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Supersymmetry, Vacuum Statistics, and the Fundamental Theorem of Algebra

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Abstract: I give an interpretation of the fundamental theorem of algebra based on supersymmetry and the Witten index. The argument gives a physical explanation of why a real polynomial of degree n need not have n real zeroes, while a complex polynomial of degree n must have n complex zeroes. This paper also addresses in a general and model-independent way the statistics of the perturbative ground states (the states which correspond to classical vacua) in supersymmetric theories with complex and with real superfields.

Supersymmetry provides some of the richest insights into the connections between physics and mathematics, with the Witten index [5] serving as one of the central tools in forging such connections. Perhaps what is most striking is the range of the applications of supersymmetry to mathematics; supersymmetry has been used to prove the Atiyah–Singer index theorem [2], to compute the topological invariants of manifolds [5, 6], and to derive a variety of results in arithmetic number theory [3]. The central role of the Witten index in these and in many other physical and mathematical applications stems from the invariance of the index under deformations of the parameters of a theory. This makes the index a powerful tool. It means that the index may be calculated reliably by simple means, as one need only find one point in parameter space where it is easily calculable to know its value at all points in parameter space; this in turn makes possible the derivation of exact, non-perturbative results about physical theories and the mathematical structures they describe (subject to certain caveats I mention below).

In this paper, using arguments that are not mathematically rigorous but which are nonetheless instructive and compelling, I extend the scope of the connections between supersymmetric physics and mathematical results by showing how one can use supersymmetry to obtain the fundamental theorem of algebra. In fact, I will use supersymmetry not only to show that an n^{th}-degree polynomial over the complex numbers always has n roots, but also to demonstrate that an n^{th}-degree polynomial over the reals has an even or odd number of real roots, according to whether

n is even or odd, respectively. (Here, and throughout this paper, I always include multiplicities when I refer to the number of roots of a polynomial.) Furthermore, our results will provide a physical interpretation of why the zeroes in the complex and real cases behave differently, of why the fundamental theorem of algebra holds for complex but not real polynomials. The central piece of the argument is to associate the zeroes of a polynomial with the classical vacuum states of a supersymmetric quantum theory, followed by the use of the Witten index to understand how the number of such states may or may not change as one changes the parameters of the theory, and hence how the number of zeroes may or may not change as the coefficients of the relevant polynomial are changed. The polynomial in question is the first derivative of the superpotential. On the physical side, our work establishes some general results regarding the statistics associated with these classical vacua, and why the statistics of these vacua exhibit different relationships in the cases of theories with complex and real superfields, respectively.

The organization of this paper is as follows. Following this introduction, I review some of the fundamental properties of the Witten index and address some basic facts regarding the zeroes of polynomials. Then I proceed to a discussion of the zeroes of polynomials over the reals by computing the Witten index of a quantum theory with real superfields. I find that the classical vacua alternate between bosonic and fermionic statistics, leading to the conclusion that the number of real zeroes of a real polynomial and its degree are equal in mod 2 arithmetic. Then I move on to a discussion of complex polynomials, and find that in the corresponding quantum theories, all the classical vacuum states have the same statistics. This, in turn, leads to the fundamental theorem of algebra, namely that an n^{th}-degree polynomial over the complex numbers has n complex roots [1]. In both cases, I show how the appearance mathematically of multiple roots corresponds physically to the existence of classical vacua with vanishing perturbative mass gap (i.e., minima with vanishing quadratic contribution to the Taylor series), and thus develop a physical understanding of how to identify and treat multiple roots.

In an appendix, I demonstrate that the invariance of the index under parameter deformations can be established without first diagonalizing the Hamiltonian. Since some treatments of diagonalization implicitly employ the fundamental theorem of algebra, this technical detail is important; it ensures us that the argument of this paper is not circular, i.e., that we are not invoking results that themselves depend on the fundamental theorem of algebra.

Let me emphasize that the point of this paper is to explore some intriguing connections between the Witten index of supersymmetric theories and the fundamental theorem of algebra, obtaining insights both into the properties of the perturbative zero-energy states of a supersymmetric theory and into the behavior of zeroes of polynomials. As such, I have intentionally focused on the physical theories that offer the most instructive insights. While this leads to a variety of interesting results, the derivation of these findings is not entirely rigorous from a mathematical point of view. The gaps in rigor stem primarily from two points: the use of field theories which have never been shown to exist formally, and the use of perturbative calculations of the index (equivalently, the use of the ultralocal limit in the functional integral to calculate the index), a method which is reasonable but which has not been rigorously proven to be valid in general. I will return to these points in the next section.

It is worth pointing out here that there are at least three ways to view the material presented in this paper. One can view this paper as a work of interpretation, recognizing a common mathematical structure at work in the behavior of the zeroes of polynomials and of the Witten index of supersymmetric theories. Such interpretation enriches our understanding of both fields, and it is this richer understanding, even in the absence of mathematical rigor, that the reader is urged to get from this paper. Second, one can view this paper as an outline of a proper derivation of the fundamental theorem of algebra; where there are gaps in rigor is clear, but how to fill in these gaps, converting an informal argument into a formal derivation, is an open task. Third, one can think of reversing the arguments presented in this paper. Since one knows that the fundamental theorem of algebra has been proven rigorously by other means, the consistency of our analysis of physical theories (e.g., the perturbative calculation of the Witten index) with the fundamental theorem of algebra provides a non-trivial (although obviously not definitive) check on those physical methods, a necessary but not sufficient test that those physical methods must pass. I have thought of this paper primarily in the spirit of the first approach, but all three approaches have relevance.

1. Index Basics

In a supersymmetric quantum theory, the Hamiltonian H is given by the square of the supercharge Q, a Hermitian fermionic operator. This implies that there are no states of negative energy. Furthermore, the operator $(-1)^F$ that measures fermion number anticommutes with the supercharge. This means that the states of positive energy come in degenerate bose-fermi pairs. Consequently, if we define the Witten index as $\mathrm{tr}(-1)^F e^{-\beta H}$ (often written more loosely as $\mathrm{tr}(-1)^F$), we see that the index calculates the difference between the number of bosonic and the number of fermionic zero energy states.

Because of the pairing of positive energy states, changes in the parameters of the theory that maintain supersymmetry can cause states to enter or leave the kernel of the supercharge (which is also the kernel of the Hamiltonian) only in bose-fermi pairs. Thus, under continuous deformations in the parameters of a theory (as long as these do not change the behavior of the potential at infinite field strength or otherwise similarly change the Hilbert space of the theory [5]), the index cannot change; it may be calculated at every point in parameter space by calculating it at one convenient point in parameter space. Furthermore, any approximation scheme that respects supersymmetry will give a correct and exact (not approximate) value for the Witten index. This is because the higher order corrections can only have the effect of moving states into and out of the kernel of the Hamiltonian in bose-fermi pairs.

The Witten index is therefore topological in nature, and so is typically exactly calculable through rather simple methods. For our purposes, we will generally calculate the index using perturbative methods. Restricting to finite volume with periodic boundary conditions, we will identify the classical ground states. We will then study the perturbative spectrum about each such state to find its contribution to the index. Summing these contributions from all the classical ground states will then give the index. One important feature of this perturbative method is that the perturbative contribution from the expansion about each ground state depends only on the properties of the potential in the neighborhood of that minimum of the potential; the other vacua have no effect on the index contribution. Tunneling between vacua is a higher order correction, and so does not change the value of the index. Note that in the functional language, this method of expanding about the perturbative zeroes is essentially equivalent to taking the ultralocal limit, in which the fields are taken to be constant, and hence the functional integral becomes an ordinary integral.

The heart of this paper lies in the application of this method, and as such it is important to understand the level of rigor of this method. Indeed, one way to present this paper is simply to say that it is a derivation of the fundamental theorem of algebra based on the assumption that the various supersymmetric physical theories employed exist and that perturbative (equivalently, ultralocal) calculation of the Witten index of such theories is valid. Of course, in the case of the supersymmetric field theories we use, it is not established that such theories exist rigorously from a mathematical point of view. And, even if they do exist, the validity of the perturbative or ultralocal approximation for calculating the index is still an open question – one can argue that such an approximation is plausible via Fourier expansions, for example, but such arguments are formal, relying on such requirements as various functional integrals being well-defined, various limits being non-singular and commuting with other operations, etc. In the case of non-relativistic quantum mechanics, the perturbative or ultralocal approximation rests on a better footing, and although there is no general proof of its applicability, at least in particular theories, with care, one can show its validity, although proper treatment of this requires careful analytical work (e.g., showing that the Hilbert space of states does not change as one includes perturbative and non-perturbative corrections). And, obviously, proving rigorously the validity of these methods in the case of field theories is an open question that will not be answered in the near future, barring dramatic progress in constructive field theory.

Note that if I removed all references to field theories in this paper, by speaking only of non-relativistic quantum mechanics and its complex generalization, this would enhance the rigor of the argument. I have nonetheless opted against this choice, favoring instead to seek greater insight into the interplay of physics and mathematics that occurs in supersymmetric theories. By using field theories, I am able to explore the behavior of perturbative ground states of supersymmetric theories, and to use this behavior to understand the behavior of the zeroes of polynomials, provided of course the physical theories in question (or at least very similar theories) exist. After all, my main goal is to use supersymmetry to enhance the understanding of the fundamental theorem of algebra. As we know, technically sound proofs of the fundamental theorem of algebra already exist; but new insights into why the theorem is true nonetheless further our understanding of the physical and mathematical structures in question. Thus the reader is urged to remember throughout this paper that the arguments presented are valuable for the insights they provide and compelling for the structures they suggest, but that the arguments from which these results are obtained are not mathematically rigorous.

Having set this context, let us return to a consideration of the perturbative calculation of the index. Suppose one considers a theory with superpotential $W(\phi)$. Let $P(\phi) = \frac{\partial W(\phi)}{\partial \phi}$. The supersymmetric vacuum states classically are given by the zeroes of $P(\phi)$ since the scalar potential is $V(\phi) = |P(\phi)|^2$. These zeroes fall into two categories. Either the perturbative excitation spectrum around the zero has no massless particles or it does have massless particles. If the classical vacuum has no massless particles in its excitation spectrum, this classical vacuum contributes either $+1$ or -1 to the Witten index. If the classical zero energy state does have massless particles in its perturbative spectrum, the situation needs to be studied more carefully. It is the connection between the Witten index and the zeroes of $P(\phi)$ that will form the basis of our interpretation of the fundamental theorem of algebra. By choosing $P(\phi)$ to be a polynomial of degree n, we can address the questions we wish to address, establishing results regarding the number of zeroes of an n^{th}-degree polynomial.

Note that any two n^{th}-degree polynomials $P(\phi)$ must produce the same Witten index (modulo the possibility in the real case for two polynomials to produce indices of the same magnitude but opposite sign; I discuss this in the next section). Changing the sub-leading coefficients in $P(\phi)$ does not change the asymptotic behavior of the potential, and hence does not change the index. Likewise, changing the leading coefficient does not change the index, as long as this leading coefficient is not made to vanish. We will use the equality of the index associated with all n^{th}-degree polynomials $P(\phi)$ within particular classes of theories (keeping track of the possible sign flip mentioned above) to relate the number of zeroes of various n^{th}-degree polynomials.

Finally, it is worth making the rather obvious remark that an n^{th}-degree polynomial cannot have more than n roots. The easiest way to see this is by observing first that if $\phi = \phi_0$ is a solution to $P(\phi) = 0$, then $\phi - \phi_0$ is a factor of $P(\phi)$ (which is obvious once one shifts variables by $\tilde{\phi} = \phi - \phi_0$), and observing that an n^{th}-degree polynomial cannot have more than n linear factors.

2. Polynomials Over the Reals

In this section, we consider the use of the index to study the zeroes of polynomials over the reals. We will first discuss this case in general terms, and then look at our results in the context of a specific model. We will use the language of field theory in the general discussion because of the useful insights one obtains, and then move on to the mathematically less precarious case of non-relativistic quantum mechanics when we turn to a specific model.

Let us consider a supersymmetric theory in which the superpotential $W(\phi)$ is a real-valued function of the real-valued field ϕ, as occurs, for example, in supersymmetric quantum mechanics and in $2+1$ dimensional supersymmetric scalar field theory. As in the previous section, define $P(\phi) = \frac{\partial W}{\partial \phi}$. To calculate the index using perturbative methods, one must identify the zeroes of $P(\phi)$, since the potential is $V(\phi) = (P(\phi))^2$. We are interested in the case that $P(\phi)$ is a polynomial in ϕ.

We now wish to find the index for a theory in which $P(\phi)$ is an n^{th}-degree polynomial. Since all polynomials of a given degree have the same index, we proceed by first considering a representative polynomial of given degree n; from this, we will learn the index associated with and gain insight into the number of zeroes of any n^{th}-degree polynomial $P(\phi)$. It turns out that we only need to consider two representative polynomials, one of even degree, and one of odd degree.

First, consider as a representative even degree polynomial $P(\phi) = \phi^n + 1$, where n is even. This polynomial manifestly has no real roots. As a consequence, the Witten index for the corresponding theory (in which $W(\phi) = \int P(\phi)\, d\phi$ is an odd degree polynomial) is zero. From this we can conclude that the Witten index of *any* theory for which $P(\phi)$ is an even degree polynomial is zero. Ignoring for the moment the situation when there are zeroes of $P(\phi)$ about which there is no mass gap, we see that, since each zero of $P(\phi)$ generically contributes either $+1$ or -1

to the Witten index, $P(\phi)$ must have an even number of zeroes (an equal number of bosonic and fermionic ones) when $P(\phi)$ is a real polynomial of even degree.

Now, as our representative odd degree polynomial, consider $P(\phi) = (\phi + 1)(\phi^{n-1} + 1)$, where n is odd. This polynomial manifestly has only one real root, about which the perturbative spectrum has a mass gap. As a consequence, the Witten index for the corresponding theory is either $+1$ or -1. (The actual sign does not matter for our purposes.) This means that any odd degree polynomial has Witten index ± 1. Hence (excepting for the moment the situation in which there are zeroes with no mass gap), an odd degree polynomial has an odd number of zeroes.

Note, incidentally, that the statement that any n^{th}-degree polynomial can be deformed continuously into any other n^{th}-degree polynomial comes with a caveat. Remember that as long as the leading asymptotic behavior of the potential does not change (keeping a canonical kinetic term throughout, of course), the index does not change. Now for a real polynomial, the leading coefficient cannot change from positive to negative without passing through zero, at which point the leading behavior of the potential is different. Thus it is possible that an n^{th}-degree polynomial will produce a different index depending on whether its leading coefficient is positive or negative. However, modifying our examples above by multiplying each representative polynomial by -1, we see that the only possible difference in the index value in the case of positive versus negative leading coefficient is a difference in sign. For even polynomials, this leaves the index as zero; for odd polynomials, we again conclude that the index is ± 1. Thus this possible change in sign, while relevant in other contexts, has no actual bearing on the arguments presented in his paper, and so we will not pursue it further here.

What happens if there are classical vacua for which there is no mass gap in the perturbative spectrum? Performing a Taylor expansion about some root, say ϕ_0, tells us that in the neighborhood of such a root, $P(\phi) \approx C(\phi - \phi_0)^r$. This is sufficient for studying the contribution to the index from the perturbative expansion about the classical vacuum point $\phi = \phi_0$. Note that if $r = 1$, there is a mass gap, and we have the case considered previously; if $r > 1$, there is no perturbative mass gap. At this point, the system looks like one in which $\partial W/\partial\phi$ is an r^{th}-degree polynomial, and so we see that the contribution to the index from the perturbative expansion about this vacuum state is simply r mod 2. Also, as is easily seen by shifting variables to $\tilde{\phi} = \phi - \phi_0$, if a polynomial $P(\phi) \approx C(\phi - \phi_0)^r$ in the neighborhood of $\phi = \phi_0$, then $(\phi - \phi_0)^r$ is a factor of $P(\phi)$.

This means that if we introduce the familiar notion of a multiple zero (i.e., counting it as r zeroes of $P(\phi)$ if $(\phi - \phi_0)^r$ is a factor of $P(\phi)$), our previous statements regarding the number of zeroes of P remain unchanged. Since an r-fold zero contributes $\pm(r \bmod 2)$ to the index, in order for a polynomial of degree n to produce the required index, namely n mod 2, that polynomial must have an even or odd number of real zeroes, respectively, according to whether n is even or odd, provided we count multiplicities when we enumerate the roots. (Of course, the total number of zeroes – indeed, of linear factors – can never exceed the degree of the polynomial.)

It is worth noting that our result for real polynomials says that, as the parameters of a real polynomial change, the number of zeroes of that polynomial can only change by two (or a multiple of two) at a time, so as to preserve the value of the index. In the physical interpretation of this result, this is the statement that classical vacua can only appear or disappear in bose-fermi pairs. There is a standard algebraic interpretation of this result, too. We can view a real polynomial as a special example of a complex polynomial (which must, as we will see below, have n complex roots).

Since the complex roots of a real polynomial must occur in complex conjugate pairs, we have recovered physically the familiar algebraic result that the only way for the number of real roots of a real polynomial to change is for its complex roots to appear or disappear in complex conjugate pairs.

As an explicit example of our construction for real polynomials, we can consider supersymmetric quantum mechanics in one spatial dimension [4]. Including the $\hbar$'s, the Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + P(x)^2 + \sigma_3 \frac{\hbar}{\sqrt{2m}}\frac{dP(x)}{dx} . \tag{2.1}$$

The operator $(-1)^F$ is simply σ_3. The classical zero energy states are given by the solutions of

$$P(x) = 0 . \tag{2.2}$$

Some of these solutions may be bosonic and some fermionic, depending on whether they are eigenstates of σ_3 with eigenvalue $+1$ or -1, respectively. As Witten showed, the exact quantum theory has either no zero energy states or exactly one zero energy state. Thus, the index is either 0 or ± 1, respectively, in accord with what we found based on general arguments; in fact, when $P(x)$ goes as x^n, Witten showed in [4] that there is no zero energy state when n is even, and that there is one zero energy state when n is odd.

In perturbation theory, these index results arise by expanding about each point where $P(x)$ vanishes. Suppose x_0 corresponds to a zero of $P(x)$ and hence to a classical vacuum state. The Hamiltonian near this point to leading order is

$$H \approx -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + ((x - x_0)P'(x_0))^2 + \sigma_3 \frac{\hbar}{\sqrt{2m}} P'(x_0) . \tag{2.3}$$

This is a harmonic oscillator potential; calculating, we see that if $P'(x_0)$ is positive (respectively, negative), this Hamiltonian has a single fermionic (respectively, bosonic) zero energy state. Hence the classical vacuum contributes an amount to the index equal to $-\mathrm{sign}(P'(x_0))$. Thus, depending on this sign, the classical vacuum is associated with either a bosonic or fermionic perturbative zero energy state. Since the fermion number is given by minus the sign of $P(x_0)$, it therefore follows that, as one proceeds along the spatial axis, the vacua one encounters are alternately bosonic and fermionic. (If $P(x)$ has vanishing slope at the classical vacuum point, we simply need to refine the argument to include the notion of multiple zeroes, just as we did earlier. There is little to be gained by doing that here, so we leave it as an exercise for any interested readers.)

Incidentally, this alternation of the fermion number of the classical vacua can be derived from more abstract index arguments. Since we can deform the location of the zeroes of the superpotential without changing the index, we can choose all but a pair of zeroes to be very far from each other, so that the physical effect of those faraway zeroes is negligible. Perturbatively, then, this theory looks just like a theory in which $P(x)$ is quadratic, plus the contributions of the faraway zeroes. Since the index in a theory with quadratic $P(x)$ is zero, the statistics of these two nearby zeroes must cancel each other. Any pair of adjacent zeroes of $P(x)$ can be isolated in the way just described above, but the order of the zeroes of $P(x)$ cannot change under smooth, non-singular deformations. Thus any pair of adjacent zeroes of $P(x)$ must have opposite fermion number, which in turn means that the classical vacua correspond, in alternating fashion, to bosonic and fermionic states. Note, too, that this gives us another way to understand the contribution to the index at a root of $P(\phi)$ which is also a point of inflection. We can model the perturbative situation of $P(x) \approx C(x - x_0)^r$ as the coalescence of r adjacent zeroes, all deformed to the point $x = x_0$. If r is even, then this coalescence will always involve an equal number of bosonic and fermionic vacua (since the fermion number alternates), producing index 0; if r is odd, by a similar argument, we see that this coalescence will produce index ± 1 about this classical vacuum.

3. Polynomials Over the Complex Numbers

The consideration of the zeroes of polynomials over the complex numbers proceeds in a way similar to what we have already done for polynomials over the reals, although the results are quite different. In short, we will see that every n^{th}-degree complex polynomial has n complex roots. In this section, I will use the language of field theory; the reader has already been alerted to the ways in which this undercuts the rigor of the argument, but we expect that the reader will nonetheless find the results instructive and insightful.

Let us consider a supersymmetric theory with complex superfields, such as supersymmetric scalar field theory in 3 + 1 dimensions. As in the real case, we define $P(\phi) = \frac{\partial W}{\partial \phi}$, where the superpotential $W(\phi)$ is now a complex-valued function of the complex field ϕ. The potential is $V(\phi) = | P(\phi) |^2$. The perturbative computation of the index can thus be achieved by determining all solutions of $P(\phi) = 0$, and then expanding about each of these. Note that any two n^{th}-degree complex polynomials $P(\phi)$ will produce the same value for the index. This result is slightly stronger than in the real case. In the complex case, the leading coefficient of $P(\phi)$ can be changed continuously from a positive to a negative number without ever passing through zero. Thus all n^{th}-degree polynomials yield exactly the same index, with the same magnitude *and* sign. This contrasts with the real case in which changing the sign of the leading coefficient could change the sign, but not the magnitude, of the index.

One additional simplification in the complex case is that we will not need to distinguish between even and odd degree polynomials, as will readily become apparent.

To proceed, then, we first pick a representative n^{th}-degree complex polynomial and calculate the index associated with it. We then will use this result to find the index associated with any n^{th}-degree polynomial, which we will then use to show that an arbitrary n^{th}-degree polynomial has exactly n roots.

As our representative n^{th}-degree polynomial, let us consider

$$P_0(\phi) = (\phi - c_1)(\phi - c_2) \cdots (\phi - c_n) , \tag{3.1}$$

with the complex constants c_j all distinct from each other. Clearly, this polynomial has n complex roots, with non-zero mass gap in the perturbative spectrum about each of these classical vacua. We must now determine which of these classical vacua correspond to bosonic states and which to fermionic states, so that we can compute the index.

Under continuous non-singular deformations of the parameters of the theory, the statistics of each individual vacuum cannot change, as the eigenvalues of $(-1)^F$ can take on only the discrete values $+1$ and -1. Let us concentrate for the moment on two of the vacuum say $\phi = c_1$ and $\phi = c_2$, as we undertake certain deformations in the parameters of the theory.

Note that one can vary $\phi - c_2$ smoothly to $\phi - c_1$, while varying $\phi - c_1$ smoothly to $\phi - c_2$. To do this, consider

$$P_\sigma(\phi) = \left(\phi - \left[\frac{c_1 + c_2}{2} + \frac{c_1 - c_2}{2} e^{i\pi\sigma}\right]\right)\left(\phi - \left[\frac{c_1 + c_2}{2} - \frac{c_1 - c_2}{2} e^{-i\pi\sigma}\right]\right)$$
$$\times (\phi - c_3) \cdots (\phi - c_n) \,. \tag{3.2}$$

As σ varies from 0 to 1, P_σ continuously deforms so that the vacua at $\phi = c_2$ and $\phi = c_1$ smoothly switch locations. Thus, by the fixed value of the statistics of each vacuum individually under continuous changes of the parameters of a theory, the statistics at $\phi = c_1$ when $\sigma = 0$ must be identical to the statistics at $\phi = c_2$ when $\sigma = 1$; and the statistics at $\phi = c_2$ when $\sigma = 0$ must be identical to the statistics at $\phi = c_1$ when $\sigma = 1$. On the other hand, the polynomial is exactly the same at $\sigma = 0$ and $\sigma = 1$. Thus the statistics of the vacuum at $\phi = c_1$ is the same whether $\sigma = 0$ or $\sigma = 1$, and the statistics of the vacuum at $\phi = c_2$ is the same whether $\sigma = 0$ or $\sigma = 1$. Putting this all together, we see that in our original polynomial, the vacua at $\phi = c_1$ and $\phi = c_2$ must have the same statistics.

Now there was nothing special about these two particular vacua. Hence we see that all the classical vacuum solutions for this polynomial have the same fermion number, and thus the Witten index is $\pm n$. (We do not need to determine the sign.)

So, in this particular theory, the index has value $\pm n$. This is a theory, however, in which $P(\phi)$ is an n^{th}-degree polynomial. As we have argued above, any two theories in which $P(\phi)$ is an n^{th}-degree polynomial over the complex numbers must produce the same index. This means that any other theory in which $P(\phi)$ is an n^{th}-degree polynomial must have a Witten index of value $\pm n$, whether we know how to write the polynomial in the factorized form in (3.1) or not.

What are the further implications of this result for general n^{th}-degree polynomials $P(\phi)$? Since the index for such a theory is $\pm n$, and since generically the zeroes of $P(\phi)$ will all exhibit mass gaps in their respective perturbative spectra, we see that a generic n^{th}-degree polynomial must have n zeroes. Since achieving an index of $\pm n$ requires at least n zeroes, and since an n^{th}-degree polynomial can have at most n zeroes, we can conclude that an n^{th}-degree complex polynomial has exactly n zeroes.

Note further that since the index is $\pm n$ and all the classical vacua where $P'(\phi) \neq 0$ can be deformed into each other and hence have the same statistics, we can rule out on index grounds alone the possibility that there are more than n zeroes for an n^{th}-degree polynomial $P(\phi)$. (In fact, using the observations developed below regarding zeroes which are also points of inflection, one can extend this to use index arguments alone to show that there are no more than n zeroes even when some are located at points where $P'(\phi)$ vanishes.)

What if there are zeroes of $P(\phi)$ about which there is perturbatively no mass gap (i.e., the first term in the Taylor series for the potential about that point is higher than quadratic)? What happens at such a point?

As in the real case, we can understand such a point independent of what is happening elsewhere in the potential. If the polynomial $P(\phi) \approx C(\phi - c)^r$ to lowest order near the zero $\phi = c$, then, calculating the contribution to the index from this point perturbatively, we see that the contribution to the index at this point is the index associated with an r^{th}-degree polynomial. Consequently, such a point can contribute an amount to the index of $\pm r$, only. Since we have ultimately to reach a total index of n, and the polynomial can have no more than n linear factors, one can infer that the sign of this index contribution must be the same as the sign of all the other index contributions from all the other perturbative vacua. (Recall that for a polynomial, if $P(\phi) \approx C(\phi - c)^r$ near $\phi = c$, then $(\phi - c)^r$ is a factor of $P(\phi)$.)

Alternatively, we can see that the sign is the same directly. We can obtain a function which is perturbatively identical to the $C(\phi - c)^r$ considered above by considering the coalescence of r vacua which have mass gaps. Since all the vacua with mass gaps contribute to the index with the same sign (the same not only among the r vacua that are coalescing, but among all the vacua), the coalescence of r such vacua must give a contribution with the same sign as all these other vacua, too (although larger in magnitude by a factor of r).

Thus, if we introduce the familiar notion of the multiplicity of a zero, we have just concluded, subject to the limitations of rigor pointed out above, that all complex n^{th}-degree polynomials have n complex roots. What the work above demonstrates is that an r^{th}-degree zero contributes an amount r to the index; all the zeroes contribute to the index with the same sign; and the total index is n for a polynomial of degree n. This means that, counting multiplicities, a complex polynomial of degree n has exactly n roots. This is the fundamental theorem of algebra. Our argument rested on our determination that any n^{th}-degree polynomial $P(\phi)$ leads to a theory with the same value for the Witten index, namely $\pm n$, which in turn implies that $P(\phi)$ has n linear factors.

Note that we have found some essential differences between the complex and real cases. The most significant is that in the complex case, the locations of the vacua in field space could be interchanged smoothly, without colliding, by a continuous change in the parameters of the theory. This is what led to an index of value $\pm n$, which in turn led to the fundamental theorem of algebra by forcing the number of zeroes to be n. In the real case, this smooth type of interchange is not possible, which makes it possible for the vacua not all to have the same fermion number. Indeed, this occurs; as we have seen, the real case always has vacua of alternating fermion number.

Before closing, it is worth noting that one approach to showing that a complex polynomial of degree n has n complex roots is first to show that any complex polynomial has at least one complex root. From this, one can proceed by induction to show that an n^{th}-degree polynomial has n roots, by pulling out a linear factor for the first root one knows about, and then studying the polynomial of degree $n - 1$ this leaves behind. We could have used the above index arguments to recognize immediately that, since any complex polynomial is associated with non-zero index, it must have at least one zero, in order for the perturbative index calculation to yield a non-zero result. One could then proceed via induction to show mathematically that such a polynomial has n complex roots. Our goal here, however, has been to show the rich set of relationships that exist between the physical properties of supersymmetric quantum theories and the mathematical results regarding the zeroes of polynomials, and thus we have chosen instead to seek an understanding of the broader behavior of these zeroes from a physical perspective.

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Appendix A. Invariance of the Index

In this appendix, I derive the topological invariance of the Witten index

$$\Delta = \mathrm{tr}(-1)^F e^{-\beta H} , \tag{A.1}$$

by which I mean that I show the index to be unchanged under parameter deformations and to be calculable exactly using approximation methods. The reason for doing this is that typically discussions of the invariance of the index invoke diagonalization of the Hamiltonian along the way, and some treatments of the diagonalization of operators implicitly use the fundamental theorem of algebra (invoked for finite dimensional matrices, followed by the taking of an appropriate limit), something we must avoid if we are to use the invariance of the Witten index to derive the fundamental theorem of algebra. Thus, in this appendix, I establish explicitly that the fundamental theorem of algebra is not in this appendix, is not necessary to derive the topological invariance of the index.

Let the Hamiltonian H and the supercharge Q be Hermitian operators related by $H = Q^2$. They act on a Hilbert space $\mathscr{H}$. There is also an operator $(-1)^F$ which anticommutes with Q and squares to the identity. Note that the operator H is positive semi-definite, and so in any state, the expectation value of H is non-negative, and it can only be zero for a state in the kernel of H. In addition, let us let $\mathscr{H}' =$ coker H; this is a subspace of $\mathscr{H}$.

Note first that $\ker Q = \ker H$.

Second, note that we can choose as a basis for $\mathscr{H}$ states which are eigenstates of $(-1)^F$. To do this, given some basis of kets $|n\rangle$, we define a new basis using the kets

$$|n\rangle \pm (-1)^F |n\rangle , \tag{A.2}$$

which have eigenvalues ± 1, respectively, under $(-1)^F$.

From now on, I use such a basis. Let B and F denote, respectively, the bosonic and fermionic subspaces of $\mathscr{H}$, and let B' and F' denote, respectively, the bosonic and fermionic subspaces of $\mathscr{H}'$.

It is straightforward to see that, given $|f'\rangle \in F', Q|f'\rangle \in B'$. Clearly, $Q|f'\rangle \in B$. Let $|b_s\rangle$ be a bosonic state in ker H. Then $\langle b_s|Q|f'\rangle = 0$, and thus we see that $QF' \subset B'$. In an entirely analogous fashion, we see that $QB' \subset F'$.

The next step is to show that, in fact, $QF' = B'$ and $QB' = F'$. This can be obtained in one of two ways.

Method 1: We start from the existence of an inverse H^{-1} on $\mathscr{H}'$. Then given a bosonic ket $|b'\rangle$, we can write it in the form $|b'\rangle = Q|f'\rangle$, by defining the fermionic ket $|f'\rangle = H^{-1}Q|b'\rangle$. This establishes that $B' \subset QF'$. Since we have both $QF' \subset B'$ (see above) and $B' \subset QF'$, we see that $QF' = B'$. Likewise, $QB' = F'$.

Method 2: We start from the (physically motivated) statement that the range of H is $\mathscr{H}'$. Now we already know that $Q|f'\rangle \in B'$. Applying Q to this, we find

$$Q^2F' \subset QB' \subset F' . \tag{A.3}$$

But in order that the range of the Hamiltonian be the full $\mathscr{H}'$, we see that $Q^2F' = F'$. Thus it follows that $QB' = F'$ and, likewise, that $QF' = B'$.

Moving on, I now consider

$$\mathrm{tr}(-1)^F e^{-\beta H} = \mathrm{tr}(-1)^F e^{-\beta H}\,|_{\ker H} + \mathrm{tr}(-1)^F e^{-\beta H}\big|_{\mathscr{H}'} . \tag{A.4}$$

(Note: If the sum is not absolutely convergent, one should take the sum in the order implicit in the bose-fermi grouping of states I use below.) Now let the kets $|b_j'\rangle$ form a basis for B'. Define

$$|f_j'\rangle = \frac{Q|b_j'\rangle}{\langle b_j'|H|b_j'\rangle^{1/2}} . \tag{A.5}$$

Clearly, since $QB' = F'$, the set of vectors in (A.5) spans F'. Do they in fact form a basis for F'? The answer is that they do. Otherwise, there would be constants c_j such that

$$\sum_j c_j|f_j'\rangle = 0 . \tag{A.6}$$

This would mean then that

$$Q\Bigl(\sum_j \alpha_j|b_j'\rangle\Bigr) = 0 , \tag{A.7}$$

(where $\alpha_j = c_j/\langle b_j'|H|b_j'\rangle^{1/2}$), which cannot be, since the $|b_j'\rangle$ form a basis for $\mathscr{H}'$, and so a linear combination of them cannot belong to ker H.

Thus we can set up a one-to-one correspondence between the basis vectors of B' and the basis vectors of F'. It is a simple exercise to show that each member of a pairing of bosonic and fermionic basis vectors (as paired above) yields the same expectation value of H or any function of H; that is,

$$\langle b_j'|H|b_j'\rangle = \langle f_j'|H|f_j'\rangle . \tag{A.8}$$

Thus,

$$\langle b_j'|(-1)^F H|b_j'\rangle + \langle f_j'|(-1)^F H|f_j'\rangle = 0 . \tag{A.9}$$

Consequently, the Witten index receives a net vanishing contribution from all the states in $\mathscr{H}'$, and thus

$$\mathrm{tr}(-1)^F e^{-\beta H} = \mathrm{tr}(-1)^F\,|_{\ker H} . \tag{A.10}$$

Note further that we have shown that the basis for $\mathscr{H}'$ can be organized in bose-fermi pairs with common values for the expectation value of the Hamiltonian. (We will refer to such pairs as degenerate because they give the same expectation value for the Hamiltonian, even though they are not necessarily eigenstates of the Hamiltonian.) If an approximation scheme erroneously determines whether a state is in ker H or $\mathscr{H}'$, it can do so only in degenerate bose-fermi pairs, as long as the approximation scheme respects supersymmetry. Thus using an approximate method to calculate the Witten index yields an exact result. Also, if we deform the parameters of the Hamiltonian, as long as this leaves the Hilbert space unchanged (which will happen generically when the asymptotic behavior of the potential is unchanged), it leaves the Witten index unchanged, as states can only move between ker H and $\mathscr{H}'$ in degenerate bose-fermi pairs.

Thus we have shown that the Witten index is indeed invariant and is indeed exactly calculable when treating the system approximately, and we have shown this without reference to diagonalizing H. This makes clear that the topological invariance of the Witten index does not depend, explicitly or implicitly, on the fundamental theorem of algebra.

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