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Controllability of Quasilinear Integrodifferential Systems in Banach Spaces

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Abstract: In this paper we establish sufficient conditions for the controllability of quasilinear delay integrodifferential systems in Banach spaces. The results are obtained using the theory of semigroup of operators and the Schauder-Tychonov theorem. The results generalize the results of [6].

Key words: Controllability, Quasilinear integrodifferential systems, Schauder-Tychonov theorem.

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1.Introduction

Controllability of nonlinear systems represented by ordinary differential equations in infinite dimensional spaces has been studied by several authors. Triggiani [20] studied the controllability problem in Banach spaces with bounded operators. The importance of the question of controllability with control constraints in abstract spaces is established in [1,11,12,19]. Using an implicit function theorem, Chukwu and Lenhart [8] showed that the nonlinear system

$$x'(t) = f(t, x(t), u(t)), \qquad x(t_0) = x_0 \tag{1}$$

is locally approximate null controllable provided that the linear operator of the system is approximately invertible and the linear approximation to (1) is locally null controllable. Naito [13-15] established the approximate controllability of semilinear control systems under simple and fundamental assumptions on the systems components. Naito and Park [16] discussed the same problem for delay Volterra control systems by using the Leray-Schauder degree theorem. Yamamoto and Park [21] established necessary and sufficient conditions for the approximate controllability of parabolic equations in a Banach space with uniformly bounded nonlinear term with the help of estimates of solutions to the nonlinear parabolic systems. Kwun et al [9] investigated the controllability and approximate controllability of delay Volterra systems by using a fixed point theorem. Balachandran et al [3-5] studied the problem for semilinear evolution systems and nonlinear integrodifferential systems in

Banach spaces. Recently Balachandran and Dauer [2] discussed the controllability of Sobolev-type integrodifferential systems in Banach spaces. In this paper we shall study the controllability of quasilinear delay integrodifferential systems by using the Schauder-Tychonov theorem. Motivation for this type of systems are found in [17,18].

2.Preliminaries

Let $L(X,Y)$ be the Banach space of all bounded linear operators from X into Y. The symbol $\|.\|$ denotes the norm of all the spaces and bounded linear operators considered in this paper. It also denotes the sup-norm of any bounded continuous function. Let $J \subset R = (-\infty,\infty)$ be a bounded interval and let the operator $A : J \times X \to Y$ be defined; then $A(.,x)$ is continuous tX-uniformly in x if for every bounded subset M of X we have

$$\lim_{t\in J;t\to t_0} \sup_{x\in M} \|A(t,x) - A(t_0,x)\| = 0 \quad \text{for every} \quad t_0 \in J. \tag{2}$$

We denote by $C(J,X)$ the space of all continuous functions from J into X with the supnorm. Let $C = C([-r,0],X)$.

Consider the quasilinear delay integrodifferential equation

$$\begin{aligned} x'(t) + A(t,x(t))x(t) &= B(t,x(t))u(t) \\ &\quad + f(t,x(t),x(t-r),\int_0^t \eta(t,s,x(s))ds),\ t,s \in J = [0,T] \\ x(t) &= \phi(t),\ t \in [-r,0] \end{aligned} \tag{3}$$

where the state $x(t)$ takes the values in the Banach space X and the control function u is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. The operators A and B are such that $A(t,x) \in L(X,X)$ and $B(t,x) \in L(U,X)$ for every $(t,x) \in J \times X$ and that A and B are compact and continuous in x. Further, the nonlinear operators $f : J \times X^3 \to X$ and $\eta : J \times J \times X \to X$ are compact and continuous in (x,y,w). For fixed $z \in C([-r,T],X)$ we let $X_z(t),\ \ t \in J = J,\ \ X_z(0) = I$, denote the fundamental operator of the equation [10]

$$x'(t) + A(t,z(t)))x(t) = 0, \quad x(0) = \phi(0).$$

Then $X_z \in C(J,L(X,X))$ and X_z is the unique continuously differentiable solution which satisfies

$$\dot{X}_z + A(t,z(t))X_z = 0, \quad t \in J,\ \ X_z(0) = I \tag{4}$$

Moreover, $X_z^{-1} \in C(J,L(X,X))$ and X_z^{-1} is the unique continuously differentiable solution of

$$\dot{X}_z^{-1} - X_z^{-1}A(t,z(t)) = 0, \quad t \in J, \quad X_z^{-1}(0) = I. \tag{5}$$

Definition: The system (3) is said to be controllable on the interval J if for every continuous initial function ϕ defined on $[-r, 0]$ and every $v \in X$ there exists a control $u \in L^2(J, U)$ such that the solution $x(t)$ of (3) satisfies $x(T) = v$.

Now for each fixed $z \in C(J, X)$, consider the system

$$\begin{aligned} x'(t) + A(t, z(t))x(t) &= B(t, z(t))u(t) \\ &\quad + f(t, z(t), z(t-r), \int_0^t \eta(t, s, z(s))ds), \quad s, t \in J \\ x(t) &= \phi(t), \quad t \in [-r, 0], \end{aligned}$$

where the operators $A(t, x)$ and $B(t, x)$ are continuous on $J \times X$ and $f(t, x, y, w)$ is continuous on $J \times X^3$.

Therefore for each controller $u(t) \in L^2(J, U)$ this equation has a unique solution $x_z(t)$ such that

$$\begin{aligned} x_z(t) &= X_z(t)\phi(0) + \int_0^t X_z(t)X_z^{-1}(s)B(s, z(s))u(s)ds \\ &+ \int_0^t X_z(t)X_z^{-1}(s)f(s, z(s), z(s-r), \int_0^s \eta(s, \tau, z(\tau))d\tau)ds, \quad t \in J \qquad (6) \\ x_z(t) &= \phi(t), \quad t \in [-r, 0]. \end{aligned}$$

We will assume the following hypotheses.

(i) There is a positive constant K such that the fundamental operator solution X_z satisfies $\|X_z(t)\| \le K$ and $\|X_z^{-1}(t)\| \le K$.

(ii) The operators $A(t, x)$ and $B(t, x)$ are compact, continuous tX -uniformly in x and satisfy equation (2) with $\|A(t, z(t))\| \le K_1$, $\|B(t, z(t))\| \le K_2$ and the operator $f(t, x, y, w)$ is compact, continuous tX -uniformly in (x, y, z) and satisfy equation(2) with $\|f(t, z(t), z(t-r), \int_0^t \eta(t, s, z(s)ds))\| \le K_3$ where K_1, K_2 and K_3 are positive constants.

(iii) The linear operator W from $L^2(J, U)$ onto X defined by

$$Wu = \int_0^T X_z(T)X_z^{-1}(s)B(s, z(s))u(s)ds$$

has an invertible operator W^{-1} which takes values in $L^2(J, U) \setminus kerW$.

3. Main Result

Theorem: If hypotheses (i)-(iii) are satisfied then the system (3) is controllable on J.

Proof: Using hypothesis (iii), define the control

$$u(t) = W^{-1}[v - X_z(T)\phi(0) - \int_0^T X_z(T)X_z^{-1}(s)f(s, z(s), z(s-r), \int_0^s \eta(s,\tau,z(\tau))d\tau)ds](t).$$

Using this control we will show that the operator defined by

$$\begin{aligned}
\Phi x_z(t) &= \phi(t), \quad \text{for } t \in [-r,0].\\
\Phi x_z(t) &= X_z(t)\phi(0)\\
&+ \int_0^t X_z(t)X_z^{-1}(s)f(s, z(s), z(s-r), \int_0^s \eta(s,\tau,z(\tau))d\tau)ds\\
&+ \int_0^t X_z(t)X_z^{-1}(s)B(s,z(s))W^{-1}[v - X_z(T)\phi(0)\\
&- \int_0^T X_z(T)X_z^{-1}(\theta)f(\theta, z(\theta), z(\theta - r), \int_0^\theta \eta(\theta,\tau,z(\tau))d\tau)d\theta](s)ds.
\end{aligned}$$

has a fixed point. This fixed point is then a solution of the equation (6).
Let $M = \{z \in C([-r,T],X): \ z(t) = \phi(t), \ t \in [-r,0], \ \|z\| \le \alpha$ and $\|z(t) - z(t')\| \le N|t - t'|, \ t,t' \in J\}$
where

$$\alpha = K\|\phi\| + K^2K_3T + LT, \quad N = KK_1\|\phi\| + K^2K_1K_3T + K^2K_3 + (1+K_1T)L,$$
$$L = K^2K_2\|W^{-1}\|\{\|v\| + K\|\phi\| + K^2K_3T\}.$$

Then M is non-empty, because the function $z : [-r,T] \to X$ with $z(t) = \phi(t), t \in [-r,0]$, and $z(t) = \phi(0), t \in J$, belongs to M. Let $\Phi : M \to C([-r,T],X)$ be the operator that maps $z \in M$ to x_z. In order to apply the Schauder-Tychonov theorem on M, we first show that $\Phi M \subset M$. In fact, given $z \in M$, we have

$$\|x_z(t)\| \le \phi(t), \quad t \in [-r,0]$$

and for $t \in J$

$$\begin{aligned}
&\|x_z(t)\|\\
&\le \|X_z(t)\|\|\phi\| + \int_0^t \|X_z(t)\|\|X_z^{-1}(s)\|\|f(s,z(s),z(s-r), \int_0^s \eta(s,\tau,z(\tau))d\tau)\|ds\\
&\quad + \int_0^t \|X_z(t)\|\|X_z^{-1}(s)\| \ \|B(s,z(s))\|\|W^{-1}\|\\
&\quad \times [\|v\| + \|X_z(T)\|\|\phi\| + \|X_z(T)\| \int_0^T \|X_z^{-1}(\theta)\|\\
&\qquad \|f(\theta, z(\theta), z(\theta - r), \int_0^\theta \eta(\theta,\tau,z(\tau))d\tau\|d\theta](s)ds\\
&\le K\|\phi\| + K^2K_3T + K^2K_2\|W^{-1}\|[\|v\| + K\|\phi\| + K^2K_3T]T.
\end{aligned}$$

It follows that $\|x_z(t)\| \leq \alpha$. Since $X_z(t)$ satisfies equation (4), we have

$$\begin{aligned}\|X_z(t) - X_z(t')\| &\leq \int_t^{t'} \|A(s, z(s))\| \|X_z(s)\| ds \\ &\leq KK_1|t - t'|.\end{aligned}$$

Using this and given $t, t' \in J$ we have

$$\begin{aligned}
&\|x_z(t) - x_z(t')\| \\
&\leq \|X_z(t) - X_z(t')\| \|\phi\| \\
&+ \|X_z(t) - X_z(t')\| \int_0^t \|X_z^{-1}(s)\| \\
&\qquad \|f(s, z(s), z(s-r), \int_0^s \eta(s, \tau, z(\tau))d\tau)\| ds \\
&+ \|X_z(t')\| \int_{t'}^t \|X_z^{-1}(s)\| \\
&\qquad \|f(s, z(s), z(s-r), \int_0^s \eta(s, \tau, z(\tau))d\tau)\| ds \\
&+ \|X_z(t) - X_z(t')\| \int_0^t \|X_z^{-1}(s)\| \|B(s, z(s))\| \|W^{-1}\| \\
&\quad [\|v\| + \|X_z(T)\| \|\phi\| + \int_0^T \|X_z(T)\| \|X_z^{-1}(\theta)\| \\
&\quad \|f(\theta, z(\theta), z(\theta - r), \int_0^\theta \eta(\theta, \tau, z(\tau))d\tau)\| d\theta](s) ds \\
&\quad + \|X_z(t')\| \int_{t'}^t \|X_z^{-1}(s)\| \|B(s, z(s))\| \|W^{-1}\| \\
&\quad + [\|v\| + \|X_z(T)\| \|\phi\| + \int_0^T \|X_z(t)\| \|X_z^{-1}(\theta)\| \\
&\quad \|f(\theta, z(\theta), z(\theta - r), \int_0^\theta \eta(\theta, \tau, z(\tau))d\tau)\| d\theta](s) ds \\
&\leq KK_1|t - t'| \|\phi\| + KK_1|t - t'| [KK_1T + K^2K_3|t - t'| \\
&\quad + KK_1|t - t'| KK_2 \|W^{-1}\| [\|v\| + K\|\phi\| + K^2K_3T]T \\
&\quad + K^2K_2\|W^{-1}\| [\|v\| + K\|\phi\| K^2K_3T]|t - t'| \\
&\leq N|t - t'|
\end{aligned}$$

Hence $\|x_z(t) - x_z(t')\| \leq N|t - t'|$. It follows that $\Phi M \subset M$. To show that Φ is continuous, let $z_n, z \in M$ be given with $\|z_n - z\| \to 0$ as $n \to \infty$. Then, using assumption (ii) with

$$\begin{aligned}
&\|X_{z_n}(t) - X_z(t)\| \\
&\leq \int_0^t \|A(s, z_n(s))X_{z_n}(s) - A(s, z(s)))X_z(s)\| ds \\
&\leq \int_0^t \|A(s, z_n(s)) - A(s, z(s))\| \|X_{z_n}(s)\| ds \\
&\quad + \int_0^t \|A(s, z(s))\| \|X_{z_n}(s) - X_z(s)\| ds
\end{aligned}$$

and Gronwall's inequality, we obtain

$$\|X_{z_n}(t) - X_z(t)\| \le Ke^{kT}\int_0^T \|A(s,z_n(s)) - A(s,z(s))\|ds$$

for every $t \in J$. This shows that $\|X_{z_n} - X_z\| \to 0$ as $n \to \infty$. Similarly, using (5), we can prove that $\|X_{z_n}^{-1} - X_z^{-1}\| \to 0$ as $n \to \infty$. From the continuity of B and f, η we see that $B(t,z_n(t))$ and $f(t,z_n(t),z_n(t-r),\int_0^t \eta(t,s,z_n(s))ds)$ converge uniformly to $B(t,x(t))$ and $f(t,z(t),z(t-r),\int_0^t \eta(t,s,z(s))ds)$ respectively. Using these facts, it is easily seen that $\|x_{z_n} - x_z\| \to 0$ as $n \to \infty$. Consequently, Φ is continuous on M. Before we show that M is relatively compact set, we first prove that the operators

$A_1 : M \to C(J, L(X,X))$ defined by $(A_1 z)(t) = A(t,z(t))$
$B_1 : M \to C(J, L(U,X))$ defined by $(B_1 z)(t) = B(t,z(t))$
$f_1 : M \to C(J,X)$ defined by

$$(f_1 z)(t) = f(t,z(t),z(t-r),\int_0^t \eta(t,s,z(s))ds)$$

are compact. For this let $\{z_n\}$ be a sequence in M. We first observe that

$$\|A(t,z_n(t))\| \le K_1, \quad t \in J, \ n = 1,2,...$$

Given $t, t_0 \in J$, we find

$$\begin{aligned}
&\|A(t,z_n(t)) - A(t_0,z_n(t_0))\| \\
&\quad \le \|A(t,z_n(t)) - A(t_0,z_n(t))\| + \|A(t_0,z_n(t)) - A(t_0,z_n(t_0))\| \\
&\quad \le \sup_{\|x\|\le\alpha} \|A(t,x) - A(t_0,x)\| + \|A(t_0,z_n(t)) - A(t_0,z_n(t_0))\| \cdot
\end{aligned}$$

Hypothesis (ii) and the uniform Lipschitz continuity of the functions z_n on $[-r,T]$ imply the equicontinuity of the functions $A_n(t) = A(t,z_n(t)), \ t \in J, \ n = 1,2,....$. Now let $t_0 \in J$ be given. Then, since $\{z_n(t_0)\}$ is a bounded sequence, the compactness of $A(t_0,x)$ in x implies the relative compactness of the set $\{A(t_0,z_n(t_0))\}$. Consequently, the operator A_1 is compact. A similar argument proves the compactness of B_1 and f_1. Thus $A \in C(J,L(X,X))$, $B \in C(J,L(U,X))$ and $f \in C(J,X)$. Therefore, given a sequence $\{z_n\} \subset M$, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $A(t,z_{n_k}(t)) \to A(t)$, $B(t,z_{n_k}(t)) \to B(t)$, $f(t,z_{n_k}(t),z_{n_k}(t-r),\int_0^t \eta(t,s,z_{n_k}(s))ds) \to f(t)$ uniformly on J as $k \to \infty$. Let $X(t)$ denote the fundamental operator for the problem

$$x'(t) + A(t)x(t) = 0, \quad x(0) = \phi(0)$$

Then,

$$x(t) = X(t)\phi(0) + \int_0^t X(t)X^{-1}(s)[B(s)u(s) + f(s)]ds, \quad t \in J.$$

is the unique solution of the problem

$$x'(t) + A(t)x(t) = B(t)u(t) + f(t), \quad t \in J, \quad x(0) = \phi(0).$$

It is easy to see now that $X_{z_{n_k}}(t) \to X(t)$ and $X_{z_{n_k}}^{-1}(t) \to X^{-1}(t)$ uniformly on J. It follows that $x_{z_{n_k}}(t) \to x(t)$ uniformly on J. Since $x_{z_{n_k}}(t) = \phi(t)$ for $t \in [-r, 0]$, we have proved the compactness of ΦM. Hence by Schauder-Tychonov theorem there exists a fixed point $x(t)$ in M such that $\Phi x(t) = x(t) = x_z(t)$ and which satisfies the condition $x(T) = x_z(T) = v$.

4. Application

Consider the Sobolev-type system of the form

$$\begin{aligned} &\frac{d}{dt}(E(t)z(t)) + A(t, z(t))z(t) \\ &\qquad\qquad = B(t, z(t))u(t) + f(t, z(t), z(t-r), \int_0^t \eta(t, s, z(s))ds), \\ &E(t)z(t) = \phi(t) \quad \text{on} \quad [-r, 0]. \end{aligned} \tag{7}$$

For motivation of the above system one can refer [7]. Assume the following additional conditions:

(i) For each $t \in [-r, T]$, $E(t)$ is linear, closed and densely defined with domain $D(E)$ (independent of t) in $D(A)$ and range Y. Moreover, for each $t \in [-r, T]$, $E^{-1}(t) : Y \to X$ exists and is compact while $E^{-1}(t)z$ is continuous in t for each $z \in Y$.

(ii) For each $(t, z) \in J \times X$, $A(t, E^{-1}(t)z)E^{-1}(t) \in L(X, X)$ is continuous in (t, z) with its continuity tX-uniform in z.

(iii) For each $(t, z) \in J \times X$, $B(t, E^{-1}(t)z) \in X$, is continuous in (t, z) with its continuity tX- uniform in z.

(iv) For each $(t, s, z) \in J \times J \times X, \eta(t, s, E^{-1}(s)z) \in X$, is continuous in (t, s, z) with its continuity t^2X-uniform in z.

(v) For each $(t, z, y, w) \in J \times X^3$,

$$f(t, E^{-1}(t)z, E^{-1}(t-r)y, \int_0^t \eta(t, s, E^{-1}(s)w)ds) \in X,$$

is continuous in (t, z, y, w) with its continuity tX^3- uniform in (z, y, w); $\phi : [-r, 0] \to X$ is a Lipschitz continuous function.

For this consider the problem

$$\begin{aligned}
&\frac{d}{dt}y(t) + A(t, E^{-1}(t)y(t))E^{-1}(t)y(t) \\
&\quad = B(t, E^{-1}(t)y(t))u(t) \\
&\qquad + f(t, E^{-1}(t)y(t), E^{-1}(t-r)y(t-r), \int_0^t \eta(t,s,E^{-1}(s)y(s))ds) \qquad (8) \\
y(t) &= \phi(t) \text{ on } [-r, 0].
\end{aligned}$$

If $y(t)$ is a solution of (8), then $z(t) = E^{-1}(t)y(t)$ satisfies equation (7). Therefore the controllability problem of (7) is equivalent to that of (8). Hence by the application of the above theorem the system (8) is controllable.

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