than one adapted canonical connection $\tilde{\nabla}$.

This is the outline of the proof of the following more detailed version of Theorem 3.1.

Theorem 3.2. *Let (M, g, ξ) be a simply connected* KTS*-space and let $\overline{\nabla}$ be the canonical connection of the isometric flow $\mathfrak{F}_\xi$. Then there is a representation of M in the form $M = G/H$ such that*

(i) *G is the transvection group of the affine reductive space $(M, \overline{\nabla})$, $G \subseteq A(M)$;*

(ii) *there is an $Ad(H)$–invariant decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ of the Lie algebra of G adapted to the naturally reductive homogeneous space $(G/H, g)$ for which $\overline{\nabla}$ is the canonical connection of the second kind.*

Next, we give a characterization of KTS–spaces inside the class of naturally reductive spaces. In the irreducible case, proceeding as in [2, Theorem 7] and using Theorem 2.2, one gets

Theorem 3.3. *Let $\mathfrak{F}_\xi$ be a normal contact flow on a simply connected Riemannian manifold (M, g). Then (M, g, ξ) is a* KTS*–space if and only if M is a naturally reductive homogeneous space with invariant unit vector field ξ.*

For the reducible case, we use the de Rham decomposition for KTS–spaces obtained in [7]:

Theorem 3.4. *If (M, g, ξ) is a simply connected* KTS*–space, then its de Rham decomposition can be written as*

$$M = M_0 \times M_1 \times \ldots \times M_k \times M_{k+1}$$

where M_0 is a Euclidean space, $M_1, \ldots, M_k$ are irreducible symmetric spaces and M_{k+1} is an irreducible KTS*–space.*

From this result and Theorem 3.3 we obtain directly

Theorem 3.5. *Let $\mathfrak{F}_\xi$ be a normal flow on a simply connected Riemannian manifold (M, g). (M, g, ξ) is a reducible* KTS*–space if and only if M is a direct product*

$$(M, g) = (M', g') \times (M'', g'')$$

where M' is a naturally reductive homogeneous space, $\mathfrak{F}_\xi$ is a contact invariant flow on it and (M'', g'') is a symmetric space.

4. Canonical connection and KTS–spaces

Let $(M,g) = G/H$ be a naturally reductive homogeneous space with reductive decomposition of the Lie algebra $\mathfrak{g}$ of G given by $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ its orthogonal complement in $\mathfrak{g}$. Denote by $\tilde{\nabla}$ the adapted canonical connection of the fixed reductive homogeneous space and by $\tilde{K}$ the torsion tensor. At the origin o, $\tilde{K}$ verifies

$$\tilde{K}_o(U,V) = -[U,V]_{\mathfrak{m}} \quad , \quad U,V \in \mathfrak{m} \tag{4.1}$$

where we use the canonical identification $\mathfrak{m} \cong T_oM$ via the natural projection $p_r : G \to G/H$.

Suppose that $(M,g) = G/H$ admits a G–invariant unit vector field ξ. From (3.1), ξ is a unit Killing vector field and moreover, parallel with respect to the connection $\tilde{\nabla}$. Consider the G–invariant tensor field T' of type (1,2) given at the origin by

$$T'_{oU}V = -\frac{1}{2}\tilde{K}_o(U,V) \quad , \quad U,V \in T_oM. \tag{4.2}$$

Then, $T' + \tilde{\nabla}$ is the Levi Civita connection ∇ and so, since $\tilde{\nabla}g = \tilde{\nabla}\tilde{R} = \tilde{\nabla}\tilde{K} = 0$, T' determines a naturally reductive structure on (M,g) (see [23]). Moreover, because $\tilde{\nabla}\xi = 0$,

$$H_oU = -\nabla_U\xi = -T'_{oU}\xi_o = \frac{1}{2}\tilde{K}_o(U,\xi_o) \tag{4.3}$$

and

$$\tilde{\nabla}H = 0.$$

From this we get that

$$(\nabla_U H)V = T'_U HV - H(T'_U V) \tag{4.4}$$

and using (2.6) and (4.2), for X, Y orthogonal to ξ_o, we have

$$R_o(X,Y,X,\xi_o) = g_o((\nabla_X H)X, Y) = -\frac{1}{2}g_o(\tilde{K}_o(X, H_oX), Y).$$

This and (4.3) yield

Proposition 4.1. *Let $(M,g) = G/H$ be a naturally reductive homogeneous space with a given $Ad(H)$–invariant decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and ξ a G–invariant unit vector field on* *it. Let $\tilde{\nabla}$ be the adapted canonical connection of G/H and $\tilde{K}$ its torsion tensor. Then $\mathfrak{F}_\xi$ is normal if and only if at the origin*

$$\tilde{K}_o(X, H_oX) = -2g_o(H_oX, H_oX)\xi_o \tag{4.5}$$

for all X on T_oM orthogonal to ξ_o.

The condition (4.5) on a naturally reductive homogeneous space with an isometric flow $\mathfrak{F}_\xi$, or equivalently, the normality condition, is a very strong condition. In fact, we can formulate the above proposition as follows using the next lemma.

Lemma 4.1. *With the same hypotheses as in* Proposition 4.1 *we have that the following statements are equivalent:*

(i) $\tilde{K}_o(U,V) = -\{g_o(\tilde{K}_o(U,\xi_o),V)\xi_o - \eta_o(V)\tilde{K}_o(U,\xi_o) + \eta_o(U)\tilde{K}_o(V,\xi_o)\}$, $U,V \in T_oM$;

(ii) $g_o(\tilde{K}_o(X,Y),Z) = 0$ *for X,Y,Z orthogonal to ξ_o.*

Proof. (ii) follows at once from (i). For X,Y orthogonal to ξ_o, (ii) implies that $\tilde{K}_o(X,Y)$ is vertical and, using (3.2), $\tilde{K}_o(X,Y) = -g_o(\tilde{K}_o(X,\xi_o),Y)\xi_o$. From (4.3), $\tilde{K}_o(U,\xi_o)$ is horizontal and it is easy to verify (i) for arbitrary vectors U,V. ■

Proposition 4.2. *Let $(M,g) = G/H$ be a naturally reductive homogeneous space with a given $Ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and ξ a G-invariant unit vector field on it. The flow $\mathfrak{F}_\xi$ on M is normal if and only if the torsion tensor field $\tilde{K}$ corresponding to the adapted canonical connection $\tilde{\nabla}$ of G/H at the origin verifies*

(i) $g_o(\tilde{K}_o(X,Y),Z) = 0$,

(ii) $\tilde{K}_o(A,B) \in \ker H_o$

for X,Y,A,B,Z horizontal on T_oM, $X,Y \notin \ker H_o$, $A,B \in \ker H_o$.

In particular, if $\mathfrak{F}_\xi$ is a contact flow, $\mathfrak{F}_\xi$ is normal if and only if $\tilde{\nabla}$ coincides with the canonical connection $\overline{\nabla}$ of the flow $\mathfrak{F}_\xi$.

Proof. It is easy to prove that $R(X,Y,X,\xi) = 0$ is equivalent to $R(Z,Y,X,\xi) = 0$ for all horizontal X,Y,Z. Using (2.6) and (4.4) we have

$$R_o(Z,Y,X,\xi_o) = g_o((\nabla_X H)Z,Y) = \frac{1}{2}g_o(H_o(\tilde{K}_o(X,Z)) - \tilde{K}_o(X,H_oZ),Y)$$

and from (4.3), the normality condition given by (4.5) is also equivalent to

$$\tilde{K}_o(X, H_o Z) = H_o(\tilde{K}_o(X, Z)) - 2g_o(H_o X, H_o Z)\xi_o. \tag{4.6}$$

From this formula and (4.3) we get

$$\tilde{K}_o(X, A) = 0 \quad , \quad \tilde{K}_o(A, B) \in \ker H_o \tag{4.7}$$

for X, A, B horizontal on $T_o M$, $A, B \in \ker H_o$, $X \notin \ker H_o$.

Now, we consider $\tilde{K}_o(X, Y)$ for X, Y orthogonal to ξ_o and $X, Y \notin \ker H_o$. Polarizing (4.5) in X, we get

$$\tilde{K}_o(X, H_o Z) + \tilde{K}_o(Z, H_o X) = -4g_o(H_o X, H_o Z)\xi_o, \tag{4.8}$$

and so,

$$H_o(\tilde{K}_o(X, H_o Z)) = H_o(\tilde{K}_o(H_o X, Z)). \tag{4.9}$$

From (4.6) we then obtain that

$$\tilde{K}_o(X, H_o^2 Z) = \tilde{K}_o(H_o^2 X, Z). \tag{4.10}$$

Suppose that the rank of the endomorphism H is $2k \leq n-1$. Using a well–known result from linear algebra, the tangent space $T_o M$ admits an orthonormal basis $\{X_1, \ldots, X_{2k}, X_{2k+1}, \ldots, X_{n-1}, \xi_o\}$ and real non–vanishing numbers $\lambda_1, \ldots, \lambda_k$ such that

$$\begin{cases} H_o X_1 = \lambda_1 X_2 \quad , \quad H_o X_2 = -\lambda_1 X_1, \\ \ldots \\ H_o X_{2k-1} = \lambda_k X_{2k} \quad , \quad H_o X_{2k} = -\lambda_k X_{2k-1}, \\ H_o X_{2k+1} = \ldots = H_o X_{n-1} = H_o \xi_o = 0. \end{cases}$$

From (4.5), $\tilde{K}_o(X_{2h-1}, X_{2h})$ is vertical. Using (4.10) we have

$$(\lambda_h^2 - \lambda_l^2)\tilde{K}_o(X_{2l-1}, X_{2h-1}) = 0 \quad , \quad 1 \leq h \neq l \leq k.$$

If $\lambda_h^2 \neq \lambda_l^2$, $\tilde{K}_o(X_{2l-1}, X_{2h-1}) = 0$ and similarly, $\tilde{K}_o(X_{2l-1}, X_{2h}) = \tilde{K}_o(X_{2l}, X_{2h}) = 0$.

If $\lambda_h^2 = \lambda_l^2$, $\tilde{K}_o(X_{2l-1}, X_{2h-1})$ belongs to the subspace generated by $\{X_{2i-1}, X_{2i}\}$ where $\lambda_i^2 = \lambda_h^2$. From (4.6),

$$\begin{aligned} g_o(\tilde{K}_o(X_{2l-1}, X_{2h-1}), X_{2i-1}) &= \frac{1}{\lambda_h^2} g_o(\tilde{K}_o(X_{2l-1}, H_o X_{2h}), H_o X_{2i}) \\ &= \frac{1}{\lambda_h^2} g_o(H_o(\tilde{K}_o(X_{2l-1}, X_{2h})), H_o X_{2i}) = g_o(\tilde{K}_o(X_{2l-1}, X_{2h}), X_{2i}). \end{aligned}$$

However, using (4.8) we have

$$\begin{cases} \tilde{K}_o(X_{2l-1}, X_{2h-1}) &= \tilde{K}_o(X_{2h}, X_{2l}), \\ \tilde{K}_o(X_{2l-1}, X_{2h}) &= \tilde{K}_o(X_{2l}, X_{2h-1}) \end{cases}$$

and from these equations

$$g_o(\tilde{K}_o(X_{2l-1}, X_{2h-1}), X_{2i-1}) = g_o(\tilde{K}_o(X_{2h-1}, X_{2i-1}), X_{2l-1}) = g_o(\tilde{K}_o(X_{2i}, X_{2h}), X_{2l-1})$$

$$= -g_o(\tilde{K}_o(X_{2l-1}, X_{2h}), X_{2i}).$$

Then $H_o(\tilde{K}_o(X,Y)) = 0$ for X, Y orthogonal to ξ_o, $X, Y \notin \ker H_o$, and (4.6) and (4.7) yield the first required result. Moreover, using (4.3), Lemma 4.1 and the fact that the tensor T defined in (2.9) is G–invariant, we get at once the result for the contact case. ∎

Next, we combine the above result and Theorems 3.3 and 3.5 to characterize the simply connected KTS–spaces in the class of naturally reductive spaces in terms of the canonical connection of the second kind. For it, we also need the following lemma

Lemma 4.2. [12] *Let $(M,g) = G/H$ be a naturally reductive space and $\tilde{\nabla}$ some of its adapted canonical connections. If either the curvature tensor $\tilde{R}$, or the torsion tensor $\tilde{K}$ vanishes, then the space (M,g) is locally symmetric.*

Theorem 4.1. *Let $\mathfrak{F}_\xi$ be an isometric flow on a simply connected Riemannian manifold (M,g). Then (M,g,ξ) is a* KTS*–space if and only if $(M,g) = G/H$ is a naturally reductive homogeneous space with G–invariant ξ and admitting an adapted canonical connection $\tilde{\nabla}$ whose torsion tensor field $\tilde{K}$ satisfies*

$$g_o(\tilde{K}_o(X,Y), Z) = 0 \tag{4.11}$$

for all horizontal X, Y, Z on the tangent space T_oM at the origin. In this case, $\tilde{\nabla}$ coincides with the canonical connection $\overline{\nabla}$ of the flow $\mathfrak{F}_\xi$.

Proof. The necessity follows from Theorem 3.2 and (2.11). For the sufficiency, note that, from Proposition 4.2, the flow $\mathfrak{F}_\xi$ is normal. If $\mathfrak{F}_\xi$ is also contact, the result follows from Theorem 3.3. If $\mathfrak{F}_\xi$ is not contact, (M,g) is a reducible naturally reductive space and proceeding as in the proof of the de Rham decomposition for KTS–spaces in [7], is easy to see that M is a direct product $(M,g) = (M',g') \times (M'',g'')$, where M' is a naturally reductive homogeneous space with invariant $\xi \in \mathfrak{X}(M')$ and contact flow $\mathfrak{F}_\xi$ on it and (M'', g'') is a naturally reductive space where the torsion tensor $\tilde{K}$ restricted to M'' vanishes. The rest follows directly from Lemma 4.2 and Theorem 3.5. ■

In [23] and [12], it was proved that for any three– and five–dimensional naturally reductive space $(M, g) = G/H$ with the adapted canonical connection $\tilde{\nabla}$ one can find an orthonormal basis of T_oM such that its torsion tensor field verifies (4.11). From this, we get (see also [5])

Theorem 4.2. *Any irreducible, non–symmetric, simply connected three– and five–dimensional naturally reductive space is a* KTS*–space.*

Using again Proposition 4.2, Theorem 4.1 can also be formulated as follows.

Theorem 4.3. *Let $\mathfrak{F}_\xi$ be an isometric flow on a complete, simply connected Riemannian manifold (M, g). Then (M, g, ξ) is a* KTS*–space if and only if $(M, g) = G/H$ is a naturally reductive homogeneous space, $\mathfrak{F}_\xi$ is a normal invariant flow and it admits an adapted canonical connection $\tilde{\nabla}$ whose torsion tensor field $\tilde{K}$ at the origin satisfies*

$$\tilde{K}_o(A, B) = 0 \quad , \quad A, B \in \ker H_o.$$

In this case, $\tilde{\nabla}$ coincides with the canonical connection $\overline{\nabla}$ of the isometric flow $\mathfrak{F}_\xi$.

Moreover, this result allows to determine KTS–spaces in the class of naturally reductive spaces by means of restrictions on the rank of H. More precisely, we have

Corollary 4.1. *Let (M, g) be a connected simply connected naturally reductive homogeneous space equipped with a normal invariant flow $\mathfrak{F}_\xi$ such that* rank $H = 2k \geq$ dim $M - 3$. *Then (M, g, ξ) is a* KTS*–space.*

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