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Fusion of the q-Vertex Operators and its Application to Solvable Vertex Models

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Abstract: We diagonalize the transfer matrix of the inhomogeneous vertex models of the 6-vertex type in the anti-ferroelectric regime directly in the infinite lattice. For this purpose we have introduced new types of q-vertex operators. The special cases of those transfer matrices were used to diagonalize the s-d exchange model [23, 1, 6]. New vertex operators are constructed from the level one vertex operators by the fusion procedure. Using this construction we determine the commutation relations among new vertex operators which play a crucial role for the diagonalization. In order to clarify the quasi-particle structure of the model we estabish new isomorphisms of crystals. The isomorphisms figure out, representation theoretically, the ground state degenerations.

1. Introduction

In [2] the anti-ferroelectric XXZ hamiltonian, or equivalently, the transfer matrix of the 6-vertex model has been diagonalized directly in the thermodynamic limit based on the quantum affine symmetry. The method is powerful enough, on the one hand, to give the integral formulas for correlation functions and form factors, on the other hand, to determine the physical space as a representation of a quantum affine algebra $U_q(\widehat{sl_2})$.

A similar approach is possible for several two dimensional lattice models such as the ABF model [11, 4]. Among them a direct generalization of the 6-vertex model is the vertex models associated with the perfect representations of any level [15, 16]. Although there are technical problems of bosonization in the case of higher levels, at least the strategy is clear and everything we need is in our hands.

In this paper I want to add one more class of vertex models which can be solved by a similar method and are not contained in the class of directly generalized models above. The models which we study here are the inhomogeneous vertex models of 6-vertex type with the inhomogeneities in the spins. Namely, on the infinite regular square lattice, with each horizontal and vertical line except a finite number of vertical lines $l_1, \dots, l_n$, we associate the vector space $\mathbf{C}^2$. With

$l_1, \ldots, l_n$ we associate $\mathbf{C}^{s_1+1}, \ldots, \mathbf{C}^{s_n+1}$ for arbitrary non-negative integers $s_1, \ldots, s_n$. To each vertex the Boltzmann weight is defined by the corresponding trigonometric R-matrix acting on $\mathbf{C}^2 \otimes \mathbf{C}^2$ or $\mathbf{C}^2 \otimes \mathbf{C}^{s_j+1}$. The rational limits of those models with $n = 1$ had been used to diagonalize the s-d exchange models (Kondo problem) [1, 23, 6, 22].

The central object in the symmetry approach is the q-vertex operator which was introduced by Frenkel–Reshetikhin [5]. In the case of the 6-vertex model the q-vertex operator makes it possible to identify the infinite tensor product $(\mathbf{C}^2)^{\otimes \mathbf{Z}_{\geqq 1}}$ with the irreducible representation $V(\Lambda_i)$ of $U_q(\widehat{sl_2})$. Using this identification, the transfer matrix, the creation-annihilation operators, correlation functions and form factors are all described in terms of q-vertex operators.

Similarly, in our case, everything is described by q-vertex operators. But here a new phenomenon appears, the degeneration of the ground states. To take this effect into consideration is crucial in the theory. To treat this situation correctly we introduce new kinds of q-vertex operators. Those new operators can be considered as a mixture of type I and type II vertex operators in the terminology of [2]. They are shown to be obtained by a fusion procedure from level one vertex operators. In particular new operators have the description by free fields. Hence physical quantities of our models can be written down in the form of integral formulas. We study these formulas in a subsequent paper.

Let us describe our idea more precisely. The total quantum space which is acted on by the transfer matrix is

$$\bigoplus_{i,j=0,1} V(\Lambda_i) \otimes V_{s_n} \otimes \cdots \otimes V_{s_1} \otimes V(\Lambda_j)^{*a}\,, \tag{1}$$

where $V_s \simeq \mathbf{C}^{s+1}$ and is considered as the representation of $U_q'(\widehat{sl_2})$. In order to give the description of the correlation function or form factors we must know the structure of eigenstates of the transfer matrix. The insight comes, as in the case of the XXZ-model [2, 7], from the decomposition of crystals

$$B(\Lambda_i) \otimes B_{s_1} \otimes \cdots \otimes B_{s_k} \otimes B(\Lambda_j)^*\,. \tag{2}$$

The result is surprisingly simple (see Corollary 1). With the aid of the decomposition we find that the physical space of our models can be written as

$$\mathbf{C}^{s_n} \otimes \cdots \otimes \mathbf{C}^{s_1} \otimes \left[\bigoplus_{m=0}^{\infty} \int_{|z_1|=1} \cdots \int_{|z_m|=1} (\mathbf{C}^2)^{\otimes m} \right]_{\mathrm{sym}}\,,$$

where sym is some symmetrization. In this tensor product the second term which is described by a bracket is isomorphic to the physical space of the XXZ model. On the other hand the former tensor component $\mathbf{C}^{s_n} \otimes \cdots \otimes \mathbf{C}^{s_1}$ describes the ground state degeneration. In the case $n = 1$ the dimension of the degeneracy of the ground states coincides with the results of Fateev–Wiegman [6] in the rational limits. This picture of the structure of the space of quasi-particles suggests that it is natural to consider the space

$$\bigoplus_{i,j=0,1} V_{s_n-1} \otimes \cdots \otimes V_{s_1-1} \otimes V(\Lambda_i) \otimes V(\Lambda_j)^{*a}\,. \tag{3}$$

The relation between two spaces (1) and (3) is given by the new vertex operators

$$ {}_{s-1}O^s(z) : (V_{s-1})_z \otimes V(\Lambda_i) \to V(\Lambda_{i+1}) \otimes (V_s)_z \,, \tag{4} $$

$$ {}^{s-1}O_s(z) : V(\Lambda_i) \otimes (V_s)_z \to (V_{s-1})_z \otimes V(\Lambda_{i+1}) \,. \tag{5} $$

On the space (3) descriptions of the model and physical quantities take very simple forms. For example the transfer matrix is, up to a scalar multiple, equal to $1 \otimes T_{\mathrm{XXZ}}(z)$, where $T_{\mathrm{XXZ}}(z)$ is the transfer matrix of the 6-vertex model acting on $\bigoplus_{i,j=0,1} V(\Lambda_i) \otimes V(\Lambda_j)^{*a}$. The peculiarity of our model comes from the definition of the local operators which are defined using vertex operators (4) and (5).

The present paper is organized in the following manner. In Sect. 2 we review necessary preliminaries and notations. In Sect. 3 we prove new isomorphisms of crystals which are considered as a generalization of the path realization of the crystals with highest weights. Applying this isomorphism we determine the decomposition of the crystal (2). In Sect. 4 we introduce new vertex operators and prove their existence. The fusion construction of the representations and R-matrices are briefly reviewed in Sect. 5. In Sect. 6 the fusion procedure is carried out for level one vertex operators and construct new vertex operators. The well definedness of the fusion procedure is the main result here. We calculate necessary commutation relations of newly defined vertex operators using the fusion construction in Sect. 7. In Sect. 8 we propose the mathematical settings for our models. In Appendix 1 the integral formulas for the highest-highest matrix element of the composition of type I and type II vertex operators are given. In Appendix 2, 3 the description of level one vertex operators in terms of bosons and their OPEs are given. These are used to derive the integral formulas in Appendix 1.

2. Notations and Preliminaries

2.1. Definition of Quantized Envelopping Algebra. Let us recall the definition of $U_q(\widehat{sl_2})$ and fix several notations related to it. Let $P = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \mathbf{Z}\delta$, $P^* = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \mathbf{Z}d$ be the weight and the dual weight lattice of $\widehat{sl_2}$ with the pairing $\langle \Lambda_i, h_j\rangle = \delta_{ij}$, $\langle \Lambda_i, d\rangle = 0$, $\langle \delta, h_i\rangle = 0$, $\langle \delta, d\rangle = 1$. Set $\alpha_1 = 2\Lambda_1 - 2\Lambda_0$, $\alpha_0 = \delta - \alpha_1$, $\rho = \Lambda_0 + \Lambda_1$. The symmetric bilinear form on P normalized as $(\alpha_i, \alpha_i) = 2$ is given by $(\Lambda_i, \Lambda_j) = \frac{\delta_{1i}\delta_{1j}}{2}$, $(\Lambda_i, \delta) = 1$, $(\delta, \delta) = 0$. Through $(\ ,\ )$ we consider P^* as a subset of P so that $2\rho = h_1 + 4d$. Let us set $F = \mathbf{Q}(q)$ with q being the complex number transcendental over the rational number field $\mathbf{Q}$. In Sect. 8, we assume that the q is real and $-1 < q < 0$.

The algebra $U_q(\widehat{sl_2})$ is the F-algebra generated by e_i, f_i, $(i = 0, 1)$, q^h $(h \in P^*)$ with the defining relations

$$ q^0 = 1, \qquad q^{h_1} q^{h_2} = q^{h_1 + h_2}, \qquad q^h e_i q^{-h} = q^{\langle h, \alpha_i\rangle} e_i, \qquad q^h f_i q^{-h} = q^{-\langle h, \alpha_i\rangle} f_i \,, $$

$$ [e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}}, \qquad \sum_{m=0}^{3} (-1)^m x_i^{(m)} x_j x_i^{(3-m)} = 0 \quad (i \neq j) \text{ for } x = e, f \,, $$

where we set $t_i = q^{h_i}$ and

$$x_i^{(m)} = \frac{x_i^m}{[m]!} \quad (x = e, f), \quad [m] = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]! = \prod_{k=1}^{m} [k],$$

$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{[n]!}{[j]![n-j]!} .$$

We denote by $U' = U_q'(\widehat{sl_2})$ the subalgebra of $U_q(\widehat{sl_2})$ generated by e_j, f_j, $t_j^{\pm 1}$ $(j = 0, 1)$ and by U_i' the subalgebra of U' generated by e_i, f_i, $t_i^{\pm 1}$ which is isomorphic to $U_q(sl_2)$.

We use the following coproduct and the anti-pode for U',

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \qquad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \qquad \Delta(q^h) = q^h \otimes q^h ,$$
$$a(e_i) = -t_i^{-1} e_i, \qquad a(f_i) = -f_i t_i, \qquad a(q^h) = q^{-h} .$$

The U' module $(V_n)_z = \bigoplus_{j=0}^n F[z, z^{-1}] v_j^{(n)}$ is defined as

$$f_1 v_j^{(n)} = [n-j] v_{j+1}^{(n)}, \qquad e_1 v_j^{(n)} = [j] v_{j-1}^{(n)}, \qquad t_1 v_j^{(n)} = q^{n-2j} v_j^{(n)} ,$$
$$f_0 = z^{-1} e_1, \qquad e_0 = z f_1, \qquad t_0 = t_1^{-1} ,$$

where z is a non-zero complex number. As a module over U_1', $(V_n)_z$ is isomorphic to the irreducible $n+1$ dimensional representation which is independent of the parameter z. We denote by V_n this representation of U_1' except in Sect. 8, where V_n is used for $(V_n)_1$. In the following sections, for the sake of simplicity, we simply write F instead of $F[z, z^{-1}]$ as far as no confusion occurs.

For a left U'-module M, we define the left module $M^{*a^{\pm 1}}$ by

$$M^{*a^{\pm 1}} = \mathrm{Hom}_F(M, F) \text{ as a linear space} ,$$

$$\langle xw, v \rangle = \langle w, a^{\pm 1}(x) v \rangle \quad \text{for } w \in M^{*a^{\pm 1}},\ v \in M \text{ and } x \in U' .$$

Here the linear dual of an integrable module with finite dimensional weight spaces should be considered the restricted dual. By definition M, $M^{*a*a^{-1}}$ and $M^{*a^{-1}*a}$ are canonically isomorphic. For these dual modules the following properties hold,

$$\mathrm{Hom}_{U'}(M_1 \otimes M_2, M_3) \simeq \mathrm{Hom}_{U'}(M_1, M_3 \otimes M_2^{*a}) , \tag{6}$$

$$\mathrm{Hom}_{U'}(M_1 \otimes M_2, M_3) \simeq \mathrm{Hom}_{U'}(M_2, M_1^{*a^{-1}} \otimes M_3) , \tag{7}$$

where $\mathrm{Hom}_{U'}(M_1, M_2)$ is the vector space of U' linear homomorphisms. Let $\{v_j^{(n)*}\}$ be the dual base of $\{v_j^{(n)}\}$, $\langle v_j^{(n)*}, v_k^{(n)} \rangle = \delta_{jk}$. Then the following isomorphisms hold,

$$C_{\pm}^{(n)} : \ (V_n)_{q^{\mp 2} z} \simeq (V_n)_z^{*a^{\pm 1}} ,$$

$$v_j^{(n)} \mapsto (-)^j q^{-j(n-j \mp 1)} \begin{bmatrix} n \\ j \end{bmatrix}^{-1} v_{n-j}^{(n)*} ,$$

$$(-)^{n-j} q^{(n-j)(j \mp 1)} \begin{bmatrix} n \\ j \end{bmatrix} v_{n-j}^{(n)} \leftarrow v_j^{(n)*} . \tag{8}$$

2.2. Level One Vertex Operators. The details of this section can be found in [2, 8]. Let $V(\Lambda_i)$ be the irreducible highest weight U'-module with highest weight Λ_i $(i = 0, 1)$, $\hat{V}(\Lambda_i)$ its weight completion $\hat{V}(\Lambda_i) = \prod_{\nu \in P} V(\Lambda_i)_\nu$, $V(\Lambda_i)_\nu = \{v \in V(\Lambda_i) | q^h v = q^{\langle h,\nu\rangle} v$ for any $h \in P\}$ and $u^*_{\Lambda_i}$ the highest weight vector of the right module $V(\Lambda_i)^*$ such that $\langle u^*_{\Lambda_i}, u_{\Lambda_i}\rangle = 1$. We often use the notations $\langle v|, |u\rangle$ for the elements of $V(\Lambda_i)^*$ and $V(\Lambda_i)$. In that case the value of the dual pairing is denoted by $\langle v|u\rangle$. For the sake of simplicity we sometimes use $\langle \Lambda_i|, |\Lambda_i\rangle$ instead of $u^*_{\Lambda_i}$, u_{Λ_i} and write $|xv\rangle$ instead of writing $x|v\rangle$ for $x \in U'$.

Let us denote $\Phi(z)$ and $\Psi(z)$ the U' intertwiners

$$\Phi(z) : V(\Lambda_i) \to V(\Lambda_{i+1}) \hat{\otimes} (V_1)_z ,$$

$$\Psi(z) : V(\Lambda_i) \to (V_1)_z \hat{\otimes} V(\Lambda_{i+1}) ,$$

normalized as

$$\langle u^*_{\Lambda_{i+1}}, \Phi(z) u_{\Lambda_i}\rangle = \langle u^*_{\Lambda_{i+1}}, \Psi(z) u_{\Lambda_i}\rangle = z^{\frac{1-2i}{4}} v^{(1)}_{1-i} .$$

Here we set $V(\Lambda_i) \hat{\otimes} (V_n)_z = (\prod_{\nu \in P} F[z, z^{-1}] \otimes V(\Lambda_i)_\nu) \bigotimes_{F[z,z^{-1}]} (V_n)_z$. In fact the images of $\Phi(z)$ and $\Psi(z)$ belong to smaller spaces [2].

The components of those operators are defined by

$$\Phi_j(z) = \langle v^{(1)*}_j, \Phi(z)\rangle, \qquad \Psi_j(z) = \langle v^{(1)*}_j, \Psi(z)\rangle .$$

We shall also introduce the intertwiners $\Phi^{V^{*a^{\pm 1}}}(z)$, $\Psi^{V^{*a^{-1}}}(z)$, $\Phi_V(z)$ and $\Psi_V(z)$

$$\Phi^{V^{*a^{\pm 1}}}(z) : V(\Lambda_i) \to V(\Lambda_{i+1}) \hat{\otimes} (V_1)^{*a^{\pm 1}}_z ,$$

$$\Psi^{V^{*a^{-1}}}(z) : V(\Lambda_i) \to (V_1)^{*a^{-1}}_z \hat{\otimes} V(\Lambda_{i+1}) ,$$

$$\Phi_V(z) : V(\Lambda_i) \otimes (V_1)_z \to \hat{V}(\Lambda_{i+1}) ,$$

$$\Psi_V(z) : (V_1)_z \otimes V(\Lambda_i) \to \hat{V}(\Lambda_{i+1}) ,$$

defined by

$$\Phi^{V^{*a^{\pm 1}}}(z) = (1 \otimes C^{(1)}_{\pm}) \Phi(q^{\mp 2} z), \qquad \Psi^{V^{*a^{-1}}}(z) = (C^{(1)}_{-} \otimes 1) \Psi(q^2 z) ,$$

$$\Phi_V(z)(u \otimes v^{(1)}_j) = \langle v^{(1)}_j, \Phi^{V^{*a}}(z) u\rangle, \qquad \Psi_V(z)(v^{(1)}_j \otimes u) = \langle v^{(1)}_j, \Psi^{V^{*a^{-1}}}(z) u\rangle .$$

The commutation relations of those vertex operators are, on $V(\Lambda_i)$,

$$-\left(\frac{z_1}{z_2}\right)^{1/2} r\left(\frac{z_1}{z_2}\right) \check{\bar{R}}\left(\frac{z_1}{z_2}\right) \Phi(z_1)\Phi(z_2) = \Phi(z_2)\Phi(z_1) ,$$

$$\left(\frac{z_1}{z_2}\right)^{1/2} r\left(\frac{z_1}{z_2}\right) \check{\bar{R}}\left(\frac{z_1}{z_2}\right) \Psi(z_2)\Psi(z_1) = \Psi(z_1)\Psi(z_2) ,$$

$$\left(\frac{z_1}{z_2}\right)^{-1/2} \frac{\theta_{q^4}(\frac{q z_1}{z_2})}{\theta_{q^4}(\frac{q z_2}{z_1})} \Psi(z_1)\Phi(z_2) = \Phi(z_2)\Psi(z_1) .$$

We shall rewrite the first and second relations for the sake of later use as

$$-q\left(\frac{z_1}{z_2}\right)^{1/2} r\left(\frac{q^2 z_1}{z_2}\right) \check{\bar{R}}_{VV^{*a}}\left(\frac{z_1}{z_2}\right) \Phi(z_1)\Phi^{V^{*a}}(z_2) = \Phi^{V^{*a}}(z_2)\Phi(z_1)\,,$$

$$q^{-1}\left(\frac{z_1}{z_2}\right)^{1/2} r\left(\frac{z_1}{q^2 z_2}\right) \check{\bar{R}}_{VV^{*a^{-1}}}\left(\frac{z_1}{z_2}\right) \Psi^{V^{*a^{-1}}}(z_2)\Psi(z_1) = \Psi(z_1)\Psi^{V^{*a^{-1}}}(z_2)\,.$$

Here $\check{\bar{R}}(z) = P\bar{R}(z)$, $\check{\bar{R}}_{VV^{*a}}(z) = P\bar{R}_{VV^{*a}}(z)$, $\check{\bar{R}}_{VV^{*a^{-1}}}(z) = P\bar{R}_{VV^{*a^{-1}}}(z)$, $P(u\otimes v) = v\otimes u$ and

$$\bar{R}(z)(v_j^{(1)}\otimes v_j^{(1)}) = v_j^{(1)}\otimes v_j^{(1)} \quad \text{for } j = 0, 1\,,$$

$$\bar{R}(z)(v_0^{(1)}\otimes v_1^{(1)}) = bv_0^{(1)}\otimes v_1^{(1)} + cv_1^{(1)}\otimes v_0^{(1)}\,,$$

$$\bar{R}(z)(v_1^{(1)}\otimes v_0^{(1)}) = c'v_0^{(1)}\otimes v_1^{(1)} + bv_1^{(1)}\otimes v_0^{(1)}\,,$$

$$b = \frac{1-z}{1-q^2 z}q, \qquad c = \frac{1-q^2}{1-q^2 z}z, \qquad c' = \frac{1-q^2}{1-q^2 z}, \qquad r(z) = \frac{(z^{-1})_\infty (q^2 z)_\infty}{(z)_\infty (q^2 z^{-1})_\infty}\,,$$

$$\bar{R}_{VV^{*a}}(z) = (1\otimes C_+^{(1)})\bar{R}(q^2 z)(1\otimes C_+^{(1)})^{-1}\,,$$

$$\bar{R}_{VV^{*a^{-1}}}(z) = (1\otimes C_-^{(1)})\bar{R}(q^{-2} z)(1\otimes C_-^{(1)})^{-1}\,,$$

where $(z)_\infty = \prod_{j=0}^{\infty}(1 - zq^{4j})$.

Let us, in general, denote by P_F^n the dual pairing map $(V_n)_z^{*a}\otimes(V_n)_z \to F$ or $(V_n)_z\otimes(V_n)_z^{*a^{-1}} \to F$ which are U' linear. Then we have

$$P_F^1 \Phi^{V^{*a}}(z)\Phi(z) = (-1)^i q^{1/2} g^{-1} \mathrm{id}_{V(\Lambda_i)}\,, \tag{9}$$

$$P_F^1 \Phi(z)\Phi^{V^{*a^{-1}}}(z) = (-1)^{i+1} q^{-1/2} g^{-1} \mathrm{id}_{V(\Lambda_i)}\,, \tag{10}$$

$$\mathrm{Res}_{z_1=z_2} \Psi(z_2)\Psi^{V^{*a^{-1}}}(z_1) = (-1)^i q^{-1/2} z_2 g (C_-^{(1)}\otimes 1) w \otimes \mathrm{id}_{V(\Lambda_i)}\,, \tag{11}$$

where $g = \frac{(q^2)_\infty}{(q^4)_\infty}$ and

$$w = v_0^{(1)}\otimes v_1^{(1)} - qv_1^{(1)}\otimes v_0^{(1)}\,. \tag{12}$$

Note that $(C_-^{(1)}\otimes 1)w = \sum_{j=0}^{1} v_j^{(1)*}\otimes v_j^{(1)}$. Equations (9) and (11) are equivalent, respectively, to

$$\begin{aligned} \Phi_V(z)\Phi(z) &= (-1)^i q^{1/2} g^{-1} \mathrm{id}_{V(\Lambda_i)}\,, \\ \mathrm{Res}_{z_1=q^2 z_2} \Psi(z_2)\Psi(z_1) &= (-1)^i q^{3/2} z_2 g w \otimes \mathrm{id}_{V(\Lambda_i)}\,. \end{aligned} \tag{13}$$

2.3. Crystal. We shall review here the definitions and fundamental properties of crystals which we need in the subsequent sections. The details of the contents in this section can be found in [15].

Definition 1. *An affine crystal B is a set B with the weight decomposition $B = \bigsqcup_{\lambda \in P} B_\lambda$ and with the maps*

$$\tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \to B \sqcup \{0\}$$

satisfying the following axioms:

(1) $\tilde{e}_i B_\lambda \subset B_{\alpha_i + \lambda} \sqcup \{0\},\ \tilde{f}_i B_\lambda \subset B_{-\alpha_i + \lambda} \sqcup \{0\}$,
(2) $\tilde{e}_i 0 = \tilde{f}_i 0 = 0$,
(3) *for any b and i, there exists n such that $\tilde{e}_i^n b = \tilde{f}_i^n b = 0$,*
(4) *for $b_1, b_2 \in B$, $b_2 = \tilde{f}_i b_1$ if and only if $b_1 = \tilde{e}_i b_2$,*
(5) *if we set*

$$\varphi_i(b) = \max\{n | \tilde{f}_i^n b \in B\}, \qquad \varepsilon_i(b) = \max\{n | \tilde{e}_i^n b \in B\}\ ,$$

then $\varphi_i(b) - \varepsilon_i(b) = \langle h_i, \lambda \rangle$ for $b \in B_\lambda$ and i.

We denote $\mathrm{wt} b = \lambda$ if $b \in B_\lambda$. Let us set $P_{\mathrm{cl}} = P/\mathbf{Z}\delta$ and cl the projection $P \to P_{\mathrm{cl}}$. Then a classical crystal is defined using P_{cl} instead of P in the definition of an affine crystal. In this paper crystal means affine or classical crystal.

A crystal has the structure of colored oriented graph by

$$b_1 \xrightarrow{i} b_2 \quad \text{if and only if } b_2 = \tilde{f}_i b_1\ .$$

A morphism $\psi : B^1 \to B^2$ of the crystals is a map $B^1 \sqcup \{0\} \to B^2 \sqcup \{0\}$ which commutes with the actions of $\tilde{e}_i$ and $\tilde{f}_i$ and satisfies $\psi(0) = 0$. A morphism of crystals is called isomorphism (injective) if the associated map is bijective (injective). A crystal B^1 is called a subcrystal of B^2 if there is an injective morphism of crystals $B^1 \to B^2$. More general definition of the concept of crystal and its morphism is introduced in [13, 14].

For a crystal B and a subset $I \subset \{0, 1\}$, the I-crystal B is the set B with the same weight decomposition as the crystal B and with the maps $\tilde{e}_j$, $\tilde{f}_j$ $(j \in I)$ which is a part of the maps of the crystal B.

For two crystals B^1, B^2 we can define the tensor product in the following manner.

Definition 2. (1) *As a set $B^1 \otimes B^2 = \bigsqcup_{\lambda \in P} (B^1 \otimes B^2)_\lambda$, $(B^1 \otimes B^2)_\lambda = \bigsqcup_{\mu + \nu = \lambda} B^1_\mu \times B^2_\nu$. We denote (b_1, b_2) by $b_1 \otimes b_2$.*
(2) *The actions of $\tilde{e}_i$ and $\tilde{f}_i$ is defined as*

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \varphi_i(b_1) \leqq \varepsilon_i(b_2) \end{cases},$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \varphi_i(b_1) \geqq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}.$$

Among the crystals we need, in this paper, three kinds of crystals. The first one is the classical crystal B_s associated with the crystal base of the representation $(V_s)_1$.

Definition 3. (1) $B_s = \left\{ \boxed{j} \,\middle|\, 0 \leqq j \leqq s \right\}$ *as a set.*
(2) $\tilde{f}_1 \boxed{j} = \boxed{j+1}$ $(0 \leqq j \leqq s-1)$, $\tilde{f}_0 \boxed{j} = \boxed{j-1}$ $(1 \leqq j \leqq s)$, $\tilde{f}_i \boxed{j} = 0$ (*otherwise*).

(3) $\mathrm{wt}\boxed{j} = (s-2j)(\Lambda_1 - \Lambda_0)$.

We often use the notations $B_1 = \{\boxed{+}, \boxed{-}\}$ by the correspondence $\boxed{+} \leftrightarrow \boxed{0}$, $\boxed{-} \leftrightarrow \boxed{1}$ and identify $\pm$ with ± 1.

The second one is the affine crystal $\mathrm{Aff}(B_s)$ which is called the affinization of B_s.

Definition 4. (1) $\mathrm{Aff}(B_s) = \{b(n) | b \in B_s,\ n \in \mathbf{Z}\}$ *as a set.*
(2) $\tilde{f}_i(b(n)) = (\tilde{f}_i b)(n + \delta_{i0})$, $\tilde{e}_i(b(n)) = (\tilde{e}_i b)(n - \delta_{i0})$, *where we set* $0(n) = 0$.
(3) $\mathrm{wt}\, b(n) = \mathrm{wt}\, b - n\delta$.

For example the graph of $\mathrm{Aff}(B_1)$ is

$$\cdots \xrightarrow{1} \circ \xrightarrow{0} \circ \xrightarrow{1} \circ \xrightarrow{0} \circ \xrightarrow{1} \circ \xrightarrow{0} \cdots .$$

The third one is the crystal $B(\Lambda_i)$ associated with the crystal base of the representation $V(\Lambda_i)$. It is known that $B(\Lambda_i)$ is described in terms of the set of paths [15, 9]. The set of paths $\mathscr{P}(\Lambda_i)$ is defined as

$$\mathscr{P}(\Lambda_i) = \{(p(j))_{j=1}^{\infty} | p(j) \in B_1,\ p(k) = (-1)^{i+k} \text{ for } k \gg 0\}$$

and has the structure of an affine crystal [9, 15].

Theorem 1. (i) *There is an isomorphism of classical crystals,*

$$B(\Lambda_i) \simeq B(\Lambda_{1-i}) \otimes B_1 . \tag{14}$$

(ii) *The isomorphism* (14) *induces the bijective map* $B(\Lambda_i) \simeq \mathscr{P}(\Lambda_i)$.

The weight of a path through the above bijection can explicitly be written in terms of the energy function [15, 9].

For a crystal B we define the dual crystal $B^\vee$ of B as

Definition 5. (i) $B^\vee = \{b^\vee | b \in B\} = \bigsqcup_{\lambda \in P} B_{-\lambda}$, $B_{-\lambda} = \{b^\vee | b \in B_\lambda\}$,

(ii) $\tilde{e}_i b^\vee = (\tilde{f}_i b)^\vee$, $\tilde{f}_i b^\vee = (\tilde{e}_i b)^\vee$, $0^\vee = 0$.

The map $(b_1 \otimes b_2)^\vee \mapsto b_2^\vee \otimes b_1^\vee$ gives the isomorphism

$$(B^1 \otimes B^2)^\vee \simeq B^{2\vee} \otimes B^{1\vee} .$$

Since $B_1^\vee \simeq B_1$ by $\boxed{+} \mapsto \boxed{-}$ and $\boxed{-} \mapsto \boxed{+}$, we have the description of $B(\Lambda_i)^\vee$ in terms of paths,

$$B(\Lambda_i)^\vee = \{(p(j))_{j=-\infty}^{0} | p(j) \in B_1,\ p(k) = (-1)^{i+k} \text{ for } k \ll 0\} ,$$

$$B_1 \otimes B(\Lambda_i)^\vee \simeq B(\Lambda_{i+1})^\vee, \quad b \otimes (p(j))_{j=-\infty}^{0} \mapsto (p'(j))_{j=-\infty}^{0} ,$$

where $p'(0) = b$, $p'(j) = p(j+1)$ $(j \leqq -1)$.

2.4. The Morphism of Crystals Induced from the Dynkin Diagram Automorphism. Let $\imath$ be the isomorphism of the $\mathbf{Z}$ module P_{cl} defined by $\imath(\Lambda_i) = \Lambda_{1-i}$ $(i = 0, 1)$. For a classical crystal B, we define the classical crystal $\imath^* B$ by

$$\imath^* B = \bigsqcup_{\lambda \in P_{\mathrm{cl}}} (\imath^* B)_\lambda, \qquad (\imath^* B)_\lambda = \{\imath^*(b) | b \in B_{\imath(\lambda)}\}, \qquad \imath(0) = 0 , \tag{15}$$

$$\tilde{f}_i \imath^*(b) = \imath^*(\tilde{f}_{1-i} b), \qquad \tilde{e}_i \imath^*(b) = \imath^*(\tilde{e}_{1-i} b) . \tag{16}$$

It is easy to prove that (15), (16) actually defines a classical crystal. For this crystal the following properties hold.

Proposition 1. (i) $\imath^* B(\Lambda_i) \simeq B(\Lambda_{1-i})$.
(ii) $\imath^* B_s \simeq B_s$ *by* $\boxed{j} \mapsto \boxed{s-j}$.
(iii) *For crystals* B^1, B^2, $B^1 \simeq B^2$ *if and only if* $\imath^* B^1 \simeq \imath^* B^2$.
(iv) *For crystals* B^1, B^2, $\imath^*(B^1 \otimes B^2) \simeq \imath^* B^1 \otimes \imath^* B^2$, *by* $\imath^*(b_1 \otimes b_2) \mapsto \imath^*(b_1) \otimes \imath^*(b_2)$.

The properties (ii)–(iv) can be checked directly using definitions. The property (i) follows from the corresponding property of the representation $V(\Lambda_i)$.

3. Isomorphisms of Crystals

The structure of the space of the eigenvectors of the XXZ hamiltonian is, in the low temperature limit, described by the decomposition of the crystals of $B(\Lambda_i) \otimes B(\Lambda_j)^*$ [2]. In this section we shall prove new isomorphisms of crystals which generalize Theorem 1(i) and give a predicted form of the structure of the space of eigenvectors of our transfer matrix in the low temperature limit.

The problem is to decompose the crystals of the form

$$B(\Lambda_i) \otimes B_{s_1} \otimes \cdots \otimes B_{s_k} \otimes B(\Lambda_j)^\vee .$$

The main results in this section are

Theorem 2. *There is an isomorphism of classical crystals,*

$$B_{s-1} \otimes B(\Lambda_{1-i}) \simeq B(\Lambda_i) \otimes B_s ,$$

for $s = 1, 2, 3, \ldots$.

Corollary 1. *For* $j = 0, 1$, *we have the isomorphism of classical crystals,*

$$\coprod_{i=0,1} B(\Lambda_i) \otimes B_{s_1} \otimes \cdots \otimes B_{s_k} \otimes B(\Lambda_j)^\vee$$

$$\simeq B_{s_1 - 1} \otimes \cdots \otimes B_{s_k - 1} \otimes \coprod_{i=0,1} B(\Lambda_i) \otimes B(\Lambda_j)^\vee .$$

The decomposition of $\coprod_{i=0,1} B(\Lambda_i) \otimes B(\Lambda_j)^\vee$ into connected components were described in [2]. Using the diagonalization of the XXZ model by vertex operators we are also able to give another description of this crystal in terms of crystalline spinons [19, 20].

We remark that the isomorphisms of Theorem 2 includes (14) as a special case $s = 1$. But the proof of Theorem 2 uses the isomorphism (14).

It is sufficient to prove the theorem for $i = 0$. Since the $i = 1$ case is obtained by applying the map $\imath$ in Subsect. 2.4.

Let us define the map

$$\psi : B_{s-1} \otimes B(\Lambda_0) \to B(\Lambda_1) \otimes B_s ,$$

first and prove that it is well defined and commutes with the action of $\tilde{e}_i$ and $\tilde{f}_j$. In order to define the map ψ we need

Lemma 1. *There is an isomorphism of* $\{0,1\}$ *crystals,*

$$\psi_1 : B_s \otimes B_1 \simeq B_1 \otimes B_s \,.$$

The isomorphism is given explicitly by

$$\boxed{j+1} \otimes \boxed{+} \to \boxed{-} \otimes \boxed{j} \quad \textit{for } 0 \leqq j \leqq s-1 \,,$$

$$\boxed{0} \otimes \boxed{+} \to \boxed{+} \otimes \boxed{0} \,,$$

$$\boxed{j} \otimes \boxed{-} \to \boxed{+} \otimes \boxed{j+1} \quad \textit{for } 0 \leqq j \leqq s-1 \,,$$

$$\boxed{s} \otimes \boxed{-} \to \boxed{-} \otimes \boxed{s} \,.$$

Using the map ψ_1 let us define the isomorphism

$$\psi_n : B_s \otimes B_1^{\otimes n} \simeq B_1^{\otimes n} \otimes B_s$$

by

$$\psi_n = (1_{n-1} \otimes \psi_1) \cdots (1_1 \otimes \psi_1 \otimes 1_{n-2})(\psi_1 \otimes 1_{n-1}) \,,$$

where 1_j is the identity map of $B_1^{\otimes j}$. We denote by τ_k the isomorphism

$$B(\Lambda_0) \otimes B_1^{\otimes k} \simeq B(\Lambda_k) \,.$$

Now let us define the map ψ in the following manner. Take any $\boxed{j}_{s-1} \otimes b \in B_{s-1} \otimes B(\Lambda_0)$. For b there exists $n \in \mathbf{Z}_{\geqq 1}$ which satisfies

$$b = (b_k)_{k=1}^{\infty}, \quad b_k = (-1)^k \text{ for } k \geqq 2n \,. \tag{17}$$

Take any such n and set

$$\psi\left(\boxed{j}_{s-1} \otimes b\right) = (\tau_{2n-1} \otimes 1)\left(b_{\Lambda_0} \otimes \psi_{2n-1}\left(\boxed{j}_s \otimes b_{2n-1} \otimes \cdots \otimes b_1\right)\right) ,$$

where b_{Λ_0} is the highest weight element of $B(\Lambda_0)$ and the subscript of $\boxed{j}$ specifies to which crystal the element belongs, $\boxed{j}_s \in B_s$. The well definedness of ψ follows from

Lemma 2. *The definition of* ψ *does not depend on the choice of* n *which satisfies the condition* (17).

Proof. It is sufficient to prove

$$\boxed{+} \otimes \boxed{-} \otimes \psi_n\left(\boxed{j}_s \otimes b'\right) = \psi_{n+2}\left(\boxed{j}_s \otimes \boxed{-} \otimes \boxed{+} \otimes b'\right) ,$$

for $0 \leqq j \leqq s-1$, $n \in \mathbf{Z}_{\geqq 1}$ and any $b' \in B_1^{\otimes n}$. These equations follow from Lemma 1. □

Lemma 3. *The map* ψ *commutes with the action of* $\tilde{e}_1$ *and* $\tilde{f}_1$.

Proof. Let B be the connected component, as a $\{1\}$-crystal, of $B_{s-1} \otimes B_1$ which contains $\boxed{0}_{s-1} \otimes \boxed{+}$. Then

$$B = \left\{ \boxed{j}_{s-1} \otimes \boxed{+} \,\middle|\, 0 \leqq j \leqq s-1 \right\} \sqcup \left\{ \boxed{s-1}_{s-1} \otimes \boxed{-} \right\}$$

and is isomorphic to B_s as a $\{1\}$-crystal by the map

$$\begin{aligned} B &\to B_s \\ \boxed{j}_{s-1} \otimes \boxed{+} &\mapsto \boxed{j}_s \quad \text{for } 0 \leqq j \leqq s-1 \\ \boxed{s-1}_{s-1} \otimes \boxed{-} &\mapsto \boxed{s}_s \,. \end{aligned}$$

Let $\boxed{j}_{s-1} \otimes b \in B_{s-1} \otimes B(\Lambda_0)$ and n be as above. Now we shall describe the restriction of ψ to the $\{1\}$-crystal connected component of $\boxed{j}_{s-1} \otimes b$ as a composition of $\{1\}$-crystal morphisms. First of all

$$\begin{aligned} B_{s-1} \otimes B(\Lambda_0) &\simeq B_{s-1} \otimes B(\Lambda_0) \otimes B_1^{\otimes 2n} \\ \boxed{j}_{s-1} \otimes b &\mapsto \boxed{j}_{s-1} \otimes b_{\Lambda_0} \otimes \boxed{+} \otimes b_{2n-1} \otimes \cdots \otimes b_1 =: \tilde{b} \end{aligned}$$

is an isomorphism of classical $\{1\}$-crystals. The crystal $B \otimes B_1^{\otimes 2n-1}$ is a sub $\{1\}$-crystal of $B_{s-1} \otimes B(\Lambda_0) \otimes B_1^{\otimes 2n}$, by the map

$$\boxed{j}_{s-1} \otimes \boxed{\varepsilon} \otimes b_{2n-1} \otimes \cdots \otimes b_1 \to \boxed{j}_{s-1} \otimes b_{\Lambda_0} \otimes \boxed{\varepsilon} \otimes b_{2n-1} \otimes \cdots \otimes b_1 \,.$$

The element $\tilde{b}$ is in this subcrystal. Next, as we already showed,

$$B \otimes B_1^{\otimes 2n-1} \simeq B_s \otimes B_1^{\otimes 2n-1}$$

as a $\{1\}$-crystal and

$$\psi_{2n-1} : B_s \otimes B_1^{\otimes 2n-1} \simeq B_1^{\otimes 2n-1} \otimes B_s$$

as a $\{0,1\}$-crystal. Finally we have the injective $\{1\}$-crystal morphism

$$\begin{aligned} B_1^{\otimes 2n-1} \otimes B_s &\to B(\Lambda_0) \otimes B_1^{\otimes 2n-1} \otimes B_s \\ b' &\mapsto b_{\Lambda_0} \otimes b' \,. \end{aligned}$$

It is easy to check that the map ψ is the composition of the above maps and $\tau_{2n-1} \otimes 1$. Since we can take sufficiently large n such that the condition (17) holds for $\boxed{j}_{s-1} \otimes b$, $\tilde{f}_1 \left(\boxed{j}_{s-1} \otimes b \right)$ and $\tilde{e}_1 \left(\boxed{j}_{s-1} \otimes b \right)$, ψ is a $\{1\}$-crystal morphism. □

Lemma 4. *The map ψ commutes with the action of $\tilde{e}_0$ and $\tilde{f}_0$.*

Proof. Let us define a map $\tilde{\psi}$ in the following manner. For $\boxed{j}_{s-1} \otimes b \in B_{s-1} \otimes B(\Lambda_0)$, take $n \in \mathbf{Z}_{\geqq 0}$ such that

$$b = (b_k)_{k=1}^{\infty}, \quad b_k = (-1)^k \text{ for } k \geqq 2n+1 \,.$$

Then

$$\tilde{\psi}\left(\boxed{j}_{s-1} \otimes b\right) = (\tau_{2n} \otimes 1)\left(b_{\Lambda_1} \otimes \psi_{2n}\left(\boxed{j+1}_s \otimes b_{2n} \otimes \cdots \otimes b_1\right)\right) .$$

In a similar manner to the ψ case, we can easily check that the definition of $\tilde{\psi}$ is independent of the choice of n.

Sublemma 1. $\psi = \tilde{\psi}$.

Proof. We use the above notations. Take n as in (17). Then

$$\psi\left(\boxed{j}_{s-1} \otimes b\right) = \tilde{\psi}\left(\boxed{j}_{s-1} \otimes b\right)$$

is equivalent to

$$\boxed{-} \otimes \psi_{2n-1}\left(\boxed{j}_s \otimes b_{2n-1} \otimes \cdots \otimes b_1\right) = \psi_{2n}\left(\boxed{j+1}_s \otimes \boxed{+} \otimes b_{2n-1} \otimes \cdots \otimes b_1\right)$$

for $0 \leqq j \leqq s-1$. This follows from Lemma 1. □

Now the commutativity of $\tilde{\psi}$ and the actions of $\tilde{f}_0$ and $\tilde{e}_0$ is similarly proved as before. Namely let us set

$$B' = \left\{\boxed{j}_{s-1} \otimes \boxed{-} \middle| 0 \leqq j \leqq s-1\right\} \sqcup \left\{\boxed{0} \otimes \boxed{+}\right\} .$$

Then this constitutes, as a $\{0\}$-crystal, a connected component of $B_{s-1} \otimes B_1$ isomorphic to B_s. The map is given by

$$\begin{aligned} &B' \longrightarrow B_s \\ &\boxed{j}_{s-1} \otimes \boxed{-} \mapsto \boxed{j+1}_s \qquad \text{for } 0 \leqq j \leqq s-1 \\ &\boxed{0}_{s-1} \otimes \boxed{+} \mapsto \boxed{0}_s . \end{aligned}$$

Using this description it is easy to show that the $\tilde{\psi}$ is described as a composition of $\{0\}$-crystal morphisms from any $\{0\}$-crystal connected component as before. Hence the lemma is proved. □

Lemma 5. *ψ is a bijection.*

Proof. We shall prove the injectivity first. Suppose that

$$\psi\left(\boxed{j}_{s-1} \otimes b\right) = \psi\left(\boxed{j'}_{s-1} \otimes b'\right) .$$

By the definition of ψ this is equivalent to

$$\psi_{2n-1}\left(\boxed{j}_s \otimes b_{2n-1} \otimes \cdots \otimes b_1\right) = \psi_{2n-1}\left(\boxed{j'}_s \otimes b'_{2n-1} \otimes \cdots \otimes b'_1\right) ,$$

for sufficiently large n. Since ψ_{2n-1} is bijective, we have

$$j = j', \qquad b_k = b'_k \quad \text{for } 1 \leqq k \leqq 2n-1 ,$$

which means $b = b'$. The surjectivity easily follows from Lemma 1:

$$\psi_1^{-1}\left(\boxed{+}\otimes\boxed{j+1}_s\right) = \boxed{j}_s\otimes\boxed{-} \quad \text{for } 0 \leqq j \leqq s-1 ,$$

$$\psi_1^{-1}\left(\boxed{-}\otimes\boxed{j}_s\right) = \boxed{j+1}_s\otimes\boxed{+} \quad \text{for } 0 \leqq j \leqq s-1 ,$$

$$\psi_1^{-1}\left(\boxed{-}\otimes\boxed{s}_s\right) = \boxed{s}_s\otimes\boxed{-} . \qquad \square$$

This lemma completes the proof of Theorem 2.

4. Existence of New Type of Vertex Operators

In this section we shall prove the existence of new types of q-vertex operators, one of which is conjectured to induce the crystal isomorphisms of Theorem 2. For sets of non-zero complex numbers $z_1,\dots,z_k$, non-negative integers $n_1,\dots,n_k$ and $(i,j) \in \{0,1\}^2$ let us define the $F[z_1^{\pm 1},\dots,z_k^{\pm 1}]$ module by

$$H_{z_1\cdots z_k}^{n_1\cdots n_k}(i,j) = \{v \in (V_{n_1})_{z_1}\otimes\cdots\otimes(V_{n_k})_{z_k}$$
$$\times\,|\mathrm{wt}(v) = \Lambda_i - \Lambda_j,\ e_l^{\langle h_l,\Lambda_j\rangle+1}v = 0 \text{ for } l = 0,1\} .$$

Our aim here is to prove

Theorem 3. (i) $H_{z_2,q^{-3}z_1}^{n,m}(i,i+1)$ *and* $H_{z_1,z_2,q^{-3}z_1}^{n+1,1,n}(i,i+1)$ *are free* $F[z_1^{\pm 1},z_2^{\pm 1}]$ *modules and their ranks are given by*

$$\operatorname{rank} H_{z_2,q^{-3}z_1}^{n,m}(i,i+1) = \delta_{|n-m|,1}\delta_{z_1,z_2} ,$$

$$\operatorname{rank} H_{z_1,z_2,q^{-3}z_1}^{n+1,1,n}(i,i+1) = 1 .$$

(ii) *There are isomorphisms of* $F[z_1^{\pm 1},z_2^{\pm 1}]$ *modules*

$$\mathrm{Hom}_{F'\otimes U'}((V_m)_{z_1}\otimes V(\Lambda_i), V(\Lambda_{i+1})\hat{\otimes}(V_n)_{z_2}) \simeq H_{z_2,q^{-3}z_1}^{n,m}(i,i+1) ,$$

$$\mathrm{Hom}_{F'\otimes U'}((V_n)_{z_1}\otimes V(\Lambda_i), V(\Lambda_{i+1})\hat{\otimes}(V_{n+1})_{z_1}\otimes(V_1)_{z_2}) \simeq H_{z_1,z_2,q^{-3}z_1}^{n+1,1,n}(i,i+1) ,$$

where $F' = F[z_1^{\pm 1}]$.

From this theorem, using (6) and (7), we have

Corollary 2.

$$\mathrm{Hom}_{U'}(V(\Lambda_i),(V_n)_{q^2z}\hat{\otimes}V(\Lambda_{i+1})\hat{\otimes}(V_{n+1})_z)$$
$$\simeq \mathrm{Hom}_{F[z^{\pm 1}]\otimes U'}(V(\Lambda_i)\otimes(V_{n+1})_z,(V_n)_z\hat{\otimes}V(\Lambda_{i+1})) \simeq F[z^{\pm 1}] .$$

Proof of Theorem 3. Let us prove the first statements of (i) and (ii). Other cases are similarly proved. The proof is similar to that of Proposition on p. 53 of [3].

Note that, from (6), (7), (8),

$$\begin{aligned}&\mathrm{Hom}_{F'\otimes U'}((V_m)_{z_1}\otimes V(\Lambda_i), V(\Lambda_{i+1})\hat{\otimes}(V_n)_{z_2})\\ &\quad\simeq \mathrm{Hom}_{U'}(V(\Lambda_i), (V_m)_{q^2z_1}\hat{\otimes} V(\Lambda_{i+1})\hat{\otimes}(V_n)_{z_2})\,.\end{aligned}$$

Let $U'(b_+)$ be the subalgebra of U' generated by e_i, $t_i^{\pm 1}$ $(i=0,1)$. Then we have

$$\begin{aligned}&\mathrm{Hom}_{U'}(V(\Lambda_i), (V_m)_{q^2z_1}\hat{\otimes} V(\Lambda_{i+1})\hat{\otimes}(V_n)_{z_2})\\ &\simeq \mathrm{Hom}_{U'(b_+)}(Fu_{\Lambda_i}, (V_m)_{q^2z_1}\hat{\otimes} V(\Lambda_{i+1})\hat{\otimes}(V_n)_{z_2})\\ &\simeq \mathrm{Hom}_{F'\otimes U'(b_+)}(V(\Lambda_{i+1})^{*a}\otimes(V_m)_{z_1}\otimes Fu_{\Lambda_i}, (V_n)_{z_2})\,.\end{aligned}$$

Here we used the following lemma which can be proved in a similar way to Lemma 3.1 in [3].

Lemma 6. *Take any i and fix it. Let $u\in(V_n)_z\hat{\otimes} V(\Lambda_i)\hat{\otimes}(V_m)_z$ be a weight vector of t_i. If u satisfies $e_i^l u=0$ for some l, then $f_i^N u=0$ for some N.*

The following lemma is easily proved.

Lemma 7. *There is an isomorphism of $U'(b_+)$-modules,*

$$(V_n)_z\otimes Fu_{\Lambda_i}\simeq Fu_{\Lambda_i}\otimes(V_n)_{q^{-1}z}$$

given by the map

$$v_j^{(n)}\otimes u_{\Lambda_i}\to q^{-ji}u_{\Lambda_i}\otimes v_j^{(n)}\,.$$

This lemma and (6) imply

$$\begin{aligned}&\mathrm{Hom}_{U'(b_+)}(V(\Lambda_{i+1})^{*a}\otimes(V_m)_{z_1}\otimes Fu_{\Lambda_i}, (V_n)_{z_2})\\ &\simeq \mathrm{Hom}_{U'(b_+)}(V(\Lambda_{i+1})^{*a}\otimes Fu_{\Lambda_i}, (V_n)_{z_2}\otimes(V_m)_{q^{-3}z_1})\\ &\simeq H^{n,m}_{z_2,q^{-3}z_1}(i,i+1)\,,\end{aligned}\tag{18}$$

which proves (ii). In order to prove (i) let us write explicitly the conditions satisfied by the vector v of $H^{n,m}_{z_2,q^{-3}z_1}(i,i+1)$ according as $i=0,1$;

$$\mathrm{wt}(v)=\Lambda_0-\Lambda_1,\quad e_1^2v=e_0v=0,\quad \text{if } i=0\,,\tag{19}$$

$$\mathrm{wt}(v)=\Lambda_1-\Lambda_0,\quad e_1v=e_0^2v=0,\quad \text{if } i=1\,.\tag{20}$$

Let us determine the vectors which satisfy the condition (19) and (20). Note first that the condition (19) or (20) implies $|n-m|=1$. In fact the vector satisfying (19) or (20) must lie in the two dimensional irreducible representation of U_i'.

Let w_j be the highest weight vectors of $V_n\otimes V_m$ with the weight $(n+m-2j)\Lambda_1$ as a U_1'-module. They are explicitly given by

$$w_j=\sum_{k=0}^{j}c_k^{(j)}(n)v_k^{(n)}\otimes v_{j-k}^{(m)}\,,\tag{21}$$

$$c_k^{(j)}(n)=(-1)^kq^{k(n+1-k)}\begin{bmatrix} j\\ k\end{bmatrix},\quad c_0^{(0)}(n)=1\,.\tag{22}$$

(1.1) $i = 0$ and $n = m + 1$ case. The vector satisfying the condition (19) is proportional to $f_1 w_m$. Let us calculate $e_0 f_1 w_m$ in the tensor product $(V_{m+1})_{z_2} \otimes (V_m)_{q^{-3}z_1}$. The result is

$$f_1 w_m = \sum_{k=0}^{m} c_k^{(m)}(m+1)q^{m-2k+1}(q^{-1}[m+1-k] - [m-k])v_{k+1}^{(m+1)} \otimes v_{m-k}^{(m)} ,$$

$$e_0 f_1 w_m = (z_2 - z_1) \sum_{k=0}^{m-1} c_k^{(m)}(m+1)q^{-k}[m-k]v_{k+2}^{(m+1)} \otimes v_{m-k}^{(m)} .$$

Hence $e_0 v = 0$ is equivalent to $z_1 = z_2$.

(1.2) $i = 0$ and $m = n + 1$ case. The vector satisfying the condition (19) is proportional to $f_1 w_n$. We have

$$f_1 w_n = \sum_{k=0}^{n} c_k^{(n)}(n)(-q^{-1}[k] + [k+1])v_k^{(n)} \otimes v_{n+1-k}^{(n+1)} ,$$

$$e_0 f_1 w_n = (z_1 - z_2) \sum_{k=1}^{n} c_k^{(n)}(n)q^{-n+3(k-1)}[k]v_k^{(n)} \otimes v_{n+2-k}^{(n+1)} .$$

Hence $e_0 f_1 w_n = 0$ is equivalent to $z_1 = z_2$.

(1.3) $i = 1$ and $n = m + 1$ case. The vector satisfying condition (20) is proportional to w_m. Then

$$e_0^2 w_m = (z_1 - z_2) \sum_{k=0}^{m-1} c_k^{(m)}(m+1)(z_1[m-1-k] - z_2[m+1-k])v_{k+2}^{(m+1)} \otimes v_{m-k}^{(m)} .$$

Therefore $e_0^2 w_m = 0$ if and only if $z_1 = z_2$.

(1.4) $i = 1$ and $m = n + 1$ case. The vector satisfying condition (20) is proportional to w_n. Then we have

$$e_0^2 w_n = q^{-6}(z_1 - z_2) \sum_{k=1}^{n} c_k^{(n)}(n)q^{-2n+4k}(z_1[k+1] - z_2[k-1])v_k^{(n)} \otimes v_{n+2-k}^{(n+1)} .$$

Consequently $e_0^2 w_n = 0$ iff $z_1 = z_2$. $\square$

Remark 1. By Theorem 3 there are uniquely determined U' intertwiners

$${}_{V_n}\Phi^{V_{n+1}}(z) : (V_n)_z \otimes V(\Lambda_i) \to V(\Lambda_{i+1}) \hat{\otimes} (V_{n+1})_z ,$$

under the normalizations

$$\langle u^*_{\Lambda_{i+1}}, {}_{V_n}\Phi^{V_{n+1}}(z)(v_j^{(n)} \otimes u_{\Lambda_i})\rangle = v_{1-i+j}^{(n+1)} .$$

I conjecture that the vertex operator ${}_{V_n}\Phi^{V_{n+1}}(z)$ preserves the crystal lattice and induces the isomorphism of crystals of Theorem 2. Some part of Miki's conjecture [18] is a special case of this conjecture.

5. Fusion of Representations

Let us briefly recall the fusion construction of representations and R-matrices in order to fix notations. Let $M_i = Fw$ be the trivial representation of U_1' in $V_1 \otimes V_1$, where w is the vector defined in (12). In $(V_1)_{q^2z}\otimes(V_1)_z$, M is the trivial representation of U' too for any z. Let us set $N_i = V_1\otimes\cdots\otimes M_i\otimes\cdots\otimes V_1\subset V_1^{\otimes n}$, where M_i is on the $i, i+1^{\text{th}}$ components. We define the U' modules

$$W_n(z) = (V_1)_{q^{n-1}z}\otimes(V_1)_{q^{n-3}z}\otimes\cdots\otimes(V_1)_{q^{-(n-1)}z}\Big/\sum_{i=1}^{n-1} N_i\,,$$

$$\tilde{W}_n(z) = U'v_0^{(1)\otimes n}\hookrightarrow (V_1)_{q^{-(n-1)}z}\otimes(V_1)_{q^{-(n-3)}z}\otimes\cdots\otimes(V_1)_{q^{n-1}z}\,.$$

Then the following proposition is well known.

Proposition 2. $W_n(z)\simeq\tilde{W}_n(z)\simeq(V_n)_z$.

In order to describe the isomorphism explicitly we shall introduce the following definitions.

Definition 6. (1) $(\varepsilon_1,\ldots,\varepsilon_n)\in\{0,1\}^n$ *is of type* j *if and only if* $\sharp\{k|\varepsilon_k=1\}=j$.

(2) *For* $(\varepsilon_1,\ldots,\varepsilon_n)$ *let us define its inversion number by*

$$\mathrm{inv}(\varepsilon_1,\ldots,\varepsilon_n) = \sum_{i:\varepsilon_i=1}\sharp\{k|\varepsilon_k=0,\ k<i\}\,.$$

Then the isomorphism is given by

$$W_n(z)\to(V_n)_z\,,$$

$$v_{\varepsilon_1}^{(1)}\otimes\cdots\otimes v_{\varepsilon_n}^{(1)}\mapsto q^{\mathrm{inv}(\varepsilon_1,\ldots,\varepsilon_n)}v_j^{(n)}$$

for $(\varepsilon_1,\ldots,\varepsilon_n)$ of type j.

Let us give the description of $\tilde{W}_n$ in terms of R-matrix for the sake of later use. Let $\check{\bar{R}}(\frac{z_1}{z_2})$ be the U' intertwiner $(V_1)_{z_1}\otimes(V_1)_{z_2}\to(V_1)_{z_2}\otimes(V_1)_{z_1}$ such that $\check{\bar{R}}(\frac{z_1}{z_2})(v_0^{(1)\otimes 2}) = v_0^{(1)\otimes 2}$. Consider the intertwiner

$$\check{\bar{R}}_n(z):\ (V_1)_{z_1}\otimes\cdots\otimes(V_1)_{z_n}\to(V_1)_{z_n}\otimes\cdots\otimes(V_1)_{z_1}\,,$$

at $z_j = q^{n-2j+1}z$ $(1\leqq j\leqq n)$ defined by the composition $\check{\bar{R}}_n(z) = \check{\bar{R}}^{n-1n}(\frac{z_{n-1}}{z_n})\cdots\check{\bar{R}}^{1n}(\frac{z_1}{z_n})\cdots\check{\bar{R}}^{12}(\frac{z_1}{z_2})$. Here

$$\check{\bar{R}}^{ij}\left(\frac{z_i}{z_j}\right) = 1\otimes\cdots\otimes\check{\bar{R}}\left(\frac{z_i}{z_j}\right)\otimes\cdots\otimes 1$$

and $\check{\bar{R}}(\frac{z_i}{z_j})$ acts on the component $(V_1)_{z_i}\otimes(V_1)_{z_j}$. We sometimes omit the upper index ij and consider, for example, $\check{\bar{R}}(\frac{z_i}{z_j})$ as the operator acting on $(V_1)_{z_i}\otimes(V_1)_{z_j}$ nontrivially as explained here. It is well known (and easily proved) that

Proposition 3. $\mathrm{Im}\,\check{\bar{R}}_n(z) = \tilde{W}_n(z)$, $\mathrm{Ker}\,\check{\bar{R}}_n(z) = \sum_{k=1}^{n-1}N_k$.

Let $\check{\bar{R}}_{n1}(\frac{z}{w}) = \check{\bar{R}}(\frac{q^{n-1}z}{w})\check{\bar{R}}(\frac{q^{n-3}z}{w})\cdots\check{\bar{R}}(\frac{q^{-(n-1)}z}{w})$ be the U' intertwiner $(V_1)_{q^{n-1}z}\otimes\cdots\otimes(V_1)_{q^{-(n-1)}z}\otimes(V_1)_w\to(V_1)_w\otimes(V_1)_{q^{n-1}z}\otimes\cdots\otimes(V_1)_{q^{-(n-1)}z}$. Then

Proposition 4. *$\check{\bar{R}}_{n1}(\frac{z}{w})$ induces the U' linear map $W_n(z) \otimes (V_1)_w \to (V_1)_w \otimes W_n(z)$ such that the following diagram is commutative:*

$$\begin{array}{ccc} (V_1)_{q^{n-1}z} \otimes \cdots \otimes (V_1)_{q^{-(n-1)}z} \otimes (V_1)_w & \xrightarrow{\check{\bar{R}}_{n1}(\frac{z}{w})} & (V_1)_w \otimes (V_1)_{q^{n-1}z} \otimes \cdots \otimes (V_1)_{q^{-(n-1)}z} \\ \downarrow & & \downarrow \\ W_n(z) \otimes (V_1)_w & \longrightarrow & (V_1)_w \otimes W_n(z)\,. \end{array}$$

Here the downarrows are the natural projections.

Proof. It is sufficient to prove

$$\check{\bar{R}}_{n1}\left(\frac{z}{w}\right)(N_j \otimes (V_1)_w) \subset (V_1)_w \otimes \sum_{k=1}^{n-1} N_k\,.$$

By Proposition 3 this is equivalent to

$$(1 \otimes \check{\bar{R}}_n(z))\check{\bar{R}}_{n1}\left(\frac{z}{w}\right)(N_j \otimes (V_1)_w) = 0\,,$$

which follows from the Yang–Baxter equation. $\square$

We use the same symbol $\check{\bar{R}}_{n1}(\frac{z}{w})$ for the induced map. This map is also characterized as the U' intertwiner $(V_n)_{z_1} \otimes (V_1)_{z_2} \to (V_1)_{z_2} \otimes (V_n)_{z_1}$ satisfying $\check{\bar{R}}_{n1}(\frac{z_1}{z_2})(v_0^{(n)} \otimes v_0^{(1)}) = v_0^{(1)} \otimes v_0^{(n)}$. Similarly let $\check{\bar{R}}_{1n}(\frac{z_1}{z_2})$ be the U' intertwiner $(V_1)_{z_1} \otimes (V_n)_{z_2} \to (V_n)_{z_2} \otimes (V_1)_{z_1}$ normalized as $\check{\bar{R}}_{1n}(\frac{z_1}{z_2})(v_0^{(1)} \otimes v_0^{(n)}) = v_0^{(n)} \otimes v_0^{(1)}$. Then $\check{\bar{R}}_{1n}(z) = \check{\bar{R}}_{n1}(z^{-1})^{-1}$. They are explicitly given by

$$\check{\bar{R}}_{n1}(z)\begin{bmatrix} v_k^{(n)} \otimes v_1^{(1)} \\ v_{k+1}^{(n)} \otimes v_0^{(1)} \end{bmatrix} = \frac{1}{1-q^{n+1}z}\begin{bmatrix} -q^{k+1}z + q^{n-k} & (1-q^{2n-2k})z \\ 1-q^{2k+2} & -q^{n-k}z + q^{k+1} \end{bmatrix}\begin{bmatrix} v_1^{(1)} \otimes v_k^{(n)} \\ v_0^{(1)} \otimes v_{k+1}^{(n)} \end{bmatrix},$$

$$\check{\bar{R}}_{1n}(z)\begin{bmatrix} v_1^{(1)} \otimes v_k^{(n)} \\ v_0^{(1)} \otimes v_{k+1}^{(n)} \end{bmatrix} = \frac{1}{1-q^{n+1}z}\begin{bmatrix} -q^{k+1}z + q^{n-k} & 1-q^{2n-2k} \\ (1-q^{2k+2})z & -q^{n-k}z + q^{k+1} \end{bmatrix}\begin{bmatrix} v_k^{(n)} \otimes v_1^{(1)} \\ v_{k+1}^{(n)} \otimes v_0^{(1)} \end{bmatrix}.$$

6. Fusion of q-Vertex Operators

In this section we shall give a construction of the U'-intertwiner

$$V(\Lambda_i) \to (V_n)_{q^2z} \otimes V(\Lambda_{i+1}) \otimes (V_{n+1})_z$$

whose existence and uniqueness up to scalars are proved in Corollary 2. For the sake of simplicity, hereafter, we omit writing the symbol $\hat{}$ of the extended tensor product. The idea is to consider the composition

$$\begin{array}{rcl} V(\Lambda_i) & \longrightarrow & (V_1)_{q^{n+1}z} \otimes \cdots \otimes (V_1)_{q^{-n+3}z} \otimes V(\Lambda_{i+1}) \otimes (V_1)_{q^nz} \otimes \cdots \otimes (V_1)_{q^{-n}z} \\ & & \downarrow \\ & & (V_n)_{q^2z} \otimes V(\Lambda_{i+1}) \otimes (V_{n+1})_z\,. \end{array}$$

The vertical arrow is the U'-linear projection defined by Proposition 2. Unfortunately the composition of vertex operators Φ and Ψ which gives the horizontal arrow is not well defined in general. So we must carefully proceed in the following manner. Let us define the operator $O(\mathbf{z}|\mathbf{u})$, $(\mathbf{z},\mathbf{u}) \in \mathbf{C}^{*n+1} \times \mathbf{C}^{*n}$, acting on $V(\Lambda_i)$ by

$$O(z_1,\dots,z_{n+1}|u_n,\dots,u_1) = \frac{1}{f}\Phi(z_1)\cdots\Phi(z_{n+1})\Psi(u_n)\cdots\Psi(u_1)\,,$$

where $\mathbf{C}^* = \{z \in \mathbf{C}|z \neq 0\}$ and

$$f(z_1,\dots,z_{n+1}|u_n,\dots,u_1) = \prod_{j<k}\frac{(\frac{q^2z_k}{z_j})_\infty}{(\frac{q^4z_k}{z_j})_\infty}\prod_{j>k}\frac{(\frac{u_k}{u_j})_\infty}{(\frac{q^2u_k}{u_j})_\infty}\prod_{j,k}\frac{(\frac{qu_j}{z_k})_\infty}{(\frac{u_j}{qz_k})_\infty}\,.$$

As usual the operator $O(\mathbf{z}|\mathbf{u})$ has a sense as a set of matrix elements which are analytically continued to meromorphic functions in $(\mathbf{z},\mathbf{u})$. The operator $O(\mathbf{z}|\mathbf{u})$ satisfies, on $V(\Lambda_i)$, the symmetry relations

$$\left(\frac{z_j}{z_{j+1}}\right)^{-1/2}\check{\bar{R}}\left(\frac{z_j}{z_{j+1}}\right)O(\mathbf{z}|\mathbf{u}) = O(\sigma_j\mathbf{z}|\mathbf{u})\,,$$

$$\left(\frac{u_j}{u_{j+1}}\right)^{1/2}\check{\bar{R}}\left(\frac{u_j}{u_{j+1}}\right)O(\mathbf{z}|\mathbf{u}) = O(\mathbf{z}|\sigma_j\mathbf{u})\,,$$

where σ_j is the permutation exchanging z_j, z_{j+1} or u_j, u_{j+1}. Let

$$Pr(z)_{jk} : (V_1)_{z_j} \otimes (V_1)_{z_k} \to V_2\,,$$

$$Pr(u)_{jk} : (V_1)_{u_j} \otimes (V_1)_{u_k} \to V_2\,,$$

$$Pr(z) : (V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}} \to V_{n+1}\,,$$

$$Pr(u) : (V_1)_{u_1} \otimes \cdots \otimes (V_1)_{u_n} \to V_n\,,$$

be the U_1'-linear projection normalized as

$$Pr(z)_{jk}(v_0^{(1)\otimes 2}) = v_0^{(2)}, \qquad Pr(z)(v_0^{(1)\otimes n+1}) = v_0^{(n+1)}\,,$$

and similarly for $Pr(u)_{jk}$, $Pr(u)$. Although those projectors are irrelevant to the arguments z and u, we write them to clarify on which space they act. Since $Pr(z)$ and $Pr(u)$ is determined uniquely under these normalizations, we have, for $j < k$,

$$Pr(z) = Pr(z)\check{\bar{R}}\left(\frac{z_j}{z_{k-1}}\right)\cdots\check{\bar{R}}\left(\frac{z_j}{z_{j+1}}\right)\,. \tag{23}$$

The $Pr(z)$ in the right-hand side is the U_1' linear projection

$$(V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_j} \otimes (V_1)_{z_k} \otimes \cdots \otimes (V_1)_{z_{n+1}} \to V_{n+1}\,.$$

To simplify the notations we use the same symbol $Pr(z)$ although the space acted by it is different from that of $Pr(z)$ in the left-hand side. Note that there is an

U_1'-linear projection $Pr(z)^{jk}$ such that

$$\begin{array}{ccc}
(V_1)_{z_1}\otimes\cdots\otimes(V_1)_{z_j}\otimes(V_1)_{z_k}\otimes\cdots\otimes(V_1)_{z_{n+1}} & \xrightarrow{Pr(z)_{jk}} & (V_1)_{z_1}\otimes\cdots\otimes V_2\otimes\cdots\otimes(V_1)_{z_{n+1}} \\
\downarrow Pr(z) & & \downarrow Pr(z)^{jk} \\
V_{n+1} & = & V_{n+1}
\end{array}$$

is a commutative diagram.

Proposition 5. (1) *The operator $O(\mathbf{z}|\mathbf{u})$ has poles at most simple at $z_j = q^2 z_k$ $(j < k)$ and $u_j = q^2 u_k$ $(j < k)$.*
(2) *The operator $Pr(z)Pr(u)O(\mathbf{z}|\mathbf{u})$ has no poles.*

Proof. (1) The integral formula of $\langle \Lambda_{i+1}|O(\mathbf{z}|\mathbf{u})|\Lambda_i\rangle$ (Appendix 1) implies that $O(\mathbf{z}|\mathbf{u})$ has poles at most at $z_j = q^2 z_k$ $(j < k)$, $u_j = q^2 u_k$ $(j < k)$ and $u_j = qz_k, q^3 z_k$ for any j,k. Because there is a possibility to occur a pinch of the integration contours only in those cases. Moreover it is easy to prove that these poles are at most simple. Hence it is sufficient to prove that there are no poles at $u_j = qz_k, q^3 z_k$ for any j,k. But again this follows easily from the integral formula of $\langle \Lambda_{i+1}|O(\mathbf{z}|\mathbf{u})|\Lambda_i\rangle$ by the following reason. Consider a component of $\langle \Lambda_{i+1}|O(\mathbf{z}|\mathbf{u})|\Lambda_i\rangle$. Let us decompose each integral as

$$\int_{C_d}\frac{d\xi_d}{2\pi i} = \int_{C_0}\frac{d\xi_d}{2\pi i} + \sum_{j=1}^{d}\mathrm{Res}_{\xi_d=u_j}, \qquad \int_{C_a}\frac{dw_a}{2\pi i} = \int_{C_\infty}\frac{dw_a}{2\pi i} - \sum_{j=1}^{a}\mathrm{Res}_{w_a=q^2 z_j},$$

where C_0, C_∞ are the small circles around 0, ∞ going anti-clockwise and clockwise direction respectively. Here, for the sake of simplicity, we omit writing the integrands. Then the integral which we consider now is a sum of terms of the form

$$\prod_{d\in D_1}\int_{C_0}\frac{d\xi_d}{2\pi i}\prod_{d\in A_1}\int_{C_\infty}\frac{dw_a}{2\pi i}\mathrm{Res}_{w_{a_r}=q^2 z_{j_r}}\cdots\mathrm{Res}_{w_{a_1}=q^2 z_{j_1}}\mathrm{Res}_{\xi_{d_l}=u_{i_l}}\cdots\mathrm{Res}_{\xi_{d_1}=u_{i_1}},$$

where D_1 and A_1 is a subset of $\{a\}$ and $\{d\}$ respectively. Since there is a term $\prod_{a<a'}(1-\frac{w_{a'}}{w_a})\prod_{d<d'}(1-\frac{\xi_{d'}}{\xi_d})$ in the numerator of the integrand, we can assume that $j_{p_1} \neq j_{p_2} (p_1 \neq p_2)$, $i_{l_1} \neq i_{l_2}$ $(l_1 \neq l_2)$. In $\mathrm{Res}_{\xi_{d_l}=u_{i_l}}\cdots\mathrm{Res}_{\xi_{d_1}=u_{i_1}}$ the possible poles at $w_a = qu_{i_k}$ are cancelled out by $\prod_a\prod_{l=1}^{n}(1-\frac{qu_l}{w_a})$. Hence after taking residues in $w'_{a_p}s$, there does not appear poles at $u_j = qz_k$. Since there is the term $\prod_d\prod_{j=1}^{n+1}(1-\frac{\xi_d}{q^3 z_j})$ in the numerator, the poles at $u_{i_p} = q^3 z_{j_k}$ which appear after taking $\mathrm{Res}_{w_{a_r}=q^2 z_{j_r}}\cdots\mathrm{Res}_{w_{a_1}=q^2 z_{j_1}}$ are also cancelled out. Finally in the remaining integral $\prod_{d\in D_1}\int_{C_0}\frac{d\xi_d}{2\pi i}\prod_{d\in A_1}\int_{C_\infty}\frac{dw_a}{2\pi i}$ there do not occur pinches of the integral contours at $u_j = qz_k, q^3 z_k$. Hence it has no singularities there.

(2) It is sufficient to prove that $Pr(z)Pr(u)O(\mathbf{z}|\mathbf{u})$ is regular at $z_j = q^2 z_k$ $(j < k)$ and $u_j = q^2 u_k$ $(j < k)$. Let us consider the composition

$$\Phi(z_1)\Phi(z_2) : V(\Lambda_i) \to V(\Lambda_i)\otimes(V_1)_{z_1}\otimes(V_1)_{z_2}.$$

By the explicit formula of $\langle \Lambda_i|\Phi(z_1)\Phi(z_2)|\Lambda_i\rangle$ (p. 116 of [2]), $\Phi(z_1)\Phi(z_2)$ is regular at $z_1 = q^2 z_2$. Since there is no non-zero U' intertwiner $V(\Lambda_i) \to \hat{V}(\Lambda_{i+1})\otimes(V_2)_z$,

$Pr(z)_{12}\Phi(q^2z_2)\Phi(z_2)=0$. Hence

$$\mathrm{Res}_{z_j=q^2z_{j+1}}\frac{1}{f}Pr(z)_{jj+1}\Phi(z_j)\Phi(z_{j+1})=0 \tag{24}$$

for any $1\leqq j\leqq n$. Using the commutation relations of the vertex operators $\Phi(z)$ and the relations (23), (24)

$$\mathrm{Res}_{z_j=q^2z_k}Pr(z)O(\mathbf{z}|\mathbf{u})$$

$$=\mathrm{Res}_{z_j=q^2z_k}\prod_{l=j+1}^{k-1}\left(\frac{z_j}{z_l}\right)^{1/2}Pr(z)\check{R}\left(\frac{z_j}{z_{j+1}}\right)^{-1}\cdots\check{R}\left(\frac{z_j}{z_{k-1}}\right)^{-1}$$

$$O(z_1,\ldots,z_j,z_k,\ldots,z_{n+1}|\mathbf{u})$$

$$=\prod_{l=j+1}^{k-1}\left(\frac{q^2z_k}{z_l}\right)^{1/2}\mathrm{Res}_{z_j=q^2z_k}Pr(z)O(z_1,\ldots,z_j,z_k,\ldots,z_{n+1}|\mathbf{u})$$

$$=\prod_{l=j+1}^{k-1}\left(\frac{q^2z_k}{z_l}\right)^{1/2}Pr(z)^{jk}\mathrm{Res}_{z_j=q^2z_k}Pr(z)_{jk}O(z_1,\ldots,z_j,z_k,\ldots,z_{n+1}|\mathbf{u})=0\,.$$

Hence $Pr(z)O(\mathbf{z}|\mathbf{u})$ is regular at $z_j=q^2z_k$ $(j<k)$. We can similarly prove that $Pr(u)O(\mathbf{z}|\mathbf{u})$ is regular at $u_j=q^2u_k$ $(j<k)$. □

Definition 7 (Fused vertex operator).

$${}^nO^{n+1}(z)=[Pr(z)Pr(u)O(\mathbf{z}|\mathbf{u})]_{z_j=q^{n-2j+2}z\ (1\leqq j\leqq n+1),u_k=q^{n-2k+3}z\ (1\leqq k\leqq n)}$$

$$=\sum_{j,k}v_j^{(n)}\otimes{}^nO^{n+1}(z)_{jk}\otimes v_k^{(n+1)}\,.$$

Theorem 4. (i) *The operator* ${}^nO^{n+1}(z)$ *is not zero as a linear map.*
(ii) *The operator* ${}^nO^{n+1}(z)$ *gives a* U'*-linear map*

$$V(\Lambda_i)\to(V_n)_{q^2z}\otimes V(\Lambda_{i+1})\otimes(V_{n+1})_z\,.$$

Proof. (i) The integral formula of $\langle\Lambda_{i+1}|O(\mathbf{z}|\mathbf{u})|\Lambda_i\rangle$ gives (see (44), (45) in Appendix 1.)

$$\langle\Lambda_1|{}^n\bar{O}^{n+1}(z)_{0,n+1}|\Lambda_0\rangle=(-1)^{[\frac{n}{2}](n-1)}(-q)^{\frac{n(n-2)}{4}-\frac{3}{8}(1-(-1)^n)}$$

$$\langle\Lambda_0|{}^n\bar{O}^{n+1}(z)_{n,0}|\Lambda_1\rangle=(-1)^{[\frac{n}{2}](n-1)}(-q)^{\frac{n}{12}(8n^2-15n+22)+\frac{3}{8}(1-(-1)^n)}z^{-\frac{n(n-1)}{2}}\,.$$

For the definition of ${}^n\bar{O}^{n+1}(z)$, see Appendix 1. Hence ${}^nO^{n+1}(z)$ is not zero as a linear map.

(2) By definition ${}^nO^{n+1}(z)$ is U_1'-linear. Therefore it is sufficient to prove that ${}^nO^{n+1}(z)$ commutes with the action of e_0 and f_0.

Let us prove the commutativity of ${}^nO^{n+1}(z)$ with e_0. The case of f_0 is similarly proved. From the intertwining properties of $O(\mathbf{z}|\mathbf{u})$ we have

$$\langle v'|O(\mathbf{z}|\mathbf{u})|e_0v\rangle = (e_0 \otimes 1)\langle v'|O(\mathbf{z}|\mathbf{u})|v\rangle + (t_0 \otimes 1)\langle v'e_0|O(\mathbf{z}|\mathbf{u})|v\rangle$$
$$+ (t_0 \otimes e_0)\langle v't_0|O(\mathbf{z}|\mathbf{u})|v\rangle \tag{25}$$

for any $|v\rangle \in V(\Lambda_i)$, $\langle v'| \in V(\Lambda_{i+1})^*$. It is sufficient to prove, modulo $\sum N_j \otimes (V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}} + (V_1)_{u_1} \otimes \cdots \otimes (V_1)_{u_{n+1}} \otimes \sum N_j$, that

$$Pr(u)Pr(z)(e_0 \otimes 1)\langle v'|O(\mathbf{z}|\mathbf{u})|v\rangle = (e_0 \otimes 1)Pr(u)Pr(z)\langle v'|O(\mathbf{z}|\mathbf{u})|v\rangle\ , \tag{26}$$

$$Pr(u)Pr(z)(t_0 \otimes 1)\langle v'e_0|O(\mathbf{z}|\mathbf{u})|v\rangle = (t_0 \otimes 1)Pr(u)Pr(z)\langle v'e_0|O(\mathbf{z}|\mathbf{u})|v\rangle\ , \tag{27}$$

$$Pr(u)Pr(z)(t_0 \otimes e_0)\langle v't_0|O(\mathbf{z}|\mathbf{u})|v\rangle = (t_0 \otimes e_0)Pr(u)Pr(z)\langle v't_0|O(\mathbf{z}|\mathbf{u})|v\rangle\ , \tag{28}$$

at $z_j = q^{n-2(j-1)}z$ $(1 \leqq j \leqq n+1)$, $u_j = q^{n-2(j-1)+1}z$ $(1 \leqq j \leqq n)$. We remark that the left-hand sides (LHS) of Eqs. (26)–(28), after removing appropriate power functions of $\{z_j, u_k\}$, are regular functions in $\{z_j, u_k\}$. This follows from Proposition 5(ii) and Eq. (25). Hence we can specialize variables as above.

Since t_0 acts on $(V_1)_{u_1} \otimes \cdots \otimes (V_1)_{u_n}$ as t_1^{-1} and $Pr(u)$ is U_1' linear, (27) holds. Let us prove Eq. (26). According as the decompositions $(V_1)_{u_1} \otimes \cdots \otimes (V_1)_{u_n} \simeq V_n \oplus \sum N_j$, $(V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}} \simeq V_{n+1} \oplus \sum N_j$ as U_1' modules, let us write

$$\langle v'|O(\mathbf{z}|\mathbf{u})|v\rangle = (A + A') \otimes (B + B')\ ,$$

$$A \in V_n, \qquad A' \in \sum N_j, \qquad B \in V_{n+1}, \qquad B' \in \sum N_j\ ,$$

where N_j is defined in the beginning of Sect. 5. Then

$$Pr(u)Pr(z)(e_0 \otimes 1)\langle v'|O(\mathbf{z}|\mathbf{u})|v\rangle - (e_0 \otimes 1)Pr(u)Pr(z)\langle v'|O(\mathbf{z}|\mathbf{u})|v\rangle$$

$$= (Pr(u)e_0A - e_0A) \otimes B + Pr(u)e_0A' \otimes B\ . \tag{29}$$

Since $Pr(u)e_0A - e_0A \equiv 0 \bmod \sum N_j$, it is sufficient to prove

$$Pr(u)e_0A' \otimes B = 0\ , \tag{30}$$

at $z_j = q^{n-2(j-1)}z$ $(1 \leqq j \leqq n+1)$, $u_j = q^{n-2(j-1)+1}z$ $(1 \leqq j \leqq n)$.

Lemma 8. *$Pr(u)e_0A' \otimes B$ has no poles.*

Proof. By Proposition 5(i) it is sufficient to prove that $Pr(u)e_0A' \otimes B$ is regular at $z_j = q^2z_k$ $(j < k)$, $u_j = q^2u_k$ $(j < k)$. The LHS and the first component of the RHS of Eq. (29) is regular at $z_j = q^2z_k$ $(j < k)$, $u_j = q^2u_k$ $(j < k)$ by the remark above and Proposition 5(ii). Hence $Pr(u)e_0A' \otimes B$ is also regular at the same place. □

Now let us decompose $\langle v'|O(\mathbf{z}|\mathbf{u})|v\rangle$ in the following manner:

$$\langle v'|O(\mathbf{z}|\mathbf{u})|v\rangle = \sum_{j=1}^{n-1} \frac{O_j(\mathbf{z}|\mathbf{u})}{u_j - q^2u_{j+1}} + \tilde{O}(\mathbf{z}|\mathbf{u})\ ,$$

$$O_j(\mathbf{z}|\mathbf{u}) = \mathrm{Res}_{u_j=q^2u_{j+1}}\left(\langle v'|O(\mathbf{z}|\mathbf{u})|v\rangle - \sum_{k=1}^{j-1}\frac{O_k(\mathbf{z}|\mathbf{u})}{u_k-q^2u_{k+1}}\right) \quad \text{for } j \geqq 2\,,$$

$$O_1(\mathbf{z}|\mathbf{u}) = \mathrm{Res}_{u_1=q^2u_2}\langle v'|O(\mathbf{z}|\mathbf{u})|v\rangle\,. \tag{31}$$

Then

Lemma 9. (i) $O_j(\mathbf{z}|\mathbf{u}) \in \sum N_k \otimes (V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}}$,

(ii) $\tilde{O}(\mathbf{z}|\mathbf{u})$ *is regular at* $u_j = q^{2(k-j)}u_k$ $(j < k)$,

(iii) $O_j(\mathbf{z}|\mathbf{u})$ *is regular at* $u_r = qz_r$ $(1 \leqq r \leqq n)$,

(iv) $O_j(\mathbf{z}|\mathbf{u})|_{u_r=qz_r\ (1\leqq r\leqq n)} = 0$.

Proof. (i) This follows from (13).

(iii) This is obvious from Proposition 5(i).

(iv) It follows from

$$\frac{1}{f}\bigg|_{u_l=q^2u_{l+1}} = g^{-1}\prod_{j<k}\frac{(\frac{q^4z_k}{z_j})_\infty}{(\frac{q^2z_k}{z_j})_\infty}\prod_{j>k,j,k\neq l,l+1}\frac{(\frac{q^2u_k}{u_j})_\infty}{(\frac{u_k}{u_j})_\infty}\prod_{j\neq l,l+1}\prod_k\frac{(\frac{u_j}{qz_k})_\infty}{(\frac{qu_j}{z_k})_\infty}$$

$$\times\frac{\prod_{k=1}^{n+1}(1-\frac{u_{l+1}}{qz_k})}{\prod_{j=l+2}^{n}(1-\frac{u_{l+1}}{u_j})}$$

and (13) that $\mathrm{Res}_{u_l=q^2u_{l+1}}\langle v'|O(\mathbf{z}|\mathbf{u})|v\rangle$ has $\prod_{r\neq l}\prod_{k=1}^{n+1}(1-\frac{u_r}{qz_k})$ as a factor of its zero divisor. Taking further residues does not produce poles at $u_s = qz_k$ $(1 \leqq s,k \leqq n)$ by Proposition 5(i). Hence $O_j(\mathbf{z}|\mathbf{u})|_{u_r=qz_r\ (1\leqq r\leqq n)} = 0$.

(ii) Let us prove, for $2 \leqq j \leqq n$, that

$$\langle v'|O(\mathbf{z}|\mathbf{u})|v\rangle - \sum_{r=1}^{j-1}\frac{O_r(\mathbf{z}|\mathbf{u})}{u_r-q^2u_{r+1}} \quad \text{is regular at } u_l = q^{2(s-l)}u_s \ (l < s, 1 \leqq l \leqq j-1)$$

by the induction on j. The $j = 2$ case is obvious from Proposition 5(ii).

Suppose that the statement is true for $1 \leqq j \leqq k$. We have

$$\langle v'|O(\mathbf{z}|\mathbf{u})|v\rangle - \sum_{r=1}^{k}\frac{O_r(\mathbf{z}|\mathbf{u})}{u_r-q^2u_{r+1}} = O^{(1)}(\mathbf{z}|\mathbf{u}) - \frac{O_k(\mathbf{z}|\mathbf{u})}{u_k-q^2u_{k+1}}\,,$$

$$O^{(1)}(\mathbf{z}|\mathbf{u}) = \langle v'|O(\mathbf{z}|\mathbf{u})|v\rangle - \sum_{r=1}^{k-1}\frac{O_r(\mathbf{z}|\mathbf{u})}{u_r-q^2u_{r+1}}\,,$$

$$O_k(\mathbf{z}|\mathbf{u}) = \mathrm{Res}_{u_k=q^2u_{k+1}}O^{(1)}(\mathbf{z}|\mathbf{u})\,.$$

By the induction hypothesis $O^{(1)}(\mathbf{z}|\mathbf{u})$ is regular at $u_l = q^{2(s-l)}u_s$ $(l < s, 1 \leqq l \leqq k-1)$. Hence $O_k(\mathbf{z}|\mathbf{u})$ and consequently $O^{(1)}(\mathbf{z}|\mathbf{u}) - \frac{O_k(\mathbf{z}|\mathbf{u})}{u_k-q^2u_{k+1}}$ are regular at $u_l = q^{2(s-l)}u_s$ $(l < s, 1 \leqq l \leqq k-1)$. The definition of a residue and Proposition 5(i) imply that $O^{(1)}(\mathbf{z}|\mathbf{u}) - \frac{O_k(\mathbf{z}|\mathbf{u})}{u_k-q^2u_{k+1}}$ is regular at $u_k = q^{2(s-k)}u_s$ $(k < s)$. Hence the statement is proved for $j = k+1$. $\square$

Using the decomposition (31) we have

$$Pr(u)e_0A' \otimes B = \sum_{j=1}^{n-1} \frac{1}{u_j - q^2 u_{j+1}} Pr(u)Pr(z)(e_0 \otimes 1)O_j(\mathbf{z}|\mathbf{u})$$

$$+ Pr(u)Pr(z)(e_0 \otimes 1)(1 - Pr(u))\tilde{O}(\mathbf{z}|\mathbf{u})\,.$$

Note that, in $(V_1)_{u_1} \otimes (V_1)_{u_2}$,

$$e_0 w = (u_1 - q^2 u_2)v_1^{(1)} \otimes v_1^{(1)}\,.$$

Since $\tilde{O}(\mathbf{z}|\mathbf{u})$ has no poles at $u_j = q^{2(s-j)}u_s$ $(j < s)$ we can conclude that

$$Pr(u)Pr(z)(e_0 \otimes 1)(1 - Pr(u))\tilde{O}(\mathbf{z}|\mathbf{u})|_{u_j = q^{n-2(j-1)+1}z} = 0\,.$$

Since each $O_j(\mathbf{z}|\mathbf{u})$ has a zero divisor of the form $\prod_{r=1}^{n+1}(1 - \frac{u_l}{qz_r})$ for some l, we have

$$\sum_{j=1}^{n-1} \frac{1}{u_j - q^2 u_{j+1}} Pr(u)Pr(z)(e_0 \otimes 1)O_j(\mathbf{z}|\mathbf{u})|_{u_j = qz_j\ (1 \leqq j \leqq n)} = 0\,.$$

Taking into account that $Pr(u)e_0A' \otimes B$ has no pole at all we can conclude that

$$Pr(u)e_0A' \otimes B|_{z_j = q^{n-2(j-1)}z\ (1 \leqq j \leqq n+1),\, u_j = q^{n-2(j-1)}z\ (1 \leqq j \leqq n)} = 0\,.$$

Hence (26) is proved. Equation (28) is similarly proved. □

7. Commutation Relations of Vertex Operators

Using the fusion construction in the previous section, we shall determine the commutation relations of new vertex operators. Here we give only commutation relations which are relevant to the later applications. We shall introduce the following variants of the vertex operator ${}^nO^{n+1}(z)$.

Definition 8. *The intertwiners*

$${}_nO^{n+1}(z) : (V_n)_z \otimes V(\Lambda_i) \to V(\Lambda_{i+1}) \otimes (V_{n+1})_z\,,$$

$${}^nO_{n+1}(z) : V(\Lambda_i) \otimes (V_{n+1})_z \to (V_n)_z \otimes V(\Lambda_{i+1})\,,$$

$${}^nO^{n+1*}(z) : V(\Lambda_i) \to (V_n)_z \otimes V(\Lambda_{i+1}) \otimes (V_{n+1})_z^{*a}\,,$$

are defined by

$${}_nO^{n+1}(z)(v_j^{(n)} \otimes \cdot\,) = \langle v_j^{(n)}, (C_-^{(n)} \otimes 1){}^nO^{n+1}(z)\rangle\,,$$

$${}^nO_{n+1}(z)(\,\cdot \otimes v_j^{(n+1)}) = \langle v_j^{(n+1)}, (1 \otimes C_+^{(n+1)}){}^nO^{n+1}(q^{-2}z)\rangle\,,$$

$${}^nO^{n+1*}(z) = (1 \otimes C_+^{(n+1)}){}^nO^{n+1}(q^{-2}z)\,.$$

Let us set $(z;p)_n = \prod_{l=0}^{n-1}(1 - zp^l)$. Recall that the highest weight vector w_n with weight zero in the U_1' module $V_n \otimes V_n$ is explicitly given in (21) and (22). Then

Theorem 5.

$$P_F^{n+1n}O^{n+1*}(z)^nO^{n+1}(z) = (-1)^{i+\frac{n(n-1)}{2}} q^{\frac{n^2+n+1}{2}} g_{n+1}^{-1} w_n \otimes \mathrm{id}_{V(\Lambda_i)}, \tag{32}$$

$$^nO_{n+1}(z)_nO^{n+1}(z) = (-1)^{i+\frac{n(n-1)}{2}} q^{\frac{n^2+n+1}{2}} g_{n+1}^{-1} \mathrm{id}_{(V_n)_z\otimes V(\Lambda_i)}, \tag{33}$$

$$(-1)^{n+1}\check{R}_{n+11}\left(\frac{z}{w}\right) {}^nO^{n+1}(z)\Phi(w) = \Phi(w)^nO^{n+1}(z), \tag{34}$$

$$(-1)^{n+1}\check{R}_{1n+1}\left(\frac{w}{z}\right) \Phi(w)_nO^{n+1}(z) = {}_nO^{n+1}(z)\Phi(w), \tag{35}$$

where

$$\check{R}_{n+11}(z) = z^{\frac{1}{2}} r_{n+1}(z)\check{\bar{R}}_{n+11}(z), \qquad \check{R}_{1n+1}(z) = z^{\frac{1}{2}} r_{n+1}(z)\check{\bar{R}}_{1n+1}(z),$$

$$g_n = \frac{(q^{2n})_\infty}{(q^{2n+2})_\infty}, \qquad r_n(z) = \frac{(q^{n+1}z)_\infty (q^{n-1}z^{-1})_\infty}{(q^{n+1}z^{-1})_\infty (q^{n-1}z)_\infty}.$$

Proof. Let us prove (32). Define

$$\tilde{O}(\mathbf{z}',\mathbf{z}|\mathbf{u}',\mathbf{u}) = \Phi^{V^{*a}}(z_1')\Phi(z_1)\cdots\Phi^{V^{*a}}(z_{n+1}')\Phi(z_{n+1})$$
$$\times \Psi(q^{-2}u_n')\Psi^{V^{*a^{-1}}}(q^{-2}u_n)\cdots\Psi(q^{-2}u_1')\Psi^{V^{*a^{-1}}}(q^{-2}u_1).$$

This is the U'-intertwiner

$$V(\Lambda_i) \to (V_1)^{*a^{-1}}_{q^{-2}u_1} \otimes (V_1)_{q^{-2}u_1'} \otimes \cdots \otimes (V_1)^{*a^{-1}}_{q^{-2}u_n} \otimes (V_1)_{q^{-2}u_n'} \otimes V(\Lambda_i)$$
$$\otimes (V_1)^{*a}_{z_1'} \otimes (V_1)_{z_1} \otimes \cdots \otimes (V_1)^{*a}_{z_{n+1}'} \otimes (V_1)_{z_{n+1}}.$$

Using the commutation relations of the vertex operators $\Phi(z)$ and $\Psi(z)$, we have

$$\frac{h}{ff'}(-1)^n q^n \prod_{j<k}\left(\frac{z_j}{z_k'}\right)^{1/2} \prod_{j<k}\left(\frac{u_j'}{u_k}\right)^{1/2} \prod_{j,k}\left(\frac{u_j'}{q^2 z_k}\right)^{1/2}$$
$$\times \check{\bar{R}}\left(\frac{u_1'}{q^2u_n}\right)\cdots\check{\bar{R}}\left(\frac{u_1'}{q^2u_2}\right)\cdots\check{\bar{R}}\left(\frac{u_{n-1}'}{q^2u_n}\right)\check{\bar{R}}\left(\frac{q^2z_1}{z_{n+1}'}\right)\cdots\check{\bar{R}}\left(\frac{q^2z_1}{z_2'}\right)\cdots\check{\bar{R}}\left(\frac{q^2z_n}{z_{n+1}'}\right)$$
$$\times (C_-^{(1)-1}\otimes 1)^{\otimes n}\otimes (C_+^{(1)-1}\otimes 1)^{\otimes n+1}\tilde{O}(\mathbf{z}',\mathbf{z}|\mathbf{u}',\mathbf{u})$$
$$= O(q^{-2}\mathbf{z}'|q^{-2}\mathbf{u}')O(\mathbf{z}|\mathbf{u}), \tag{36}$$

$$h = \prod_{j<k} r\left(\frac{q^2z_j}{z_k'}\right)\prod_{j<k} r\left(\frac{u_j'}{q^2u_k}\right)\prod_{j,k}\frac{\theta_{q^4}\left(\frac{q^3z_j}{u_k'}\right)}{\theta_{q^4}\left(\frac{u_k'}{qz_j}\right)},$$

where $f' = f(\mathbf{z}'|\mathbf{u}')$, $q^{-2}\mathbf{z}' = (q^{-2}z_1', \ldots, q^{-2}z_{n+1}')$ etc. Note that

$$h|_{u_j = q^3 z_{j+1}, u_j' = q^3 z_{j+1}' \ (1 \leqq j \leqq n)} = \prod_{j=2}^{n+1} \left(1 - \frac{z_j}{z_j'}\right) \tilde{h} ,$$

$$\tilde{h} = q^{-n(n+1)} \prod_{2 \leqq j < k \leqq n+1} \frac{\left(\frac{q^4 z_j}{z_k'}\right)_\infty^2 \left(\frac{q^4 z_j'}{z_k}\right)_\infty \left(\frac{z_j'}{z_k}\right)_\infty \left(1 - \frac{z_k}{z_j'}\right)}{\left(\frac{q^2 z_j}{z_k'}\right)_\infty \left(\frac{q^6 z_j}{z_k'}\right)_\infty \left(\frac{z_j'}{q^2 z_k}\right)_\infty \left(\frac{q^2 z_j'}{z_k}\right)_\infty}$$

$$\times \prod_{j=2}^{n+1} \frac{\left(\frac{q^4 z_1}{z_j'}\right)_\infty^2 \left(\frac{q^4 z_j}{z_j'}\right)_\infty \left(\frac{q^4 z_j'}{z_j}\right)_\infty}{\left(\frac{q^6 z_1}{z_j'}\right)_\infty \left(\frac{q^2 z_1}{z_j'}\right)_\infty \left(\frac{q^2 z_j'}{z_j}\right)_\infty \left(\frac{q^2 z_j}{z_j'}\right)_\infty} .$$

Specializing the variables to $u_j = q^3 z_{j+1}$, $u_j' = q^3 z_{j+1}'$ $(1 \leqq j \leqq n)$ in both sides of Eq. (36), after that setting $z_j = z_j'$ $(1 \leqq j \leqq n+1)$ and using (see (11))

$$\lim_{z_j \to z_j'} \left(1 - \frac{z_j}{z_j'}\right) (C_-^{(1)-1} \otimes 1) \Psi(z_j') \Psi^{V^{*a^{-1}}}(z_j) = (-1)^{i+1} q^{-1/2} g w \otimes \mathrm{id}_{V(\Lambda_i)} ,$$

we have

$$\frac{\tilde{\tilde{h}}}{\tilde{f}^2} (-1)^{ni} q^{n/2} g^n \prod_{1 \leqq j < k \leqq n+1} \left(\frac{z_j}{z_k}\right)^{1/2} \prod_{2 \leqq j < k \leqq n+1} \left(\frac{z_j}{z_k}\right)^{1/2}$$

$$\times \prod_{j=2}^{n+1} \prod_{k=1}^{n+1} \left(\frac{q z_j}{z_k}\right)^{1/2} \prod_{2 \leqq j < k \leqq n+1} \frac{1}{1 - \frac{z_j}{q^2 z_k}} R_n(z) w^{\otimes n}$$

$$\otimes (C_+^{(1)\otimes n+1} \otimes 1)^{-1} \tilde{R}_{n+1}^*(z) \Phi^{V^{*a}}(z_1) \Phi(z_1) \cdots \Phi^{V^{*a}}(z_{n+1}) \Phi(z_{n+1})$$

$$= O(q^{-2}\mathbf{z}|q z_{n+1}, \ldots, q z_2) O(\mathbf{z}|q^3 z_{n+1}, \ldots, q^3 z_2) , \tag{37}$$

where

$$R_n(\mathbf{z}) = \check{\bar{R}}\left(\frac{z_2}{q^2 z_{n+1}}\right) \cdots \check{\bar{R}}\left(\frac{z_2}{q^2 z_3}\right) \cdots \check{\bar{R}}\left(\frac{z_n}{q^2 z_{n+1}}\right) ,$$

$$\tilde{R}_{n+1}^*(\mathbf{z}) = (C_+^{(1)\otimes n+1} \otimes 1) \tilde{R}_{n+1}(\mathbf{z}) (C_+^{(1)-1} \otimes 1)^{\otimes n+1} ,$$

$$\tilde{R}_{n+1}(\mathbf{z}) = \check{\bar{R}}\left(\frac{q^2 z_1}{z_{n+1}}\right) \cdots \check{\bar{R}}\left(\frac{q^2 z_1}{z_2}\right) \cdots \check{\bar{R}}\left(\frac{q^2 z_n}{z_{n+1}}\right) ,$$

$$\tilde{\tilde{h}} = q^{-n(n+1)} g^{-2n} \prod_{2\leqq j<k\leqq n+1} \frac{\left(\frac{q^4 z_j}{z_k}\right)_\infty^3 \left(\frac{z_j}{z_k}\right)_\infty \left(1-\frac{z_k}{z_j}\right)}{\left(\frac{q^2 z_j}{z_k}\right)_\infty^3 \left(\frac{q^6 z_j}{z_k}\right)_\infty} \prod_{j=2}^{n+1} \frac{\left(\frac{q^4 z_1}{z_j}\right)_\infty^2}{\left(\frac{q^6 z_1}{z_j}\right)_\infty \left(\frac{q^2 z_1}{z_j}\right)_\infty},$$

$$\tilde{f} = g^{-n} \prod_{2\leqq j<k\leqq n+1} \frac{\left(\frac{z_j}{z_k}\right)_\infty \left(\frac{q^4 z_j}{z_k}\right)_\infty}{\left(\frac{q^2 z_j}{z_k}\right)_\infty^2},$$

and $w = v_0^{(1)} \otimes v_1^{(1)} - q v_1^{(1)} \otimes v_0^{(1)}$.

Lemma 10. *Let Pr_n be the U_1' linear projection $V_1^{\otimes n} \otimes V_1^{\otimes n} \to V_n \otimes V_n$ normalized as $Pr_n(v_0^{(1)\otimes 2n}) = v_0^{(n)\otimes 2}$. Then we have*

$$Pr_n R_n(\mathbf{z}) w^{\otimes n} = q^{\frac{n(n-1)}{2}} \prod_{2\leqq j<k\leqq n+1} \frac{1-\frac{z_j}{q^2 z_k}}{1-\frac{z_j}{z_k}} w_n .$$

Proof. Since $Pr_n R_n(\mathbf{z}) w^{\otimes n}$ is in the trivial representation of $V_n \otimes V_n$, we have $Pr_n R_n(\mathbf{z}) w^{\otimes n} = c w_n$ for some scalar function c. The function c is the coefficient of $v_0^{(n)} \otimes v_n^{(n)}$ in the right-hand side. Let us calculate the coefficient of $v_0^{(1)\otimes n} \otimes v_1^{(1)\otimes n}$ in $R_n(\mathbf{z}) w^{\otimes n}$. It is easy to see that this coefficient is the same as that of $v_0^{(1)\otimes n} \otimes v_1^{(1)\otimes n}$ in $R_n(\mathbf{z})(v_0^{(1)} \otimes v_1^{(1)})^{\otimes n}$. The latter coefficient is easily calculated and coincides with the function in the statement of the lemma. $\square$

Let $(P_F^1)^{\otimes(n+1)}$ be the U' linear map $(V_1)_{z_1}^{*a} \otimes (V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}}^{*a} \otimes (V_1)_{z_{n+1}} \to F$ defined by $(P_F^1)^{\otimes(n+1)}(\bigotimes_{l=1}^{n+1}(v_{j_l}^{(1)*} \otimes v_{k_l}^{(1)})) = \prod_{l=1}^{n+1} \delta_{j_l,k_l}$ and P_F^{n+1} the dual pairing map $(V_{n+1})_z^{*a} \otimes (V_{n+1})_z \to F$. We set

$$\tilde{Pr}_{n+1} = (C_+^{(n+1)} \otimes 1) Pr_{n+1} (C_+^{(1)\otimes n+1} \otimes 1)^{-1} .$$

Lemma 11. *There is an equation*

$$P_F^{n+1} \tilde{Pr}_{n+1} \tilde{R}_{n+1}^*(\mathbf{z}) = c (P_F^1)^{\otimes(n+1)}, \qquad c = q^{\frac{n(n+1)}{2}} \frac{(1-q^2)^{n+1}}{(q^2;q^2)_{n+1}}$$

at $z_j = q^{n-2(j-1)} z$ $(1 \leqq j \leqq n+1)$.

Note that the R-matrix $\bar{R}(\frac{q^2 z_j}{z_k})$ $(j<k)$ is regular at $\frac{z_j}{z_k} = q^{2(k-j)}$ and $\bar{R}(\frac{q^2 z_j}{z_k})^{-1} = \bar{R}(\frac{z_k}{q^2 z_j})$ which is also regular at $\frac{z_j}{z_k} = q^{2(k-j)}$. Hence there exists the inverse of $\tilde{R}_{n+1}(\mathbf{z})$ which is regular at $z_j = q^{n-2(j-1)} z$ $(1 \leqq j \leqq n+1)$. Let us set $\varphi(\mathbf{z}) = (P_F^1)^{\otimes(n+1)} \tilde{R}_{n+1}^{-1}(\mathbf{z})$,

$$\begin{array}{ccc}
(V_1)_{z_1}^{*a} \otimes (V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}}^{*a} \otimes (V_1)_{z_{n+1}} & \xrightarrow{(P_F^1)^{\otimes(n+1)}} & F \\
\Big\downarrow \tilde{R}_{n+1}(\mathbf{z}) & & \Big\downarrow \mathrm{id} \\
(V_1)_{z_1}^{*a} \otimes \cdots \otimes (V_1)_{z_{n+1}}^{*a} \otimes (V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}} & \xrightarrow{\varphi(\mathbf{z})} & F .
\end{array}$$

If we set $N_j^* = C_+^{(1)\otimes n+1} N_j$, we have

$$(V_{n+1})_z^{*a} \simeq (V_1)_{q^n z}^{*a} \otimes \cdots \otimes (V_1)_{q^{-n}z}^{*a} \Big/ \sum_{j=1}^{n} N_j^*$$

using the isomorphism

$$(C_+^{(1)-1})^{\otimes n+1} : \ (V_1)_{q^n z}^{*a} \otimes \cdots \otimes (V_1)_{q^{-n}z}^{*a} \simeq (V_1)_{q^{n-2}z} \otimes \cdots \otimes (V_1)_{q^{-n-2}z} .$$

Then

Sublemma 1.

$$\varphi(\mathbf{z})(N_j^* \otimes V_{q^n z} \otimes \cdots \otimes V_{q^{-n}z}) = \varphi(\mathbf{z})(V_{q^n z}^{*a} \otimes \cdots \otimes V_{q^{-n}z}^{*a} \otimes N_j) = 0$$

for all $1 \leqq j \leqq n+1$.

Proof. Since $\varphi(\mathbf{z})$ is a U' linear map we have

$$\begin{aligned}
&\varphi(\mathbf{z})(v_{j_1}^{(1)*} \otimes \cdots \otimes v_{j_{n+1}}^{(1)*} \otimes v_{k_1}^{(1)} \otimes \cdots \otimes v_{k_{n+1}}^{(1)}) \\
&\quad = \beta \langle v_{j_{n+1}}^{(1)*} \otimes \cdots \otimes v_{j_1}^{(1)*}, \tilde{\tilde{R}}_{n+1}(z)(v_{k_1}^{(1)} \otimes \cdots \otimes v_{k_{n+1}}^{(1)}) \rangle , \qquad (38)
\end{aligned}$$

for some scalar function β. Here $\tilde{\tilde{R}}_{n+1}(z)$ is defined by setting $z_j = q^{n-2(j-1)}z$ $(1 \leqq j \leqq n+1)$ in the U' intertwiner $(V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}} \to (V_1)_{z_{n+1}} \otimes \cdots \otimes (V_1)_{z_1}$ normalized as $\tilde{\tilde{R}}_{n+1}(\mathbf{z})(v_0^{(1)\otimes n}) = v_0^{(1)\otimes n}$. In fact, for generic values of z_j's for which $(V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}}$ is irreducible, the U' linear map $(V_1)_{z_1}^{*a} \otimes \cdots \otimes (V_1)_{z_{n+1}}^{*a} \otimes (V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}} \to F$ is unique up to a scalar factor and given by $\tilde{\tilde{R}}_{n+1}(\mathbf{z})$ as in the right-hand side of (38). Since $\beta = \varphi(\mathbf{z})(v_0^{(1)*\otimes(n+1)} \otimes v_0^{(1)\otimes(n+1)})$ and $\tilde{R}_{n+1}(\mathbf{z})$ is regular at $z_j = q^{2(k-j)}z_k$ $(j < k)$, β is also regular at $z_j = q^{2(k-j)}z_k$ $(j < k)$. Hence (38) holds at $z_j = q^{n-2(j-1)}z$ $(1 \leqq j \leqq n+1)$. By Proposition 3 we have $\tilde{\tilde{R}}_{n+1}(\mathbf{z})(N_j) = 0$ and hence $\varphi(\mathbf{z})(V_{q^n z}^{*a} \otimes \cdots \otimes V_{q^{-n}z}^{*a} \otimes N_j) = 0$.

Let us prove the remaining equation. Note that the base of the trivial representation in $V_u^{*a} \otimes V_{q^{-2}u}^{*a}$ is given by $v_1^{(1)*} \otimes v_0^{(1)*} - qv_0^{(1)*} \otimes v_1^{(1)*}$. Taking into account the fact that, in the left part of the right-hand side of the equality (38), the order of the tensor product is reversed, we set $w^* = v_0^{(1)*} \otimes v_1^{(1)*} - qv_1^{(1)*} \otimes v_0^{(1)*}$. Then, by calculations, we have

$$\langle w^*, f_1^k v_0^{(1)\otimes 2} \rangle = 0 \quad \text{for } 0 \leqq k \leqq 2 .$$

Since, by Proposition 3, $\operatorname{Im} \tilde{\tilde{R}}_{n+1}(\mathbf{z}) \simeq (V_{n+1})_z$ which is generated by $v_0^{(1)\otimes(n+1)}$ over U_1', we have

$$\varphi(\mathbf{z})(N_j^* \otimes V_{q^n z} \otimes \cdots \otimes V_{q^{-n}z}) = 0 . \qquad \square$$

Let us continue the proof of the lemma. By the sublemma the map $\varphi(\mathbf{z})$ induces the U' linear map

$$(V_{n+1})_z^{*a} \otimes (V_{n+1})_z \to F .$$

Hence $\varphi(\mathbf{z})$ is a scalar multiple of the canonical pairing map P_F^{n+1}, that is, $\varphi(\mathbf{z}) = cP_F^{n+1}\tilde{P}r_{n+1}$. Let us determine the scalar c. Note that $c = \varphi(\mathbf{z})((v_1^{(1)*})^{\otimes(n+1)} \otimes (v_1^{(1)})^{\otimes(n+1)})$. We can prove easily that

$$\varphi(\mathbf{z})((v_0^{(1)})^{\otimes(n+1)} \otimes (v_1^{(1)})^{\otimes(n+1)})$$

$$= \langle (v_0^{(1)*} \otimes v_1^{(1)*})^{\otimes(n+1)}, \tilde{R}_{n+1}^{-1}(z)((v_0^{(1)})^{\otimes(n+1)} \otimes (v_1^{(1)})^{\otimes(n+1)})\rangle .$$

Recall that

$$\tilde{R}_{n+1}^{-1}(z) = \check{\bar{R}}\left(\frac{z_{n+1}}{q^2 z_n}\right) \cdots \check{\bar{R}}\left(\frac{z_2}{q^2 z_1}\right) \cdots \check{\bar{R}}\left(\frac{z_{n+1}}{q^2 z_1}\right)$$

with $z_j = q^{n-2(j-1)}z$ $(1 \leqq j \leqq n+1)$.

From those descriptions we have

$$c = q^{\frac{n(n+1)}{2}} \prod_{1\leqq j<k\leqq n+1} \frac{1-\frac{z_k}{q^2 z_j}}{1-\frac{z_k}{z_j}} = q^{\frac{n(n+1)}{2}} \frac{\prod_{l=1}^{n+1}(1-q^{-2l})}{(1-q^{-2})^{n+1}} . \qquad \square$$

Now taking $(1\otimes P_F^{n+1})(1\otimes (C_+^{(n+1)}\otimes 1))((Pr_n \otimes Pr_{n+1})$ in both sides of Eq. (37) and using Lemma 10, Lemma 11, Eq. (9), we have the Eq. (32).

Next let us prove (34). Using the commutation relations of $\Phi(z)$ and $\Psi(z)$ we have

$$(-1)^{n+1} \prod_{l=1}^{n} \left(\frac{u_l}{w}\right)^{-1/2} \prod_{j=1}^{n+1} \left(\frac{z_j}{w}\right)^{1/2} \prod_{j=1}^{n+1} r\left(\frac{z_j}{w}\right) \prod_{l=1}^{n} \frac{\theta_{q^4}(\frac{qu_l}{w})}{\theta_{q^4}(\frac{qw}{u_l})}$$

$$\times \check{\bar{R}}\left(\frac{z_1}{w}\right) \cdots \check{\bar{R}}\left(\frac{z_{n+1}}{w}\right) O(\mathbf{z}|\mathbf{u})\Phi(w) = \Phi(w)O(\mathbf{z}|\mathbf{u}) . \qquad (39)$$

Similarly to the proof of Proposition 5 and Theorem 4, we can prove that both of the operators $(Pr(z)\otimes Pr(u))\check{\bar{R}}(\frac{z_1}{w}) \cdots \check{\bar{R}}(\frac{z_{n+1}}{w})\, O(\mathbf{z}|\mathbf{u})\Phi(w)$ and $(Pr(z)\otimes Pr(u))$ $\Phi(w)O(\mathbf{z}|\mathbf{u})$ give well-defined U'-intertwiners at $z_j = q^{n-2j+2}z$, $u_j = q^{n-2j+3}z$. Hence, by Theorem 3, we have

$$\left[(Pr(z)\otimes Pr(u))\check{\bar{R}}\left(\frac{z_1}{w}\right) \cdots \check{\bar{R}}\left(\frac{z_{n+1}}{w}\right) O(\mathbf{z}|\mathbf{u})\Phi(w)\right]_{z_j=q^{n-2j+2}z,\, u_j=q^{n-2j+3}z} , \qquad (40)$$

$$= c(z,w)\check{\bar{R}}_{n+1,1}\left(\frac{z}{w}\right) [(Pr(z)\otimes Pr(u))O(\mathbf{z}|\mathbf{u})\Phi(w)]_{z_j=q^{n-2j+2}z,\, u_j=q^{n-2j+3}z} \qquad (41)$$

for some scalar function $c(z,w)$. Comparing the coefficient of $v_0^{(1)} \otimes v_0^{(n)}$ we conclude that $c(z,w) \equiv 1$. Taking $Pr(z)\otimes Pr(u)$ of both sides of Eq. (39) and substituting $z_j = q^{n-2(j-1)}z$ $(1 \leqq j \leqq n+1)$, $u_j = q^{n-2j+3}z$ $(1 \leqq j \leqq n)$, we obtain the desired equation. Equations (33) and (35) follow from (32) and (34) respectively. $\square$

8. Inhomogeneous Vertex Models of 6-Vertex Type

In this section we denote $(V_s)_1$ by V_s for the sake of simplicity and assume $-1 < q < 0$, $1 < z < q^{-2}$ which corresponds to the antiferroelectric regime. Let us consider the two dimensional regular square infinite lattice. Fix a positive integer N non-negative integers $s_1, \dots, s_N$ and vertical lines $l_1, \dots, l_N$. Then the vertex model which we study here is defined in the following way. We associate the representation V_1 with each edge on horizontal lines and on vertical lines except $l_1, \dots, l_N$. With each edge on the line l_j we associate the vector space V_{s_j}. For each vertex the Boltzmann weight is given by the corresponding R-matrix, $R_{11}(z)$, $R_{1\Lambda_j}(z)$. We can assume that the lines $l_1, \dots, l_N$ are successive by including 1 in the set of s_j. Let us first give the mathematical objects and after that explain the validity of them.

The representation theoretical formulation of the model is given by

Space. The space acted by the transfer matrix is

$$\mathscr{H} = \bigoplus_{i,j=0,1} \mathscr{H}_{s_N \cdots s_1, ij} ,$$

$$\mathscr{H}_{s_N \cdots s_1, ij} = V_{s_N - 1} \otimes \cdots \otimes V_{s_1 - 1} \otimes V(\Lambda_i) \otimes V(\Lambda_j)^{*a} .$$

Transfer matrix. The transfer matrix is given by

$$T(z) = \mathrm{id} \otimes T_{\mathrm{XXZ}}(z) ,$$

where $T_{\mathrm{XXZ}}(z)$ is the transfer matrix of the 6-vertex model acting on $\bigoplus_{i,j=0,1} V(\Lambda_i) \otimes V(\Lambda_j)^{*a}$. Explicitly, on $V(\Lambda_i) \otimes V(\Lambda_j)^{*a}$,

$$T_{\mathrm{XXZ}}(z) = (-1)^{j+1} q^{1/2} g \, {}^t\Phi^{V^{*a^{-1}}}(z) \Phi(z) ,$$

where ${}^t\Phi^{V^{*a^{-1}}}(z) : (V_1)_z \otimes V(\Lambda_i)^{*a} \to V(\Lambda_{i+1})^{*a}$ is the transposition of $\Phi^{V^{*a^{-1}}}(z)$.

Ground state. The space of vacuum vectors V_{vac} is

$$V_{\mathrm{vac}} = \bigoplus_{i,j=0,1} V_{s_N - 1} \otimes \cdots \otimes V_{s_1 - 1} \otimes F |\mathrm{vac}\rangle_{\mathrm{XXZ},i} ,$$

where $|\mathrm{vac}\rangle_{\mathrm{XXZ},i}$ is the vacuum vector of the XXZ-model [2] in $V(\Lambda_i) \hat{\otimes} V(\Lambda_i)^{*a}$. As an element of $\mathrm{End}_F(V(\Lambda_i))$ we have

$$|\mathrm{vac}\rangle_{\mathrm{XXZ},i} = \mathrm{id}_{V(\Lambda_i)} .$$

Excited states. The creation and annihilation operators are given by

$$\varphi_j^*(z) = 1 \otimes \varphi_{j,\mathrm{XXZ}}^*(z), \qquad \varphi_j(z) = 1 \otimes \varphi_{j,\mathrm{XXZ}}(z) ,$$

where $\varphi_{j,\mathrm{XXZ}}^*(z)$, $\varphi_{j,\mathrm{XXZ}}(z)$ are the creation and annihilation operators of the XXZ model,

$$\varphi_{j,\mathrm{XXZ}}^*(z) = \langle v_j^{(1)}, \Psi^{V^{*a^{-1}}}(z) \rangle, \qquad \varphi_{j,\mathrm{XXZ}}(z) = \langle v_j^{(1)*}, \Psi(z) \rangle .$$

Local operators. For $L \in \mathrm{End}(V_{s_N} \otimes \cdots \otimes V_{s_1})$ the corresponding local operator $\mathscr{L}$ is defined by

$$\mathscr{L} = \Phi^{s_N,\dots,s_1}(1,\dots,1)^{-1}(1 \otimes L)\Phi^{s_N,\dots,s_1}(1,\dots,1),$$

$$\Phi^{s_N,\dots,s_1}(z_N,\dots,z_1) = {}_{s_N-1}O^{s_N}(z_N)\cdots {}_{s_1-1}O^{s_1}(z_1),$$

$$\Phi^{s_N,\dots,s_1}(z_N,\dots,z_1)^{-1} = c_{i,N}(s)^{s_1-1}O_{s_1}(z_1)\cdots {}^{s_N-1}O_{s_N}(z_N),$$

$$c_{i,N}(s) = (-1)^{iN+\frac{N(N-1)}{2}+\sum_{j=1}^N\frac{(s_j-1)(s_j-2)}{2}} q^{-\sum_{j=1}^N\frac{s_j^2-s_j+1}{2}} \prod_{j=1}^N g_{s_j}^{-1}.$$

Here $\Phi^{s_N,\dots,s_1}(z_N,\dots,z_1)^{-1}$ is defined on

$$V(\Lambda_{i+N}) \otimes V_{s_N} \otimes \cdots \otimes V_{s_1} \otimes V(\Lambda_j)^{*a}.$$

Correlation functions. The expectation values of the local operator $\mathscr{L}$ is given by

$$\langle \mathscr{L} \rangle_i = \frac{\mathrm{tr}_{V_{s_N-1}\otimes\cdots\otimes V_{s_1-1}\otimes V(\Lambda_i)}((1 \otimes q^{-2\rho})\mathscr{L})}{s_1\cdots s_N \mathrm{tr}_{V(\Lambda_i)}(q^{-2\rho})},$$

where $\rho = \Lambda_0 + \Lambda_1$ and 1 is the identity operator acting on $V_{s_N-1} \otimes \cdots \otimes V_{s_1-1}$.

Let us explain why we have given the mathematical setting as above. The less obvious definition is that of the transfer matrix. If it is accepted then others are rather natural from the formulation of the case of the XXZ model [2]. So we shall explain the reason for our definition of the transfer matrix. Since we consider $V(\Lambda_i)$ and $V(\Lambda_j)^{*a}$ as half infinite tensor products $V_1^{\otimes|\mathbf{Z}_{\geqq 1}|}$ and $V_1^{\otimes|\mathbf{Z}_{\leqq 0}|}$, the natural space on which our transfer matrix acts is

$$\bigoplus_{i,j=0,1} V(\Lambda_i) \otimes V_{s_N} \otimes \cdots \otimes V_{s_1} \otimes V(\Lambda_j)^{*a}, \tag{42}$$

and the natural definition of the transfer matrix $T(z)$ on this space is

$$T(z) = {}^t\Phi^{V^{*a^{-1}}}(z)\check{R}_{1s_N}(z)\cdots\check{R}_{1s_1}(z)\Phi(z).$$

We identify the space $V(\Lambda_i) \otimes V_{s_N} \otimes \cdots \otimes V_{s_1} \otimes V(\Lambda_j)^{*a}$ with $\mathscr{H}_{s_N\cdots s_1,ij}$ by the map $\Phi^{s_N,\dots,s_1}(1,\dots,1)$ and its inverse. Let us determine the map $\tilde{T}(z)$ for which

$$\begin{array}{ccc}
\mathscr{H}_{s_N\cdots s_1,ij} & \xrightarrow{\Phi^{s_N,\dots,s_1}(1,\dots,1)} & V(\Lambda_i) \otimes V_{s_N} \otimes \cdots \otimes V_{s_1} \otimes V(\Lambda_j)^{*a} \\
\downarrow \tilde{T}(z) & & \downarrow T(z) \\
\mathscr{H}_{s_N\cdots s_1,i+1j+1} & \xrightarrow{\Phi^{s_N,\dots,s_1}(1,\dots,1)} & V(\Lambda_{i+1}) \otimes V_{s_N} \otimes \cdots \otimes V_{s_1} \otimes V(\Lambda_{j+1})^{*a}
\end{array}$$

is a commutative diagram. Using the commutation relations (35) we have

$$\begin{aligned}
\tilde{T}(z) &= \Phi^{s_N,\dots,s_1}(1,\dots,1)^{-1}T(z)\Phi^{s_N,\dots,s_1}(1,\dots,1) \\
&= \Phi^{s_N,\dots,s_1}(1,\dots,1)^{-1\,t}\Phi^{V^{*a^{-1}}}(z)\check{R}_{1s_N}(z)\cdots\check{R}_{1s_1}(z)\Phi(z)\Phi^{s_N,\dots,s_1}(1,\dots,1) \\
&= (-1)^{\sum_{j=1}^N s_j\,t}\Phi^{V^{*a^{-1}}}(z)\Phi^{s_N,\dots,s_1}(1,\dots,1)^{-1}\Phi^{s_N,\dots,s_1}(1,\dots,1)\Phi(z) \\
&= (-1)^{\sum_{j=1}^N s_j}(1 \otimes T_{\mathrm{XXZ}}(z)).
\end{aligned}$$

Hence, up to a scalar factor, the transfer matrix coincides with $1 \otimes T_{\mathrm{XXZ}}(z)$. If we normalize the eigenvalue of the vacuum vectors is equal to one, then the transfer matrix is given by $1 \otimes T_{\mathrm{XXZ}}(z)$.

Now we summarize about the eigen-vectors and eigen-values of our transfer matrix as

$$T(z)(v^{s_N \cdots s_1}_{j_N \cdots j_1} \otimes |\mathrm{vac}\rangle^{\pm}) = \pm(v^{s_N \cdots s_1}_{j_N \cdots j_1} \otimes |\mathrm{vac}\rangle^{\pm}) \,,$$

$$T(z)\varphi^*_j(z') = \tau\left(\frac{z}{z'}\right)\varphi^*_j(z')T(z) \,,$$

$$v^{s_N \cdots s_1}_{j_N \cdots j_1} = v^{(s_N - 1)}_{j_N} \otimes \cdots \otimes v^{(s_1 - 1)}_{j_1} \,,$$

$$|\mathrm{vac}\rangle^{\pm} = |\mathrm{vac}\rangle_{\mathrm{XXZ},i} \pm |\mathrm{vac}\rangle_{\mathrm{XXZ},1-i}, \qquad \tau(z) = z^{-1/2}\frac{\theta_{q^4}(qz)}{\theta_{q^4}(qz^{-1})} \,.$$

Remark 2. If $n \geqq 2$ and $-1 < q < 0$, there is no value of the parameter z for which every coefficient of $R_{1n}(z)$ is positive. Hence it will be better to regard our model as an inhomogeneous XXZ spin chain rather than a two dimensional vertex model. Then the hamiltonian H should be defined by

$$H = -(q - q^{-1})\frac{d}{dz}\log T(z)|_{z=1} \,.$$

Since we consider the thermodynamic limit and almost all spins of this spin chain is of $1/2$, H can be written as a sum of local hamiltonians. Our calculation shows that the excitation energies of H over the ground state in the thermodynamic limit coincide with those of the antiferromagnetic XXZ model. This is consistent with the results of Bethe–Ansatz [1, 6, 22, 23].

9. Discussion

In this paper we introduce new kinds of q-vertex operators and using them propose the mathematical model of the inhomogeneous vertex models of the 6-vertex type. One of our vertex operators ${}_nO^{n+1}(z)$ already appeared in Miki's paper [18] in the simplest non-trivial form $n = 1$ in a different context.

It follows from our mathematical setting of the models that the excitation energies over the ground states are the same as that of the 6-vertex model. In our approach the impurity contributions to the several physical quantities may be calculated through the correlation functions. In the case $N = 1$ and $s_1 \geqq 1$, our results on the dimension of the degenerate ground states coincide with the known results [6, 22].

As in the case of the other solvable lattice models [4, 10] the trace of the compositions of the new vertex operators satisfy certain q-difference equations. Except the case of the form $\mathrm{tr}_{V(\Lambda_i)}(q^{-2\rho}\Phi(z_1)\cdots\Phi(z_k)_{V_{s-1}}\Phi^{V_s}(z))$, those equations are different from the q-KZ equation with mixed spins. Hence the situation is rather unexpected from the point of view by the rough pictorial arguments [10, 4].

The new vertex operators can be considered as non-local operators acting on the physical space of the XXZ-model. This fact may open the door to study the fusion model [21, 17] of the 6 vertex model using the vertex operators defined here.

Obviously we can introduce the inhomogeneities in the spectral parameter (or the rapidities). This corresponds to consider the space $(V_{s_N-1})_{z_N}\otimes\cdots\otimes(V_{s_1-1})_{z_1}\otimes V(\Lambda_i)\otimes V(\Lambda_j)^{*a}$, etc.

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A. Appendix 1

In this section we give the integral formula for the matrix element $\langle\Lambda_{i+1}|O(\mathbf{z}|\mathbf{u})|\Lambda_i\rangle$. Let us set, on $V(\Lambda_i)$,

$$\tilde{\Phi}(z)=z^{\frac{-1+2i}{4}}\Phi(z),\qquad \tilde{\Psi}(z)=z^{\frac{-1+2i}{4}}\Psi(z),$$

and

$$\bar{O}(\mathbf{z}|\mathbf{u})=\frac{1}{f}\tilde{\Phi}_{\varepsilon_1}(z_1)\cdots\tilde{\Phi}_{\varepsilon_{n+1}}(z_{n+1})\tilde{\Psi}_{\mu_n}(u_n)\cdots\tilde{\Psi}_{\mu_1}(u_1).$$

Then we have, on $V(\Lambda_i)$,

$$O(\mathbf{z}|\mathbf{u})=\prod_{j=1}^{n+1}z_j^{-\frac{1}{4}+\frac{1}{2}\overline{(i+j-1)}}\prod_{j=1}^{n}u_j^{-\frac{1}{4}+\frac{1}{2}\overline{(i+j-1)}}\bar{O}(\mathbf{z}|\mathbf{u}),$$

where $\bar{k}=0,1$ according as k is even or odd. We set

$$\begin{aligned}{}^{n}\bar{O}^{n+1}(z)&=[Pr(z)Pr(u)\bar{O}(\mathbf{z}|\mathbf{u})]_{z_j=q^{n-2j+2}z\ (1\leqq j\leqq n+1),u_k=q^{n-2k+3}z\ (1\leqq k\leqq n)}\\&=\sum_{j,k}v_j^{(n)}\otimes{}^{n}\bar{O}^{n+1}(z)_{jk}\otimes v_k^{(n+1)}.\end{aligned}$$

Then

$${}^{n}O^{n+1}(z)=z^{\frac{n+1-i}{4}}q^{\frac{1}{4}(n+i)(n+2)+\frac{5}{4}n-\frac{3}{2}[\frac{n+i}{2}]}{}^{n}\bar{O}^{n+1}(z).$$

We have

$$\langle\Lambda_{i+1}|\bar{O}(\mathbf{z}|\mathbf{u})|\Lambda_i\rangle=\frac{1}{f}\langle\Lambda_{i+1}|\tilde{\Phi}_{\varepsilon_1}(z_1)\cdots\tilde{\Phi}_{\varepsilon_{n+1}}(z_{n+1})\tilde{\Psi}_{\mu_n}(u_n)\cdots\tilde{\Psi}_{\mu_1}(u_1)|\Lambda_i\rangle$$

$$=(-1)^{s_2}(q-q^{-1})^{r_1+s_2}(-q)^{-i[\frac{n}{2}]+(i-1)[\frac{n+1}{2}]}\prod_{j:\text{odd}}^{n+1}(-q^3z_j)^i$$

$$\times\prod_{j:\text{even}}^{n+1}(-q^3z_j)^{\frac{1}{2}}\prod_{j=1}^{n+1}(-q^3z_j)^{\frac{s_2-s_1}{2}}\prod_{k:\text{even}}^{n}(-qu_k)^{\frac{1}{2}-i}\prod_a(q^2z_a)^{-1}\prod_b\prod_{j<b}(-q^3z_j)^{\frac{1}{2}}$$

$$\times\prod_a\prod_{j<a}(-q^3z_j)^{-\frac{1}{2}}\prod_c\prod_{j>c}(-qu_j)^{\frac{1}{2}}\prod_d\prod_{j>d}(-qu_j)^{-\frac{1}{2}}\prod_a\int_{C_a}\frac{dw_a}{2\pi i}\prod_d\int_{C_d}\frac{d\xi_d}{2\pi i}$$

$$\times\prod_a w_a^{-i+s_1-s_2}\prod_{a<b}w_a^{-1}\prod_{a<a'}w_a\prod_d\xi_d^{i-1}\prod_{d>c}\xi_d^{-1}\prod_{d>d'}\xi_d$$

$$\times \frac{\prod_d \prod_{j=1}^{n+1}(1-\frac{\xi_d}{q^3 z_j})\prod_a \prod_{l=1}^{n}(1-\frac{q u_l}{w_a})}{\prod_a \prod_{j\leqq a}(1-\frac{w_a}{q^2 z_j})\prod_a \prod_{j\geqq a}(1-\frac{q^4 z_j}{w_a})\prod_d \prod_{k\leqq d}(1-\frac{u_k}{\xi_d})\prod_d \prod_{k\geqq d}(1-\frac{\xi_d}{q^2 u_k})}$$

$$\times \frac{\prod_{a<a'}(1-\frac{w_{a'}}{w_a})(1-\frac{q^2 w_{a'}}{w_a})\prod_{d<d'}(1-\frac{\xi_{d'}}{\xi_d})(1-\frac{\xi_{d'}}{q^2\xi_d})}{\prod_{a,d}(1-\frac{q\xi_d}{w_a})(1-\frac{\xi_d}{q w_a})}. \tag{43}$$

Here $r_1, r_2, s_1, s_2, a, b, c, d$ is defined as follows.

$$\{a\}=\{j|\varepsilon_j=0\},\quad \{b\}=\{j|\varepsilon_j=1\},\quad \{c\}=\{j|\mu_j=0\},\quad \{d\}=\{j|\mu_j=1\},$$

$$r_1=\#\{a\},\qquad r_2=\#\{b\},\qquad s_1=\#\{c\},\qquad s_2=\#\{d\}.$$

w_a and ξ_d are the integral variables. The integral contour C_a and C_d are taken in the following manner:

C_a : $q^4 z_j$ $(j \geqq a)$ and $q^{\pm 1}\xi_d$ (all d) are inside,

: $q^2 z_j$ $(j \leqq a)$ are outside.

C_d : u_k $(k \leqq d)$ are inside,

: $q^2 u_k$ $(k \geqq d)$ and $q^{\pm 1} w_a$ (all a) are outside.

The special components are given by

$$\langle \Lambda_1|\bar{O}(\mathbf{z}|\mathbf{u})_{1\cdots 1,0\cdots 0}|\Lambda_0\rangle = (-q)^{-[\frac{n+1}{2}]}\prod_{j:\mathrm{even}}(-q^3 z_j)^{\frac{1}{2}}\prod_{j=1}^{n+1}(-q^3 z_j)^{\frac{1-j}{2}}$$

$$\times \prod_{k:\mathrm{even}}(-q u_k)^{\frac{1}{2}}\prod_{k=1}^{n}(-q u_k)^{\frac{k-1}{2}}, \tag{44}$$

$$\langle \Lambda_0|\bar{O}(\mathbf{z}|\mathbf{u})_{0\cdots 0,1\cdots 1}|\Lambda_1\rangle = (-q)^{-[\frac{n}{2}]+\frac{n(n+1)}{2}}\prod_{j:\mathrm{odd}}(-q^3 z_j)^{\frac{1}{2}}\prod_{j=1}^{n+1}(-q^3 z_j)^{-\frac{j}{2}}$$

$$\times \prod_{k:\mathrm{odd}}(-q u_k)^{\frac{1}{2}}\prod_{k=1}^{n}(-q u_k)^{1-\frac{k}{2}}. \tag{45}$$

B. Appendix 2

We give the description of the level one vertex operators $\tilde{\Phi}(z)$ and $\tilde{\Psi}(z)$ on the free field realization of the representations [8].

$$\tilde{\Phi}_1(z) = \exp \sum_{n=1}^{\infty} \left(\frac{a_{-n}}{[2n]} q^{\frac{7n}{2}} z^n \right) \exp \sum_{n=1}^{\infty} \left(-\frac{a_n}{[2n]} q^{-\frac{5n}{2}} z^{-n} \right) e^{\frac{\alpha}{2}} (-q^3 z)^{\frac{\partial_\alpha + i}{2}} ,$$

$$\tilde{\Phi}_0(z,w) = \frac{(q - q^{-1})(q^2 z)^{-1}}{(1 - \frac{w}{q^2 z})(1 - \frac{q^4 z}{w})} \exp \sum_{n=1}^{\infty} \left(\frac{a_{-n}}{[2n]} q^{\frac{7n}{2}} z^n - \frac{a_{-n}}{[n]} q^{\frac{n}{2}} w^n \right)$$

$$\times \exp \sum_{n=1}^{\infty} \left(-\frac{a_n}{[2n]} q^{-\frac{5n}{2}} z^{-n} + \frac{a_n}{[n]} q^{\frac{n}{2}} w^{-n} \right) e^{-\frac{\alpha}{2}} w^{-\partial_\alpha} (-q^3 z)^{\frac{\partial_\alpha + i}{2}} ,$$

$$\tilde{\Psi}_0(u) = \exp \sum_{n=1}^{\infty} \left(-\frac{a_{-n}}{[2n]} q^{\frac{n}{2}} u^n \right) \exp \sum_{n=1}^{\infty} \left(\frac{a_n}{[2n]} q^{-\frac{3n}{2}} u^{-n} \right)$$

$$\times e^{-\frac{\alpha}{2}} (-qu)^{\frac{-\partial_\alpha + i}{2}} (-q)^{-1+i} ,$$

$$\tilde{\Psi}_1(u,\xi) = \frac{-(q - q^{-1})\xi^{-1}}{(1 - \frac{u}{\xi})(1 - \frac{\xi}{q^2 u})} \exp \sum_{n=1}^{\infty} \left(-\frac{a_{-n}}{[2n]} q^{\frac{n}{2}} u^n + \frac{a_{-n}}{[n]} q^{-\frac{n}{2}} \xi^n \right)$$

$$\times \exp \sum_{n=1}^{\infty} \left(\frac{a_n}{[2n]} q^{-\frac{3n}{2}} u^{-n} - \frac{a_n}{[n]} q^{-\frac{n}{2}} \xi^{-n} \right) e^{\frac{\alpha}{2}} \xi^{\partial_\alpha} (-qu)^{\frac{-\partial_\alpha + i}{2}} (-q)^{-1+i} ,$$

$$\tilde{\Phi}_0(z) = \int_{C_1} \frac{dw}{2\pi i} \tilde{\Phi}_0(z,w) , \qquad \tilde{\Psi}_1(u) = \int_{C_2} \frac{d\xi}{2\pi i} \tilde{\Psi}_1(u,\xi) ,$$

where the contour C_1 and C_2 are specified by

$$C_1 : \ q^4 z \text{ is inside and } q^2 z \text{ is outside} ,$$

$$C_2 : \ u \text{ is inside and } q^2 u \text{ is outside} .$$

C. Appendix 3

Here we give the OPE of the level one vertex operators. Notations are the same as that in [8] except that the normal orderings are carried out for $e^{n\alpha}$ and ∂_α,

$$\tilde{\Phi}_1(z_1)\tilde{\Phi}_1(z_2) = \gamma \left(\frac{z_1}{z_2} \right) (-q^3 z_1)^{\frac{1}{2}} : \tilde{\Phi}_1(z_1)\tilde{\Phi}_1(z_2) : ,$$

$$\tilde{\Phi}_1(z_1)\tilde{\Phi}_0(z_2,w) = \gamma\left(\frac{z_1}{z_2}\right)\frac{(-q^3z_1)^{-\frac{1}{2}}}{1-\frac{w}{q^2z_1}} : \tilde{\Phi}_1(z_1)\tilde{\Phi}_1(z_2,w) : ,$$

$$\tilde{\Phi}_0(z_1,w)\tilde{\Phi}_1(z_2) = \gamma\left(\frac{z_1}{z_2}\right)\frac{w^{-1}(-q^3z_1)^{\frac{1}{2}}}{1-\frac{q^4z_2}{w}} : \tilde{\Phi}_0(z_1,w)\tilde{\Phi}_1(z_2) : ,$$

$$\tilde{\Phi}_0(z_1,w_1)\tilde{\Phi}_0(z_2,w_2) = \gamma\left(\frac{z_1}{z_2}\right) w_1(-q^3z_1)^{-\frac{1}{2}}\frac{(1-\frac{w_2}{w_1})(1-\frac{q^2w_2}{w_1})}{(1-\frac{w_2}{q^2z_1})(1-\frac{q^4z_2}{w_1})} : \tilde{\Phi}_0(z_1,w_1)\tilde{\Phi}_0(z_2,w_2) : ,$$

$$\tilde{\Psi}_0(u_1)\tilde{\Psi}_0(u_2) = \beta\left(\frac{u_1}{u_2}\right)(-qu_1)^{\frac{1}{2}} : \tilde{\Psi}_0(u_1)\tilde{\Psi}_0(u_2) : ,$$

$$\tilde{\Psi}_0(u_1)\tilde{\Psi}_1(u_2,\xi) = \beta\left(\frac{u_1}{u_2}\right)\frac{(-qu_1)^{-\frac{1}{2}}}{1-\frac{\xi}{q^2u_1}} : \tilde{\Psi}_0(u_1)\tilde{\Psi}_1(u_2,\xi) : ,$$

$$\tilde{\Psi}_1(u_1,\xi)\tilde{\Psi}_0(u_2) = \beta\left(\frac{u_1}{u_2}\right)\frac{\xi^{-1}(-qu_1)^{\frac{1}{2}}}{1-\frac{u_2}{\xi}} : \tilde{\Psi}_1(u_1,\xi)\tilde{\Psi}_0(u_2) : ,$$

$$\tilde{\Psi}_1(u_1,\xi_1)\tilde{\Psi}_1(u_2,\xi_2) = \beta\left(\frac{u_1}{u_2}\right)\xi_1(-qu_1)^{-\frac{1}{2}}\frac{(1-\frac{\xi_2}{\xi_1})(1-\frac{\xi_2}{q^2\xi_1})}{(1-\frac{\xi_2}{q^2u_1})(1-\frac{u_2}{\xi_1})} : \tilde{\Psi}_1(u_1,\xi_1)\tilde{\Psi}_1(u_2,\xi_2) : ,$$

$$\tilde{\Phi}_1(z)\tilde{\Psi}_0(u) = \alpha\left(\frac{z}{u}\right)(-q^3z)^{-\frac{1}{2}} : \tilde{\Phi}_1(z)\tilde{\Psi}_0(u) : ,$$

$$\tilde{\Phi}_1(z)\tilde{\Psi}_1(u,\xi) = \alpha\left(\frac{z}{u}\right)(-q^3z)^{\frac{1}{2}}\left(1-\frac{\xi}{q^3z}\right) : \tilde{\Phi}_1(z)\tilde{\Psi}_1(u,\xi) : ,$$

$$\tilde{\Phi}_0(z,w)\tilde{\Psi}_0(u) = \alpha\left(\frac{z}{u}\right)w(-q^3z)^{-\frac{1}{2}}\left(1-\frac{qu}{w}\right) : \tilde{\Phi}_0(z,w)\tilde{\Psi}_0(u) : ,$$

$$\tilde{\Phi}_0(z,w)\tilde{\Psi}_1(u,\xi) = \alpha\left(\frac{z}{u}\right)w^{-1}(-q^3z)^{\frac{1}{2}}\frac{(1-\frac{qu}{w})(1-\frac{\xi}{q^3z})}{(1-\frac{q\xi}{w})(1-\frac{\xi}{qw})} : \tilde{\Phi}_0(z,w)\tilde{\Psi}_1(u,\xi) : ,$$

$$\tilde{\Psi}_0(u)\tilde{\Phi}_1(z) = \omega\left(\frac{u}{z}\right)(-qu)^{-\frac{1}{2}} : \tilde{\Psi}_0(u)\tilde{\Phi}_1(z) : ,$$

$$\tilde{\Psi}_0(u)\tilde{\Phi}_0(z,w) = \omega\left(\frac{u}{z}\right)(-qu)^{\frac{1}{2}}\left(1-\frac{w}{qu}\right) : \tilde{\Psi}_0(u)\tilde{\Phi}_0(z,w) : ,$$

$$\tilde{\Psi}_1(u,\xi)\tilde{\Phi}_1(z) = \omega\left(\frac{u}{z}\right)\xi(-qu)^{-\frac{1}{2}}\left(1-\frac{q^3z}{\xi}\right) : \tilde{\Psi}_1(u,\xi)\tilde{\Phi}_1(z) : ,$$

$$\tilde{\Psi}_1(u,\xi)\tilde{\Phi}_0(z,w) = \omega\left(\frac{u}{z}\right)\xi^{-1}(-qu)^{\frac{1}{2}}\frac{(1-\frac{q^3z}{\xi})(1-\frac{w}{qu})}{(1-\frac{qw}{\xi})(1-\frac{w}{q\xi})} : \tilde{\Psi}_1(u,\xi)\tilde{\Phi}_0(z,w) : .$$

Here

$$\gamma(z) = \frac{(q^2z^{-1})_\infty}{(q^4z^{-1})_\infty}, \quad \beta(z) = \frac{(z^{-1})_\infty}{(q^2z^{-1})_\infty}, \quad \alpha(z) = \frac{(qz^{-1})_\infty}{(q^{-1}z^{-1})_\infty}, \quad \omega(z) = \frac{(q^5z^{-1})_\infty}{(q^3z^{-1})_\infty}.$$

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