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Continuity Properties of the Electronic Spectrum of $1D$ Quasicrystals

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Abstract. In this paper we consider operators $H(\alpha, x)$ defined on $l^2(\mathbb{Z})$ by

$$H(\alpha, x)\psi(n) = \sum_{m \in \mathbb{Z}} t_m \circ \phi^{-n}(\alpha, x)\psi(n-m),$$

where $\phi(\alpha, x) = (\alpha, x-\alpha)$, t_m is in the algebra of bounded periodic functions on $\mathbb{R}^2$ generated by the characteristic functions of the sets

$$\phi^n\{(\alpha, x) \in \mathbb{R}^2 \mid 1-\alpha \leqq x < \alpha \ (\mathrm{mod}\ 1)\}.$$

This class of hamiltonian includes the Kohmoto model numerically computed by Ostlund and Kim, where the potential is given by

$$v_{\alpha,x}(n) = \lambda\chi_{[1-\alpha, 1[}(x+n\alpha), \qquad n \in \mathbb{Z},\ x,\ \lambda,\ \alpha \in \mathbb{R}$$

(see [B.I.S.T.]). We prove that the spectrum (as a set) of $H(\alpha, x)$ varies continuously with respect to α near each irrational, for any x. We also show that the various strong limits obtained as α converges to a rational number $\frac{p}{q}$ describe either a periodic medium or a periodic medium with a localized impurity. The corresponding spectrum has eigenvalues in the gaps and the right and left limits as $\alpha \to \frac{p}{q}$ do not coincide, for the Kohmoto model. The results are obtained through C^*-algebra techniques.

1. Introduction

Let us consider the following discrete one dimensional Schrödinger hamiltonian with quasiperiodic potential, acting on $l^2(\mathbb{Z})$ and given by

$$H(\alpha, x, \lambda)\psi(n) = \psi(n+1) + \psi(n-1) + \lambda v_{\alpha,x}(n)\psi(n), \tag{1}$$

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with:

$$v_{\alpha,x}(n) = \chi_{[1-\alpha,1[}(x+n\alpha),$$

where $\chi_{[1-\alpha,1[}$ is the characteristic function of the interval $[1-\alpha, 1[\subset \mathbb{T} = [0, 1[$, the numbers x and α are in $\mathbb{T}$, and λ (the coupling constant) is in $\mathbb{R}$. This model was considered first by Kadanoff, Kohmoto, and Tang [K.K.T.], for $\alpha = \frac{1}{2}(\sqrt{5}-1)$. They used a renormalization group analysis and transfer matrices to construct the energy spectrum and the wave functions. Later on, Ostlund and Kim [O.K.] gave a numerical algorithm to compute the spectrum for any rational value of α.

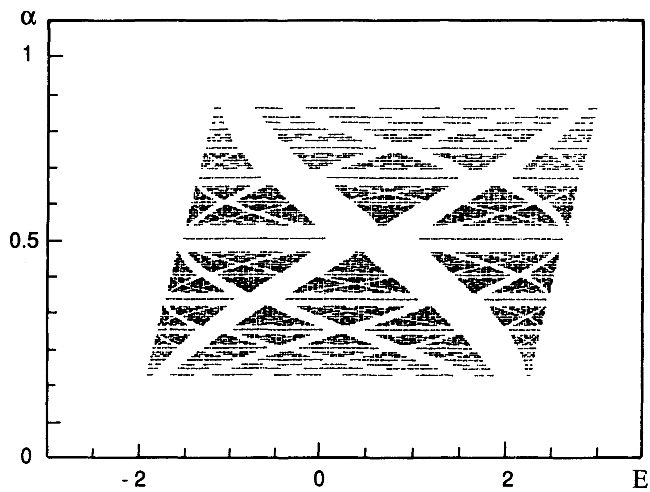

Fig. 1. The energy E has been plotted as a function of α if it is in the spectrum of $H(\alpha, x, \lambda)$. The coupling constant λ is 1 and $x=0$

The beautiful fractal nature of the picture reflects the properties of the renormalization group.

The model was interpreted by Luck and Petritis [L.P.] as describing the phonon spectra in a one dimensional quasicrystal. In this latter case, the cut and projection method based on a periodic two-dimensional structure gives rise to Eq. (1) where α is the irrational slope of a strip and x is the position of this strip. The fact that the potential is discontinuous is justified in some problems of quasicrystals: for instance, the spectrum of surface states of electrons on a crystal face with large Miller indices, and electrons on a dislocation the direction of which is incommensurate with the lattice periods.

This model has also been related to the problem of Peierls instability for one dimensional chains. In this respect, the work by Machida-Nakano [M.N.], based upon a mean field approach to the Fröhlich hamiltonian, gives rise to a one electron energy spectrum very much reminiscent of Fig. 1. This fact seems to indicate that the effective one electron hamiltonian belongs to the class of operators we consider in this paper. It is interesting to remark that α represents the product of the modulation frequency of the charge density wave by the period of the chain. Hence it can be modified by changing the charge carrier density. So α appears as a physical parameter. The same is true for x which is related to phason modes if one takes into account the fluctuation of the phonon groundstates.

Model (1) can be also used for describing the quasisuperlattices grown according to a role given by the Fibonacci sequence whenever $\alpha=\frac{1}{2}(\sqrt{5}-1)$. Such a device leads to a number of interesting questions: computation of the electrical resistivity, optical transmission, effective impedance, Raman scattering from acoustic phonons, interface polariton modes, critical plasmons... [M.].

The aim of this paper is to give a mathematical explanation of the Ostlund-Kim spectrum (see Fig. 1). We shall especially address the following questions:

– The numerical computations involved only rational values of α. Is the picture relevant for irrational α's? In other words, is the spectrum continuous in the vicinity of an irrational number?

– We can easily see from this figure that the spectrum is discontinuous at $\alpha=\frac{p}{q}$, where $p, q \in \mathbb{N}^*$. In particular the right and left limits do not coincide and they both differ from the spectrum at $\frac{p}{q}$. More precisely the difference is given by isolated eigenvalues located in each gap. Is there a physical interpretation of this observation?

The main result of this paper is Theorem 1 below according to which the spectrum is a continuous function of α in the vicinity of irrational values. This result actually applies to more general hamiltonians acting on $l^2(\mathbb{Z})$ as follows:

$$H(\alpha,x)\psi(n)=\sum_{m\in\mathbb{Z}} t_m\circ\phi^{-n}(\alpha,x)\psi(n-m), \tag{2}$$

where the t_m's are in the subset defined below, of periodic bounded functions of period one.

As a byproduct of the method we use here, we will get an explanation of the discontinuity of the spectrum at rational values of α. In particular we will show that isolated eigenvalues showing up in gaps of the right (left) limits of the spectrum near rational numbers come from a localized impurity appearing in the hamiltonian by taking a strong right (left) limit with respect to α.

The usual description of the hamiltonian (1) goes through the transfer matrix formulation [K.K.T., O.P.R.S.S., C., L.] and leads to the result that the spectrum is a Cantor set of zero Lebesgue measure for any irrational α and any $\lambda \neq 0$ [S., B.I.S.T.]. We will rather use a somewhat different approach (see however Theorem 7 below). The reason is that the transfer matrix method is essentially limited to nearest neighbour interactions, whereas many results still hold for long range interactions as well.

Given $H(\alpha,x)$ like in (2), we introduce the unital C^*-algebra $\mathscr{A}_\alpha$ generated by the family $\{T^n H(\alpha,x)T^{*n} \mid n\in\mathbb{Z}\}$, where T is the translation operator. This is natural since the system described by $H(\alpha,x)$ is macroscopically translation invariant. Therefore, translating the origin in the lattice will give as good as description as the previous one (see [Be.] where the homogeneity in space is discussed). So $\mathscr{A}_\alpha$ contains no more information than the energy and the homogeneity properties of the system.

Our Theorem 1 can be rephrased by proving that $\alpha\to\mathscr{A}_\alpha$ is a continuous field of C^*-algebras [D., T.] near any irrational number.

To prove this, we will go one step further in the abstract setup. We will construct a "universal" algebra $\mathscr{A}$ which is, roughly speaking, the disjoint union $\bigcup_\alpha \mathscr{A}_\alpha$. To define $\mathscr{A}$, one remarks that $H(\alpha,x)$ is generated algebraically by two kinds of operators: First of all, the translation operator T acting on $l^2(\mathbb{Z})$ as

$$(T\psi)(n)=\psi(n-1),$$

and, second, the multiplication by the function χ given by

$$\chi(\alpha, x)=\chi_{[1-\alpha, 1[}(x),$$

or more precisely, by the functions $v_{\alpha,x}(n)=\chi(\alpha, x+n\alpha)$, x, $\alpha\in\mathbb{R}$. Actually, we get

$$T\chi T^*=\chi\circ\phi, \tag{3}$$

where $\phi(\alpha, x)=(\alpha, x-\alpha)$.

So we can rather consider the abelian unital C^*-algebra $\mathscr{B}$ generated by the functions $\chi_n=\chi\circ\phi^{-n}$ in $l^\infty(\mathbb{T}^2)$. By Gelfand's theorem [D.], this is isomorphic to $C(\Omega)$, the set of continuous functions on some compact Hausdorff space Ω. The map ϕ defines a $*$-automorphism on the algebra generated by the functions χ_n, and so can be extended as a homeomorphism of Ω which will be denoted also by ϕ.

The C^*-algebra $\mathscr{A}$ is nothing but the C^*-crossed product of $C(\Omega)$ by the group $\mathbb{Z}$ acting through ϕ [P.]. Namely, every element of $\mathscr{A}$ can be approximated in norm by finite sums $\sum_{m=-N}^{N} f_m U^m$, where f_m is in $C(\Omega)$ and U is an abstract unitary implementing ϕ. Ω appears as a compactification of the set

$$\Lambda=\{(\alpha, x)\in\mathbb{T}^2 \mid \alpha\notin\mathbb{Q} \text{ and } x\notin\mathbb{Z}\alpha\}$$

endowed with the weakest topology making all the χ_n's continuous.

One then remarks that the restriction of ϕ to Λ does not change the value of the coordinate α. So that if we define p as the map $\Lambda\to\mathbb{I}:=\mathbb{T}\backslash\mathbb{Q}$, $p(\alpha, x)=\alpha$, then $p^{-1}\{\alpha\}$ is ϕ-invariant. We will show that p extends as a continuous function from Ω onto $\mathbb{T}$.

Given $\alpha\in\mathbb{I}$, let J_α be the closed two sided ideal generated by the sums $\sum_{m=-N}^{m=N} f_m U^m$, where the f_m's vanish on $p^{-1}\{\alpha\}$. Then $\mathscr{A}_\alpha$ is the C^*-quotient $\mathscr{A}/J_\alpha$. We denote by η_α the quotient mapping.

As a corollary of the continuity of the C^*-field $\alpha\to\mathscr{A}_\alpha$ at irrational numbers, if $h=h^*\in\mathscr{A}$ and if $\eta_\alpha(h)$ is the representative of h in $\mathscr{A}_\alpha$, the gap edges of the spectrum of $\eta_\alpha(h)$ are continuous functions of α. Then our construction will show that the topology of $\mathbb{I}$ coincides with the topology of $\mathbb{T}$ in the vicinity of any irrational number α, proving the main result (Theorem 1) in the first part of this paper.

Unfortunately the constructed map p is not open, so it gives no information on the behaviour of the spectrum near a rational number. To overcome this difficulty we have explicitly described, in the second part of this paper, a compact Hausdorff space $\Gamma\supset\mathbb{I}$ and a continuous open map from Ω onto Γ. As a consequence of this, it is seen that the topology of Ω explains the qualitative nature of the discontinuities of the spectrum of $H(\alpha, x)$ near the rational values of α. Moreover, a point ω in Ω can be viewed as a limit point of a sequence (α_n, x_n) in Λ. Correspondingly, one can construct an operator $H(\omega)$ as a strong limit of $H(\alpha_n, x_n)$. We will prove (Theorem 2) that if $\alpha_n\to\frac{p}{q}$ in the usual topology, the right and left limits exist for the spectrum. This means that there are limit points ω for which $H(\omega)$ is a periodic operator of the type given in (1) perturbed by a localized impurity (Theorem 4 and its Corollary). $H(\omega)$ admits a band spectrum and in addition a finite number of eigenvalues in the gaps, as shown in Fig. 1.

The paper is organized as follows. In Sect. 2 we describe precisely the results. Section 3 is devoted to a proof of Theorem 3 concerning an abstract continuity result in the algebraic set-up. A proof of Theorem 1 is given in Sect. 4 which concludes part one. Section 5 concerns the construction and the properties of the spaces Ω and Λ and the map p. It ends with a proof of Theorem 2. Some details on the spectrum around a rational number and the proof of its discontinuity at such a point for the Kohmoto model are given in the last section.

2. Notations and Main Results

The spectrum (respectively the absolutely continuous part of the spectrum, the essential spectrum, the discrete spectrum) of a selfadjoint operator A will always be denoted by $\sigma(A)$ [respectively $\sigma_{\mathrm{ac}}(A)$, $\sigma_{\mathrm{ess}}(A)$, $\sigma_{\mathrm{discrete}}(A)$].

We consider the following maps: $((\alpha, x) \in \mathbb{T}^2)$

$$\phi(\alpha, x) = (\alpha, x-\alpha) \in \mathbb{T}^2,$$

$$\chi_0(\alpha, x) = \chi_{[1-\alpha, 1[}(x) \in \{0, 1\}.$$

These two maps are obviously related to model (1): Denoting by $\chi_n = \chi_0 \circ \phi^{-n}$, $n \in \mathbb{Z}$, the translates of χ_0 through ϕ^n, we have

$$\chi_n(\alpha, x) = v_{\alpha, x}(n).$$

Thus the map $x \to \chi_n(\alpha, x)$ is right-continuous.

Let $\mathscr{B}(\mathbb{T}^2)$ denote the C^*-algebra of all complex valued bounded functions on $\mathbb{T}^2$ with the norm given by the supremum and $\mathscr{B}$ the C^*-subalgebra generated by the functions χ_n.

Let $(t_m)_{m \in \mathbb{Z}}$ be a family in $\mathscr{B}$. We define formally the hamiltonian $H(\alpha, x)$ by

$$H(\alpha, x) = \sum_{m \in \mathbb{Z}} t_{m,\alpha,x} T^m, \tag{4}$$

where T is the shift on $l^2(\mathbb{Z})$, $t_{m,\alpha,x}$ is the multiplication by $t_m \circ \phi^{-n}(\alpha, x)$ and we assume that the $t_{m,\alpha,x}$'s are such that the sum converges in norm and defines a bounded selfadjoint operator.

Definition 1. Let $H(\alpha, x)$ be as in (4). The total spectrum at α of H is

$$\sigma_\alpha = \overline{\bigcup_x \sigma(H(\alpha, x))}.$$

In this case, the total spectrum at each α coincides with the spectrum:

Proposition 1. *Let $\alpha \in \mathbb{T}$ and let $H(\alpha, x)$ be as in (4). Then the spectrum of $H(\alpha, x)$ is independent of x and coincides with the total spectrum.*

Proof. With the notations of (4),

$$\chi_{0,\alpha,x+k\alpha} = T^k \chi_{0,\alpha,x} T^{*k}, \qquad x \in [0, 1[, \ k \in \mathbb{Z}.$$

Assume first $\alpha = \frac{p}{q}$. $H(\alpha, x)$ and $H(\alpha, 0)$ are unitarily equivalent for any x: It is easily checked that $v_{p/q,\varepsilon}(n) = v_{p/q,0}(n)$ for all $n \in \mathbb{Z}$ if $\varepsilon \in \left[0, \frac{1}{q}\right[$. Given x, there exists $k \in \mathbb{Z}$

satisfying $x+k\frac{p}{q}\in\left[0,\frac{1}{q}\right[$. It follows that $\chi_{0,p/q,x}=T^k\chi_{0,p/q,0}T^{*k}$, and by extension to $\mathscr{B}$, the claim is proved.

Assume now that $\alpha\notin\mathbb{Q}$. Let x, y be in $[0,1[$. Then, there exists a sequence of integers n_k such that $0\leqq x+n_k\alpha-y\to 0$ when $k\to\infty$. The map $x\to t_m(\alpha,x)$ being right-continuous, $t_{m,\alpha,y}$ is the strong limit of $t_{m,\alpha,x+n_k\alpha}=T^k t_{m,\alpha,x}T^{*k}$. Thus $\sigma(H(\alpha,y))$ is included in $\sigma(H(\alpha,x))$ ([R.S.] p. 290).

A reasonable definition of the continuity property of the spectrum as a function of α is that the gap boundaries are continuous functions of α:

Definition 2. Let $\{\Sigma_\beta\}_\beta$ be a family of subsets of $\mathbb{R}$ indexed by $\beta\in\,]0,1]$. This family is said to be outer-continuous (respectively left outer-continuous, right outer-continuous) at the point $\alpha\in]0,1]$ if for any closed interval F in $\mathbb{R}$ such that $\Sigma_\alpha\cap F=\emptyset$, there exists $\varepsilon>0$ such that $\Sigma_\beta\cap F=\emptyset$ if $\beta\in]\alpha-\varepsilon,\alpha+\varepsilon[$ (respectively $\beta\in[\alpha-\varepsilon,\alpha[$, $\beta\in[\alpha,\alpha+\varepsilon[$).

Similarly it is said to be inner-continuous (respectively left inner-continuous, right inner-continuous) at the point $\alpha\in]0,1]$ if, for any open interval O in $\mathbb{R}$ such that $\Sigma_\alpha\cap O\neq\emptyset$, there exists $\varepsilon>0$ such that $\Sigma_\beta\cap O\neq\emptyset$ whenever $\beta\in]\alpha-\varepsilon,\alpha+\varepsilon[$ (respectively $\beta\in]\alpha-\varepsilon,\alpha]$, $\beta\in[\alpha,\alpha+\varepsilon[$).

When the family is outer-continuous and inner-continuous (respectively left outer-continuous and left inner-continuous, respectively right outer-continuous and right inner-continuous), we simply say it is continuous (respectively left continuous, respectively right continuous) and we write:

$$\Sigma_\alpha=\lim_{\beta\to\alpha}\Sigma_\beta$$

$$\left(\text{respectively } \Sigma_\alpha^-=\lim_{\alpha>\beta\to\alpha}\Sigma_\beta,\ \text{respectively } \Sigma_\alpha^+=\lim_{\alpha<\beta\to\alpha}\Sigma_\beta\right).$$

The main result is:

Theorem 1. *Let $H(\alpha,x)$ be as in* (4). *The map $\alpha\in[0,1[\to\sigma_\alpha$ is continuous at each irrational number.*

Theorem 2. *Let $H(\alpha,x)$ be of type* (4). *The sets $\sigma_{p/q}^+$ and $\sigma_{p/q}^-$ exist at each rational $\alpha=\frac{p}{q}$.*

The proof of Theorem 1 uses the following Theorem 3. Let us introduce first some notation:

Let Ω be Hausdorff compact metrizable spaces, ϕ be a homeomorphism of Ω and p be a continuous surjective map from Ω onto a compact space Γ such that $p\circ\phi=p$. Denote by $\mathscr{A}$ the C^*-crossed product $C(\Omega)\times_\phi\mathbb{Z}$ of the complex valued continuous functions on Ω by the action of $\mathbb{Z}$ through ϕ. The map ϕ is implemented by a unitary U in $\mathscr{A}$. For ω in Ω, we define the representation Π_ω of $\mathscr{A}$ by $\Pi_\omega(f)\psi(n)=f(\phi^{-n}\omega)\psi(n)$ and $\Pi_\omega(U)\psi(n)=\psi(n-1)$ when $\psi\in l^2(\mathbb{Z})$. For $\gamma\in\Gamma$, let J_γ denote the norm-closed ideal in $\mathscr{A}$.

$$J_\gamma=\{a\in\mathscr{A}\mid\Pi_\omega(a)=0,\ \omega\in p^{-1}(\gamma)\},$$

and η_γ the canonical map from $\mathscr{A}$ onto the quotient C^*-algebra $\mathscr{A}_\gamma=\mathscr{A}/J_\gamma$.

Definition 3. We say that $\omega \in \Omega$ is p-isolated whenever there is an open set U in Ω containing ω, and a sequence γ_n in Γ converging to $\gamma = p(\omega)$ such that U intersects none of the $p^{-1}\{\gamma_n\}$.

Note that the set of p-isolated points is open in $p^{-1}\{\gamma\}$.

Theorem 3. *Let (Ω, Γ, p) be as before and $h = h^*$ be in $C(\Omega) \times_\phi \mathbb{Z}$.*
(i) *The spectrum $\sigma(\eta_\gamma(h))$ is outer-continuous at every point γ in Γ.*
(ii) *Let γ in Γ be such that the fiber $p^{-1}\{\gamma\}$ contains no p-isolated points. Then the spectrum $\sigma(\eta_\gamma(h))$ is inner-continuous at γ.*

This theorem is very close to Theorem 3.1 of [T.] and Theorem 4 of [Le.]. However, we do not require that the decomposition of the structure space of $C(\Omega) \times_\phi \mathbb{Z}$ by means of η_γ be Hausdorff (see Lemma 9 and Remarks 1). This is why p-isolated points may create discontinuities in the spectrum.

As in the introduction, let $\mathscr{B}$ denote the abelian C^*-algebra generated by the χ_n's. Let Ω denote the spectrum of $\mathscr{B}$, so that $\mathscr{B}$ is identified with $C(\Omega)$, and consider the homeomorphism ϕ of Ω corresponding to the translation $(\alpha, x) \in \mathbb{T}^2 \to (\alpha, x - \alpha)$ through $\mathscr{B}$. Consider the crossed product $\mathscr{A}$ of $\mathscr{B}$ by $\mathbb{Z}$ via the action ϕ.

Proposition 2. (i) *The set $\Lambda = \{(\alpha, x) \in \mathbb{T}^2 \mid \alpha \notin \mathbb{Q}$ and $x \notin \mathbb{Z}\alpha\}$ can be canonically identified (via evaluation) with a dense subset of Ω.*
(ii) *The points $\left(\frac{p}{q}, \frac{r}{q}\right)$, $p, r \in \{0, \dots, q-1\}$, are also in Ω.*

Proposition 3. *Let $H(\alpha, x)$ be of type (4) and h be the element of $C(\Omega) \times_\phi \mathbb{Z}$ defined by $h = \sum_{m \in \mathbb{Z}} t_m U^m$. If $\alpha \notin \mathbb{Q}$, then:*
(i) *$H(\alpha, x) = \Pi_{(\alpha, x)}(h)$ for any $x \notin \mathbb{Z}\alpha$.*
(ii) *The total spectrum σ_α of H coincides with the spectrum of $\eta_\alpha(h)$ in $\mathscr{A}_\alpha$.*
(iii) *$\Pi_{(p/q, r/q)}(h) = H\left(\frac{p}{q}, \frac{r}{q}\right)$, $p, r \in \{0, \dots, q-1\}$.*

Proposition 2 indicates that the "irrational points" of the square are in Ω. More generally, every point in $\mathbb{T}^2$ gives rise to at least one character. But for some points, one can get more than one and Ω appears as a non-locally trivial fiber bundle on $\mathbb{T}^2$. This desingularization of $\mathbb{T}^2$ is at the origin of the continuity and discontinuity properties of the map $\alpha \to \sigma_\alpha$. For instance, when $\alpha \notin \mathbb{Q}$, $x = m\alpha \in \mathbb{Z}\alpha$, there are two characters corresponding to the point (α, x) in $\mathbb{T}^2$, representing the right and left limits as x converges to $m\alpha$. More complicated is the situation where α is a rational number $\frac{p}{q}$ and $x = \frac{r}{q}$ for $r \in \{0, \dots, q-1\}$. Here, three possibilities for α coexist, $\frac{p}{q}, \frac{p}{q} - 0$ and $\frac{p}{q} + 0$, where ± 0 refers to the right and left limits. The first case gives rise to the usual periodic hamiltonian, the spectrum of which contains q bands. The two other cases correspond to periodic operators with an impurity producing eigenvalues in the gaps. This explains, first, the discontinuity of the spectrum at each rational α and, second, the shape of the spectrum in [O.K.]. More precisely, given $\left(\frac{p}{q}, \frac{r}{q}\right)$, we define two elements of Ω by $\omega_{j,\pm} = \left(\frac{p}{q}, \frac{r}{q}, j, \pm\right)$, where $j \in \mathbb{Z}$ refers to the wedge bounded by two lines with integral slope passing through $\left(\frac{p}{q}, \frac{r}{q}\right)$. This character is the limit of points converging to $\left(\frac{p}{q}, \frac{r}{q}\right)$ within the wedge, respectively from the right $(+)$ and from the left $(-)$.

Theorem 4. *Let $h=h^* \in \mathscr{A}=C(\Omega)\times_\phi \mathbb{Z}$.*
(i) *$\Pi_{\omega_{j,\pm}}(h)$ converges strongly to the q-periodic operator $\Pi_{(p/q, r/q)}(h)$ as j tends to $\pm\infty$.*
(ii) *$\sigma(\Pi_{\omega_{j,\pm}}(h)) \supset \sigma(\Pi_{(p/q,r/q)}(h))$ and*

$$\sigma_{\text{discrete}}(\Pi_{\omega_{j,\pm}}(h)) = \sigma(\Pi_{\omega_{j,\pm}}(h)) \setminus \sigma(\Pi_{(p/q,r/q)}(h)).$$

Corollary 1. *Let $H(\alpha, x)$ be of type* (4) *and $h \in \mathscr{A}$ be the associated operator. Then $\Pi_{\omega_{j,\pm}}(h)$ is equal to $S_\pm^* H\left(\frac{p}{q}, \frac{r}{q}\right) S_\pm$ modulo a finite rank operator, where $S_\pm$ is a partial isometry associated to the impurity domain of $\Pi_{\omega_{j,\pm}}(h)$. Moreover,*

$$\sigma_{p/q}^{\pm} = \sigma(\Pi_{(p/q,0,0,\pm)}(h))$$

and

$$\sigma_{\text{ess}}(\Pi_{\omega_{j,\pm}}(h)) = \sigma_{\text{ac}}(\Pi_{\omega_{j,\pm}}(h)) = \sigma\left(H\left(\frac{p}{q}, \frac{r}{q}\right)\right).$$

Since the size of the impurity domains for $\Pi_{(p/q,0,0,+)}(h)$ and $\Pi_{(p/q,0,0,-)}(h)$ are different, the spectra $\sigma_{p/q}^{+}$ and $\sigma_{p/q}^{-}$ generally differ from each other. More precisely:

Theorem 7. *Let $H(\alpha, x)$ be of type* (1) *and $h \in \mathscr{A}$ be the associated operator. Then:*

$$\sigma_{\text{discrete}}(\Pi_{(p/q,0,0,-)}(h)) \neq \sigma_{\text{discrete}}(\Pi_{(p/q,0,0,+)}(h)).$$

3. The Abstract Continuity Theorem

Let $C(\Omega)$ be the C^*-algebra of continuous functions on a compact metrizable space Ω. Let ϕ be a homeomorphism of Ω and $\mathscr{A}=C(\Omega)\times_\phi \mathbb{Z}$ be the C^*-algebra defined as the crossed product of $C(\Omega)$ by the group $\mathbb{Z}$ acting on $C(\Omega)$ by ϕ. This action is implemented by a unitary U in $\mathscr{A}$. We consider the dense subalgebra $\mathscr{A}_0$ whose elements are of the form

$$a = \sum_{n=-N}^{N} a_n U^n,$$

where $a_n \in C(\Omega)$.

To each $\omega \in \Omega$ corresponds a representation Π_ω of $\mathscr{A}$ on $l^2(\mathbb{Z})$ defined on the generators by

$$\begin{cases} \Pi_\omega(f) \text{ is the multiplication by } \Pi_\omega(f)(n) = f(\phi^{-n}\omega) \\ \Pi_\omega(U) = T. \end{cases}$$

By definition of the crossed product [P.], the map

$$a \in \mathscr{A} \to \oplus_{\omega\in\Omega} \Pi_\omega(a)$$

is an isometry and $\omega \in \Omega \to \Pi_\omega(a)$ is strongly continuous for $a \in \mathscr{A}$.

It is well known that the torus $\mathbb{T}$ (the dual group of $\mathbb{Z}$) acts by automorphisms ϱ_t on $\mathscr{A}$ (the "dual action" on the crossed product): $\forall t \in \mathbb{T}$, $\forall f \in C(\Omega)$

$$\varrho_t(f) = f, \quad \varrho_t(U) = \exp(i2\pi t)U.$$

This gives ($a \in \mathscr{A}$)

$$\Pi_\omega(\varrho_t(a)) = V_t \Pi_\omega(a) V_t^*, \tag{5}$$

where $(V_t\psi)(n) = \exp(i2\pi nt)\psi(n)$, $\psi \in l^2(\mathbb{Z})$.

Given a in $\mathscr{A}$, we now want to construct an explicit sequence $(a_N)_N$ of elements in $\mathscr{A}_0$, converging in norm to a. Let g be in $L^1(\mathbb{T})$ and let

$$\varrho_g(a) = \int_{t\in\mathbb{T}} g(t)\varrho_t(a)dt$$

(Bochner integral). ϱ_g is a continuous linear operator on $\mathscr{A}$ with norm less than the L^1-norm of g. Taking a sequence $\{g_N\}_N$ such that $g_N \geqq 0$, $\|g_N\|_{L^1} = 1$ and, for any $\varepsilon > 0$,

$$\lim_N \left(\int_{|t|>\varepsilon} g_N(t)dt \right) = 0,$$

we have that $a_N = \varrho_{g_N}(a)$ converges in norm to a. If, moreover, the g_N's are Fourier transforms of functions with compact support, then $\varrho_{g_N}(a)$ belongs to $\mathscr{A}_0$.

The Space of Orbits. Let us assume that there exists a continuous surjective map p from Ω onto a compact space Γ such that $p \circ \phi = p$.

We introduce for any $\gamma \in \Gamma$, $a \in \mathscr{A}$, the seminorm

$$\|a\|_\gamma = \sup_{\omega \in p^{-1}(\gamma)} \|\Pi_\omega(a)\|.$$

Clearly, $\|a\|_\gamma \leqq \|a\|$. The set $J_\gamma = \{a \in \mathscr{A} \mid \|a\|_\gamma = 0\}$ is a closed two sided ideal in $\mathscr{A}$. We define the quotient C^*-algebra $\mathscr{A}_\gamma = \mathscr{A}/J_\gamma$ with the canonical surjective morphism η_γ from $\mathscr{A}$ onto $\mathscr{A}_\gamma$. Using (5) we get $\varrho_t(J_\gamma) = J_\gamma$, $\forall \gamma \in \Gamma$, so by extension $\varrho_g(J_\gamma) = J_\gamma$, $\forall g \in L^1(\mathbb{T})$. This implies:

Lemma 1. *$J_\gamma \cap \mathscr{A}_0$ is dense in J_γ.*

The following lemma is a generalization of a result of Elliott [E.].

Lemma 2. *Let $a \in J_\gamma$. Then* $\lim_{\mu\to\gamma} \|\eta_\mu(a)\| = 0$.

Proof. We may assume that $a \in J_\gamma \cap \mathscr{A}_0$ because $J_\gamma \cap \mathscr{A}_0$ is dense in J_γ and $\|a\|_\gamma \leqq \|a\|$. So $a = \sum_{n=-N}^{n=N} a_n U^n$, where $a_n \in C(\Omega)$. Using

$$a_n = \int_{t\in\mathbb{T}} \varrho_t(U^{-n}a)dt$$

it follows that $a_n \in J_\gamma \cap C(\Omega)$.

Since $\|\Pi_\omega(a)\| \leqq \sum_{n=-N}^{n=N} \|\Pi_\omega(a_n)\|$, we may suppose that $a \in J_\gamma \cap C(\Omega)$. So $\Pi_\omega(a)$ is a diagonal operator in the canonical basis of $l^2(\mathbb{Z})$ and $a(\omega) = 0$, $\forall \omega \in p^{-1}(\gamma)$.

Let us assume that $\sup_{\omega \in p^{-1}(\mu)} |a(\omega)|$ does not tend to zero when μ tends to γ. There exist $c > 0$ and a sequence $\{\omega_k\}_k$ in Ω such that $p(\omega_k) \to \gamma$ and $|a(\omega_k)| > c$. Ω being metrizable and compact, there is a convergent subsequence, also denoted by ω_k, with the same properties. Let ω denote its limit in Ω. Thus $|a(\omega)| \geqq c$ and $p(\omega) = \gamma$, a contradiction. Moreover, for any $\omega \in p^{-1}(\mu)$, $a \in C(\Omega)$,

$$\|\Pi_\omega(a)\| \leqq \sup_{\omega' \in p^{-1}(\mu)} |a(\omega')|.$$

This gives the result.

Proof of Theorem 3. Let F be a closed interval in $\mathbb{R}$ such that $\sigma(\eta_\gamma(h)\cap F=\emptyset$. By Urysohn's lemma there exists a continuous function g with $0\leqq g\leqq 1$, equal to one on F and zero on $\sigma(\eta_\gamma(h))$. Thus, $g(h)\in\mathscr{A}$ and $\eta_\gamma(g(h))=0$. By Lemma 2, there is $\varepsilon>0$ such that if $|\beta-\gamma|<\varepsilon$ then $\|g(\eta_\beta(h))\|=\|\eta_\beta(g(h))\|<1/2$. Assuming $\sigma(\eta_\beta(h))\cap F\neq\emptyset$ for such β, we get a contradiction since g equals one on F. This proves the outer-continuity.

We claim that for $\gamma\in\Gamma$,

$$\sigma(\eta_\gamma(h))=\overline{\bigcup_{\omega\in p^{-1}(\gamma)}\sigma(\Pi_\omega(h))}\,. \tag{6}$$

In fact, for a in $\mathscr{A}$, $\eta_\gamma(a)=0$ if and only if $\Pi_\omega(a)=0$ for all $\omega\in p^{-1}(\gamma)$. [Recall that a real E is not in the spectrum of a selfadjoint bounded operator A if and only if there exists a continuous function g on $\mathbb{R}$ satisfying $0\leqq g\leqq 1$, $g(E)=1$ and $g(A)=0$.]

Now let γ be as in (ii) and O be an open interval in $\mathbb{R}$ such that $O\cap\sigma(\eta_\gamma(h))\neq\emptyset$. By (6) there exists $\omega\in p^{-1}(\gamma)$ such that $O\cap\sigma(\Pi_\omega(h))\neq\emptyset$. Since $p^{-1}(\gamma)$ contains no p-isolated points, for any open set $\mathscr{V}$ containing ω, and any sequence γ_n converging to γ, the fibers $p^{-1}(\gamma_n)$ (or a subsequence) meet $\mathscr{V}$ for large n. Suppose that there is a sequence γ_n converging to γ such that $O\cap\sigma(\eta_{\gamma_n}(h))=\emptyset$. Then, there exists a sequence ω_n converging to ω with $O\cap\sigma(\Pi_{\omega_n}(h))=\emptyset$ for all n. Since the map $\omega\in\Omega\to\Pi_\omega(a)$ is strongly continuous, it follows that $O\cap\sigma(\Pi_\omega(h))=\emptyset$ [R.S., p. 290] and we get a contradiction.

4. Proof of Theorem 1

In this section we describe partially the spectrum (also called the character space) of the C^*-algebra generated by $H(\alpha,x)$ and its translates, using a geometrical partition of $\mathbb{T}^2$. Actually, the knowledge of a dense subset is sufficient for proving Theorem 1.

Let $H(\alpha,x)$ be of type (4).

Lemma 3. *If* $[\;]$ *denotes the integer part, then for all* n *in* $\mathbb{Z}$,

$$\chi_n(\alpha,x)=[x+(n+1)\alpha]-[x+n\alpha]\,.$$

Proof. We have

$$\begin{aligned}\chi_n(\alpha,x)=1&\Leftrightarrow 1-\alpha\leqq(x+n\alpha)-[x+n\alpha]<1\,.\\&\Leftrightarrow\exists m\in\mathbb{Z}\text{ such that } 1-\alpha\leqq(x+n\alpha)-m<1\,.\\&\Leftrightarrow\exists m\in\mathbb{Z}\text{ such that } (x+n\alpha)<m+1\leqq x+(n+1)\alpha\,.\end{aligned}$$

Moreover, $[x+(n+1)\alpha]-[x+n\alpha]\in\{0,1\}$. Actually,

$$\begin{aligned}0\leqq[x+(n+1)\alpha]-[x+n\alpha]&\leqq x+n\alpha+\alpha-[x+n\alpha]\\&=\{x+n\alpha\}+\alpha<2\,,\end{aligned}$$

where $\{\;\}$ is the fractional part.

On the other hand,

$$\begin{aligned}&[x+(n+1)\alpha]-[x+n\alpha]=1\\&\quad\Leftrightarrow\exists k\in\mathbb{Z}\text{ such that } x+n\alpha<k\leqq x+(n+1)\alpha\,.\end{aligned}$$

Now let us consider the geometry on $\mathbb{T}^2$ determined by the generators of $\mathscr{B}$.

Definition 4. A band in $\mathbb{T}^2$ is a set of the form

$$T_{n,k}=A_{n,k}/A_{n,k+1},$$

where $A_{n,k}$ is a half-plane in $\mathbb{T}^2$,

$$A_{n,k}=\{(\alpha,x)\in\mathbb{R}^2 \mid -n\alpha+k\leqq x\}\cap\mathbb{T}^2,$$

and $(n,k)\in\mathbb{Z}^2$.

The interest of this definition stems from the following

Lemma 4. *Let $\sigma_{n,k}$ (respectively $\varrho_{n,k}$) denote the characteristic function of $T_{n,k}$ (respectively $A_{n,k}$). Then $\mathscr{B}$ is generated by the $\sigma_{n,k}$'s or by the $\varrho_{n,k}$'s.*

Proof. $\sigma_{n,k}$ (respectively $\varrho_{n,k}$) can be expressed as a finite linear combination of $\varrho_{n,k}$ (respectively $\sigma_{n,k}$) and it is sufficient to prove the statement concerning the $\varrho_{n,k}$'s. A typical generator of $\mathscr{B}$ is of the form (Lemma 3)

$$\chi_n(\alpha,x)=[x+(n+1)\alpha]-[x+n\alpha].$$

Notice that this is equivalent to

$$[x+n\alpha]=\sum_{k=0}^{n-1}\chi_k(\alpha,x).$$

So $\mathscr{B}$ is generated by the functions $f_n:(\alpha,x)\to[x+ n\alpha]$, $n\in\mathbb{Z}$. For $n\geqq 0$, such functions are valued in $\{0,\dots,n\}$. Let $P_{n,k}$ be a polynomial of degree $n+1$ such that for $m\in\{0,\dots,n\}$,

$$P_{n,k}(m)=1 \quad \text{if} \quad k\geqq m,$$

$$P_{n,k}(m)=0 \quad \text{if} \quad k<m.$$

Thus, $P_{n,k}([x+n\alpha])\in\{0,1\}$ and

$$P_{n,k}([x+n\alpha])=1\Leftrightarrow k\leqq[x+n\alpha]\Leftrightarrow -n\alpha+k\leqq x.$$

It follows that $P_{n,k}(f_n)=\varrho_{n,k}$ is in $\mathscr{B}$.

The case $n<0$ is similar.

It is immediate to check that

$$[x+n\alpha]=\sum_{k=-\infty}^{+\infty}k\sigma_{n,k}(\alpha,x),$$

where only a finite number of terms are not zero in the sum. This proves the assertion.

Let $\mathscr{T}$ be the smallest set of subsets of $\mathbb{T}^2$ containing all the bands $T_{n,k}$, which is stable under taking finite intersections, finite unions and complements.

The $T_{n,k}$'s give a partition of $\mathbb{T}^2$ by lines

$$D_n=\{(\alpha,n\alpha)\in\mathbb{T}^2 \mid \alpha\in[0,1[\}, \qquad n\in\mathbb{Z}.$$

Here the lines are taken modulo 1. If $\mathscr{D}$ is the set of such lines, we remark first that

$$\left(\frac{p}{q},x\right)\in\mathscr{D}\Leftrightarrow x=\frac{r}{q} \quad \text{for some} \quad r\in\{0,\dots,q-1\}.$$

[We use the convention that $\frac{p}{q}$ always defines an irreducible fraction so that there is a unique couple (p', q') such that $1 \leqq q' \leqq q-1$, $p' \in \mathbb{Z}$ and $pq' = 1 + qp'$. Actually, given p, q, r we get $\left(\frac{p}{q}, \frac{r}{q}\right) \in D_n$ with $n = rq'$, where q' is the inverse of p modulo q; the converse is immediate.]

There are many lines passing through the point $\left(\frac{p}{q}, \frac{r}{q}\right)$, namely,

$$\left(\frac{p}{q}, \frac{r}{q}\right) \in \bigcap_{j \in \mathbb{Z}} D_{rq' + jq},$$

and conversely, all lines passing through that point are of the form $D_{rq'+jq}$.

Finally we note that if $(\alpha, x) \in \mathscr{D}$ for $\alpha \notin \mathbb{Q}$, then $(\alpha, x) \in D_n$ for a unique $n \in \mathbb{Z}$. In particular, the partition of $\mathbb{T}^2$ defined by $\mathscr{D}$ is given by polygons whose vertices are of the form $\left(\frac{p}{q}, \frac{r}{q}\right)$. These polygons are the atoms of the collection $\mathscr{T}$. In particular, every element of $\mathscr{T}$ has a nonempty interior. Note that χ_T is a projection of $\mathscr{B}$ for any $T \in \mathscr{T}$.

In the following figure some lines passing through $(\frac{2}{3}, \frac{1}{3})$ are drawn.

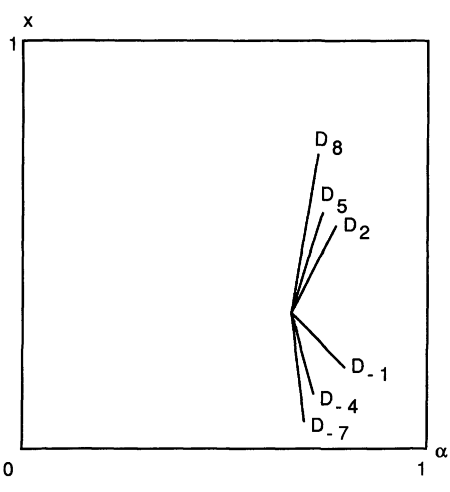

Fig. 2

It will be useful to define the set

$$I_{p/q, r} = \left\{\frac{p}{q}\right\} \times \left[\frac{r}{q}, \frac{r+1}{q}\right[.$$

Notice that for $T \in \mathscr{T}$, either T contains $I_{p/q,r}$ or $T \cap I_{p/q,r} = \emptyset$.
Recall that Ω is defined as the set of characters of $\mathscr{B}$. Then we get:

Lemma 5. *Given $\varepsilon > 0$ and $\omega \in \Omega$, there exists a finite partition of $\mathbb{T}^2$ by bands $(T_i)_{i \in I}$ in $\mathscr{T}$ satisfying:*
– *The diameter of the first projection of T_i is less than ε for all i (such bands will be called ε-bands).*
– *There is a unique j in I such that $\omega(\chi_{T_j}) = 1$.*

Proof. Let us take $n > \frac{2}{\varepsilon}$. $\{T_{-n,k} \mid -n \leqq k \leqq 0\}$ is a partition of $\mathbb{T}^2$ by ε-bands. Since

$$\sum_{k=-n}^{k=0} \sigma_{-n,k} = \mathbb{1},$$

there is k_0 such that $\omega(\sigma_{-n,k_0}) \neq 0$, σ_{-n,k_0} being a projection in $\mathscr{B}$, it follows that $\omega(\sigma_{-n,k_0}) = 1$ and therefore $\omega(\sigma_{-n,k}) = 0$ if $k \neq k_0$.

Let $\mathscr{B}_0$ be the algebra made of finite linear combinations of finite products of χ_n. Then $\mathscr{B}(\mathbb{T}^2) \supset \mathscr{B}_0$ and

Lemma 6. *If* $\Lambda = \{(\alpha, x) \in \mathbb{T}^2 \mid \alpha \notin \mathbb{Q}$ *and* $x \notin \mathbb{Z}\alpha\}$, *then for any* f *in* $\mathscr{B}$,

$$\|f\| = \sup_{(\alpha,x)\in\Lambda} |f(\alpha,x)|.$$

Proof. The equality is satisfied when f is in $\mathscr{B}_0$. Let now f in $\mathscr{B}$ and a sequence $\{f_n\}_n$ in $\mathscr{B}_0$ be such that $f_n \to f$. For $(\alpha, x) \in \mathbb{T}^2$,

$$|f(\alpha,x)| \leqq |f(\alpha,x) - f_n(\alpha,x)| + |f_n(\alpha,x)|.$$

Thus,

$$\|f\| = \sup_{(\alpha,x)\in\mathbb{T}^2} |f(\alpha,x)| \leqq \|f - f_n\| + \sup_{(\alpha,x)\in\mathbb{T}^2} |f_n(\alpha,x)|.$$

Since

$$\begin{aligned} \sup_{(\alpha,x)\in\mathbb{T}^2} |f_n(\alpha,x)| &= \sup_{(\alpha,x)\in\Lambda} |f_n(\alpha,x)| \\ &\leqq \sup_{(\alpha,x)\in\Lambda} |(f_n - f)(\alpha,x)| \\ &\quad + \sup_{(\alpha,x)\in\Lambda} |f(\alpha,x)|, \end{aligned}$$

it follows that

$$\|f\| \leqq 2\|f - f_n\| + \sup_{(\alpha,x)\in\Lambda} |f(\alpha,x)|,$$

and the lemma is proved.

$\mathscr{B}$ is an abelian C^*-algebra with unit. Thus, $\mathscr{B}$ is identified with $C(\Omega(\mathscr{B}))$, where $\Omega(\mathscr{B})$ is a compact metrizable space. By the Gelfand transform, the set Λ is embedded in $\Omega(\mathscr{B})$.

Proof of Proposition 2. (i) Let ω be in $\Omega(\mathscr{B})$ but not in the closure of Λ. By Urysohn's lemma, there exists a non zero f in $\mathscr{B}$ with values 1 on ω and 0 on Λ. This contradicts Lemma 6.

(ii) follows from the next lemma.

For $\omega \in \Omega(\mathscr{B})$, we define

$$\mathscr{F}_\omega = \{T \in \mathscr{T} \mid \omega(\chi_T) = 1\}.$$

Since ω is a homomorphism, $\mathscr{F}_\omega$ is a filter on $\mathscr{T}$ and even an ultrafilter because T or its complement T^c is in $\mathscr{F}_\omega$ for any T in $\mathscr{T}$. Let

$$S(\omega) = \cap\{\bar{T} \mid T \in \mathscr{F}_\omega\},$$

where $\bar{T}$ is the closure of T in $\mathbb{T}^2$ for the usual topology. $S(\omega)$ is not empty because the family $\{\bar{T} \mid T \in \mathscr{F}_\omega\}$ has the finite intersection property and the $\bar{T}$'s are closed subsets of the compact set $\mathbb{T}^2$.

Lemma 7. *S is a map from $\Omega(\mathscr{B})$ into the subsets of $\mathbb{T}^2$ with the following properties.*
(i) *Let $\omega \in \Omega(\mathscr{B})$. If $S(\omega)$ contains more than one point then*

$$S(\omega)=I_{p/q,r} \quad \textit{for some} \quad r\in\{0,\dots,q-1\}.$$

(ii) *S coincides on Λ with the inverse of the Gelfand transform.*
(iii) *$S(\Omega(\mathscr{B}))$ is the union of the following sets:*

$$\bigcup_{x\in\mathbb{T}} \{(\alpha,x)\mid \alpha\notin\mathbb{Q}\},$$

$$\bigcup_{p\in\mathbb{N},q\in\mathbb{N}^*,r\in\{0,\dots,q-1\}} \left\{\left(\frac{p}{q},\frac{r}{q}\right)\right\},$$

$$\bigcup_{p\in\mathbb{N},q\in\mathbb{N}^*,r\in\{0,\dots,q-1\}} \overline{I_{p/q,r}}.$$

Proof. (i) Let (α,x), (α',x') be two different points of $S(\omega)$.
(a) If $\alpha\neq\alpha'$, there exists a line in $\mathscr{D}$ which separates these two points. Thus there is $T\in\mathscr{T}$ such that $(\alpha,x)\in T$ and $(\alpha',x')\in\overline{T}^c$. If $T\in\mathscr{F}_\omega$ then $(\alpha',x')\notin\overline{T}$, and thus $(\alpha',x')\notin S(\omega)$. If $T\notin\mathscr{F}_\omega$, then T^c is in $\mathscr{F}_\omega$ and $(\alpha,x)\notin\overline{T^c}$, so we obtain again a contradiction.
(b) If $\alpha=\alpha'\notin\mathbb{Q}$, the same situation occurs. The same is true when $\alpha=\alpha'=\frac{p}{q}$ and x,x' are not in the same segment $I_{p/q,r}$.
(c) The last case is $\alpha=\alpha'=\frac{p}{q}$ and $x,x'\in I_{p/q,r}$. For each T in $\mathscr{F}_\omega$ (α,x) and (α,x') are in $\overline{T}$, so by construction $\overline{T}$ contains $\overline{I_{p/q,r}}$. Since there are no lines between $\left(\frac{p}{q},\frac{r}{q}\right)$ and $\left(\frac{p}{q},\frac{r+1}{q}\right)$, so also does $S(\omega)$. By (a), $S(\omega)$ contains $\left\{\frac{p}{q}\right\}\times[0,1[$. By (b), $S(\omega)=\overline{I_{p/q,r}}$.

Clearly if $\alpha\notin\mathbb{Q}$, $S(\omega)$ is a point.
(ii) Let $\omega=(\alpha,a)\in\Lambda$. For each T in $\mathscr{F}_\omega$, (α,x) is in the interior of T, so $(\alpha,x)\in S(\omega)$. Now apply (i).
(iii)
$$S(\Omega(\mathscr{B}))\supset \bigcup_{x\in\mathbb{T}}\{(\alpha,x)\}:$$

Let $\alpha\notin\mathbb{Q}$. The map

$$\omega:\chi_T\in\mathscr{B}_0\to \lim_{0<\varepsilon\to 0}\chi_T(\alpha,k\alpha+\varepsilon),$$

where $k\in\mathbb{Z}$ defines a character on the algebra $\mathscr{B}_0$. Its (unique) extension to the closure $\mathscr{B}$ is in Ω. It is easily checked that if $T\in\mathscr{F}_\omega$ then $(\alpha,k\alpha)\in T$ and thus $S(\omega)=(\alpha,k\alpha)$ by (i).

$$S(\Omega(\mathscr{B}))\supset \bigcup_{p\in\mathbb{N},q\in\mathbb{N}^*,r\in\{0,\dots,q-1\}} \left\{\left(\frac{p}{q},\frac{r}{q}\right)\right\}:$$

Given p, q, r, consider the extension to $\mathscr{B}$ of the character

$$\omega:\chi_T\in\mathscr{B}_0\to \lim_{0<\varepsilon\to 0}\chi_T\left(\frac{p}{q}+\varepsilon,\frac{r}{q}+rq'\varepsilon\right),$$

where q' is the inverse of p modulo q ($pq'=1+p'q$ with $p'\in\mathbb{Z}$). If $T\in\mathscr{F}_\omega$, $\left(\frac{p}{q},\frac{r}{q}\right)\in\overline{T}$ and thus $S(\omega)$ contains $\left(\frac{p}{q},\frac{r}{q}\right)$. The set

$$T=A_{rq',-rp'}\cap(A_{rq'+q,-rp'-p})^c$$

is in $\mathscr{T}$ (Lemma 4) and in fact in $\mathscr{F}_\omega$ by definition of ω. Moreover, $\left(\frac{p}{q}, \frac{r+1}{q}\right)$ is not in $\bar{T}$. On applying (i), $S(\omega)$ is reduced to $\left(\frac{p}{q}, \frac{r}{q}\right)$,

$$S(\Omega(\mathscr{B})) \supset \bigcup_{p \in \mathbb{N}, q \in \mathbb{N}^*, r \in \{0, \ldots, q-1\}} \overline{I_{p/q,r}}:$$

For $\delta \in]0,1[$, we define $\left(\frac{p}{q}, \frac{r+\delta}{q}\right)$ as the extension of the character

$$\omega : \chi_T \in \mathscr{B}_0 \to \chi_T\left(\frac{p}{q}, \frac{r+\delta}{q}\right).$$

If $T \in \mathscr{F}_\omega$, $\left(\frac{p}{q}, \frac{r+\delta}{q}\right) \in T$ and T contains $\left\{\frac{p}{q}\right\} \times \left]\frac{r}{q}, \frac{r+1}{q}\right[$ by construction. Hence $S(\omega) = \overline{I_{p/q,r}}$, on using (i) again.

$\Omega(\mathscr{B})$ is a compact metrizable (since $\mathscr{B}$ is separable) space for the $\sigma(\mathscr{B}^*, \mathscr{B})$-topology. Indeed this topology is equivalent to a natural topology associated to the family $\mathscr{T}$. For $\omega \in \Omega(\mathscr{B})$ and $T \in \mathscr{F}_\omega$ define $V_T = \{\omega' \in \Omega(\mathscr{B}) \mid T \in \mathscr{F}_{\omega'}\}$. The family $\mathscr{V}(\omega) = \{V_T \mid T \in \mathscr{F}_\omega\}$ satisfies the axioms for a fundamental basis of neighborhoods of ω. Actually

$$\bigcap_{i \in \{1, \ldots, n\}} V_{T_i} = V_{\bigcap_{i \in \{1, \ldots, n\}} T_i}$$

when $T_i \in \mathscr{F}_\omega$ and if $\omega' \in V_T \in \mathscr{V}(\omega)$, $V_T \in \mathscr{V}(\omega')$. The $\mathscr{T}$-topology defined this way is Hausdorff for when ω and ω' are different points of $\Omega(\mathscr{B})$, there exists $T \in \mathscr{T}$ satisfying $T \in \mathscr{F}_\omega$ and $T^c \in \mathscr{F}_{\omega'}$. Thus $\omega \in V_T$, $\omega' \in V_{T^c}$ and $V_T \cap V_{T^c} = \emptyset$.

Lemma 8. *The $\mathscr{T}$-topology and the $\sigma(\mathscr{B}^*, \mathscr{B})$-topology are equivalent on $\Omega(\mathscr{B})$.*

Proof. Let $\{\omega_\beta\}_\beta$ be a net in $\Omega(\mathscr{B})$, $\mathscr{T}$-converging to ω. To prove that it $\sigma(\mathscr{B}^*, \mathscr{B})$-converges, it is sufficient to check that $\omega_\beta(\chi_T) \to \omega(\chi_T)$ for any $T \in \mathscr{F}_\omega$. By hypothesis, for $T \in \mathscr{F}_\omega$ there exists β_0 such that if $\beta > \beta_0$, then $\omega_\beta \in V_T$. Hence $T \in \mathscr{F}_{\omega_\beta}$ and $\omega_\beta(\chi_T) = 1 \to 1 = \omega(\chi_T)$.

Conversely, let ω be the $\sigma(\mathscr{B}^*, \mathscr{B})$-limit of a net $\{\omega_\beta\}_\beta$ included in $\Omega(\mathscr{B})$. For $T \in \mathscr{F}_\omega$, $\omega_\beta(\chi_T) \to \omega(\chi_T) = 1$ and there is a β_0 satisfying $T \in \mathscr{F}_{\omega_\beta}$ for $\beta > \beta_0$. So ω is the $\mathscr{T}$-limit of ω_β.

Lemma 9. *The map $\underline{p} = pr_1 \circ S$ is a continuous surjection from $\Omega(\mathscr{B})$ onto $\mathbb{T}$* satisfying

$$\underline{p} \circ \phi = \underline{p}.$$

Proof. $\underline{p}$ is a well defined and surjective map by Lemma 7.

It is sufficient to prove that $\underline{p}(\omega_n)$ converges to $\underline{p}(\omega)$ for any sequence $\{\omega_n\}_n$ in $\Omega(\mathscr{B})$ $\mathscr{T}$-converging to ω. If $T \in \mathscr{F}_\omega$ and $\omega_n(\chi_T) \to \omega(\chi_T) = 1$, then $\omega_n(\chi_T) = 1$ and $S(\omega_n)$ is contained in $\bar{T}$ for n large enough. By Lemma 5, for any ε there exists an ε-band T_ε in $\mathscr{F}_\omega$. Since $S(\omega)$ and $S(\omega_n)$ are in $\overline{T_\varepsilon}$, $|\underline{p}(\omega_n) - \underline{p}(\omega)| < \varepsilon$.

Since $S \circ \phi = S$ is the identity on Λ, we get $\underline{p} \circ \phi = \underline{p}$ by continuity of $\underline{p}$ and density of Λ.

Lemma 10. *Let $\alpha \notin \mathbb{Q}$.*

(i) *If $\omega \in \underline{p}^{-1}(\alpha)$, then $S(\omega) = (\alpha, x)$ for some $x \in [0, 1[$.*
(ii) *$\Lambda_\alpha = \underline{p}^{-1}(\alpha) \cap \Lambda$ is dense in $\underline{p}^{-1}(\alpha)$.*
(iii) *$\underline{p}^{-1}(\alpha)$ has no p-isolated points.*

Proof. (i) Let $\omega \in p^{-1}(\alpha)$ and suppose that $(\beta, x) \in S(\omega)$. Let T_ε be an ε-band in $\mathscr{F}_\omega$ (Lemma 5) and (α_n, x_n) be a sequence in Λ which $\mathscr{T}$-converges to ω. Then $(\beta, x) \in \bar{T}_\varepsilon$ and $|\beta - \alpha_n| < \varepsilon$. The continuity of p gives $\alpha_n = p(\alpha_n, x_n) \to p(\omega) = \alpha$. ε being arbitrary, $\beta = \alpha$. Lemma 7 shows that $S(\omega)$ is reduced to a point.
(ii) It is sufficient to prove that given $\omega \in p^{-1}(\alpha)$ and $T \in \mathscr{F}_\omega$, there exists $\omega' \in \Lambda_\alpha \cap V_T$ (Lemma 8). By (i), $S(\omega) = (\alpha, x)$ for x in $[0, 1[$. Since (α, x) is in $\bar{T}$ but is not a vertex of T and since T has a non-empty interior, $\{(\alpha, y) \mid y \notin \mathbb{Z}\alpha\} \cap T \neq \emptyset$. If we choose (α, y) in this intersection, then $(\alpha, y) \in \Lambda_\alpha \cap V_T$.
(iii) We need only to show that Λ_α has no p-isolated point. Let $(\alpha, x) \in \Lambda_\alpha$ be a p-isolated point. There exist T in $\mathscr{F}_\omega$, a sequence α_n in $\mathbb{T}$ which converges to α in the ordinary topology of $\mathbb{T}$ such that $p^{-1}(\alpha_n) \cap V_T = \emptyset$ for each n. Since (α, x) is not a vertex of T, α is in the interior of $p(T)$ and so are α_n for large n.

Case $\alpha_n \notin \mathbb{Q}$: Clearly $\{(\alpha_n, x) \mid x \notin \mathbb{Z}\alpha_n\} \cap T \neq \emptyset$. Choosing ω_n in this intersection, we get $\omega_n \in p^{-1}(\alpha_n)$ and $\omega_n \in V_T$, in contradiction with the hypothesis.

Case $\alpha_n = \frac{p_n}{q_n}$: Since $\alpha_n \to \alpha$, we may suppose that $\frac{1}{q_n}$ is arbitrarily small. In particular, there exists $r_n \in \{0, \ldots, q_n - 1\}$ such that $\overline{I_{p_n/q_n, r_n}}$ is in the interior of T. Choose ω_n in $\Omega(\mathscr{B})$ such that $S(\omega_n) = \overline{I_{p_n/q_n, r_n}}$ (Lemma 5) and again we obtain the contradiction $\omega_n \in p^{-1}(\alpha_n)$ hence $\omega_n \in V_T$.

Proof of Proposition 3. It is immediate to verify that

$$\begin{cases} \Pi_{(\alpha, x)}(h) = H(\alpha, x), & x \notin \mathbb{Z}\alpha \\ \Pi_{(p/q, r/q)}(h) = H\left(\frac{p}{q}, \frac{r}{q}\right). \end{cases} \tag{7}$$

Moreover, when $\alpha \notin \mathbb{Q}$,

$$\sigma(\eta_\alpha(h)) = \overline{\bigcup_{x \notin \mathbb{Z}\alpha} \sigma(\Pi_{(\alpha, x)}(h))}. \tag{8}$$

Actually, if E is in $\sigma(\eta_\alpha(h))$, $E \in \overline{\bigcup_{\omega \in p^{-1}(\alpha)} \sigma(\Pi_\omega(h))}$ by (6). Hence E is in $\overline{\bigcup_{x \notin \mathbb{Z}\alpha} \sigma(\Pi_{(\alpha, x)}(h))}$ by density of Λ_α in $p^{-1}(\alpha)$ [R.S., p. 290]. Using Proposition 1, (7) and (8) we get $\sigma_\alpha = \sigma(\eta_\alpha(h))$.

Now Theorem 1 is a consequence of Theorem 3, Proposition 3, and Lemma 10.

5. The Spectrum of the C^*-Algebra Associated to the Hamiltonian

In this section we describe completely the spectrum of the C^*-algebra generated by the translates of $H(\alpha, x)$ of type (4), using the geometrical partition of $\mathbb{T}^2$.

We now write the different parts of the spectrum:

$$\Lambda = \{(\alpha, x) \in \mathbb{T}^2 \mid \alpha \notin \mathbb{Q} \text{ and } x \notin \mathbb{Z}\alpha\},$$

$$\Omega^1 = \bigcup_{\alpha \notin \mathbb{Q}, k \in \mathbb{Z}, s \in \{+, -\}} \left\{(\alpha, k\alpha, s) : f \in \mathscr{B} \to \lim_{0 < \varepsilon \to 0} f(\alpha, k\alpha + s\varepsilon)\right\},$$

$$\Omega^2_{p/q, s} = \bigcup_{j \in \mathbb{Z}, r \in \{0, \ldots, q-1\}} \left\{\left(\frac{p}{q}, \frac{r}{q}, j, s\right)\right\},$$

where $s \in \{+, -\}$ and $\left(\frac{p}{q}, \frac{r}{q}, j, s\right)$ is the character

$$f \in \mathscr{B} \to \lim_{0<\varepsilon\to 0} f\left(\frac{p}{q} + s\varepsilon, \frac{r}{q} + (rq' + jq + \delta)s\varepsilon\right), \quad \forall \delta \in [0, q[,$$

$$\Omega^3_{p/q} = \bigcup_{r \in \{0, \ldots, q-1\}} \left\{\left(\frac{p}{q}, \frac{r}{q}\right) : f \in \mathscr{B} \to f\left(\frac{p}{q}, \frac{r+\delta}{q}\right), \ \forall \delta \in [0, 1[\right\}.$$

As can be seen from Fig. 2, the character $\left(\frac{p}{q}, \frac{r}{q}, j, \pm\right)$ is nothing else than the limit of points going to $\left(\frac{p}{q}, \frac{r}{q}\right)$ within the edge between the lines $D_{rq'+jq}$ and $D_{rq'+(j+1)q}$ respectively from the right $(+)$ and from the left $(-)$. When $j = \infty$ the sign $+$ or $-$ is not determined. This justifies the introduction of the space $\Omega^3_{p/q}$ because for each r, the characters $\left(\frac{p}{q}, \frac{r+\delta}{q}\right)$, $\delta \in [0, 1[$, cannot be separated by left or right limits.

For later convenience, we introduce the notation

$$\Omega^2 = \Omega^2_+ \cup \Omega^2_-$$

where

$$\Omega^2_\pm = \bigcup_{p, q \in \mathbb{N}^*} \Omega^2_{p/q, \pm},$$

$$\Omega^3 = \bigcup_{p, q \in \mathbb{N}^*} \Omega^3_{p/q}.$$

Theorem 5. *The spectrum $\Omega(\mathscr{B})$ of $\mathscr{B}$ can be identified with*

$$\Omega = \Lambda \cup \Omega^1 \cup \Omega^2 \cup \Omega^3.$$

Proof. $\Omega(\mathscr{B}) \supset \Omega$: Clearly the points of Ω define characters on the algebra $\mathscr{B}_0$. Their (unique) extensions to the closure $\mathscr{B}$ are in $\Omega(\mathscr{B})$. It is easily checked that all these characters in Ω are different.

$\Omega \supset \Omega(\mathscr{B})$: For proving that $\omega \in \Omega(\mathscr{B})$ is in Ω, it is sufficient to find an element ω' of Ω which coincides with ω on the projections χ_T, $T \in \mathscr{F}_\omega$: Actually, if $T \in \mathscr{T}$ then $\omega(\chi_T) \in \{0, 1\}$. Thus, if $T \notin \mathscr{F}_\omega$, then

$$\omega(\chi_T) = 0 = 1 - \omega(\chi_{T^c}) = 1 - \omega'(\chi_{T^c}) = \omega'(\chi_T).$$

A 3ε-type argument shows that $\omega(f) = \omega'(f)$ $\forall f \in \mathscr{B}$.

We exhaust all possible cases for $S(\omega)$ (Lemma 7):

(i) $S(\omega) = \{(\alpha, x)\}$, $\alpha \notin \mathbb{Q}$ and $z \notin \mathbb{Z}\alpha$: $\omega \in \Lambda$ by Lemma 7 (ii).

(ii) $S(\omega) = \{(\alpha, k\alpha)\}$, $\alpha \notin \mathbb{Q}$ and $k \in \mathbb{Z}$: We assert that $\omega \in \Omega^1$.

For $T \in \mathscr{T}$, define

$$T_+ = T \cap A_{-k,0} = \{(\beta, y) \in T \mid y \geqq k\beta\}$$

$$(\text{respectively } T_- = T \cap A^c_{-k,0} = \{(\beta, y) \in T \mid y < k\beta\}).$$

When $T \in \mathscr{F}_\omega$, either T_+ or T_- is in $\mathscr{F}_\omega$. Actually we have $T_+ \cup T_- = T$ and $T_+ \cap T_- = \emptyset$. This fixes a sign $+$ or $-$ because $\emptyset = T_+ \cap R_- \in \mathscr{F}_\omega$ is impossible when R, T are in $\mathscr{F}_\omega$.

Suppose it is $+$ and let $T \in \mathscr{F}_\omega$. Thus $(\alpha, k\alpha) \in \overline{T_+}$ and there exists $\varepsilon_0(T)$ such that $(\alpha, k\alpha+\varepsilon)$ is in the interior of T for $\varepsilon \in]0, \varepsilon_0(T)[$. So, for these ε,

$$\omega(\chi_T) = 1 = \chi_T(\alpha, k\alpha+\varepsilon) = (\alpha, k\alpha, +)(\chi_T).$$

Similarly for the sign $-$.

(iii) $S(\omega) = \left\{\left(\frac{p}{q}, \frac{r}{q}\right)\right\}$ with $r \in \{0, \ldots, q-1\}$: We assert that in that case, $\omega \in \Omega^2_{p/q,s}$.

For $j \in \mathbb{Z}$, define the sectors

$$S_{p,q,r,j,+} = (A_{-rq'-jq,\,-rp'-jp}) \cap (A_{-rq'-(j+1)q,\,-rp'-(j+1)p})^c,$$
$$S_{p,q,r,j,-} = (A_{-rq'-(j+1)q,\,-rp'-(j+1)p}) \cap (A_{-rq'-jq,\,-rp'-jp})^c.$$

Recall that $pq' = 1 + p'q$.

Note that $\left(\frac{p}{q}, \frac{r}{q}\right)$ belongs to none of these sectors.

There exists a unique sector $S_{p,q,r,j,s}$ in $\mathscr{F}_\omega$, where $j \in \mathbb{Z}$ and $s \in \{+,-\}$:

Let $T_1 \in \mathscr{F}_\omega$ be such that $T_1 \cap I_{p/q,r} = \emptyset$. Such a set exists, otherwise the inclusion $S(\omega) \supset \overline{I_{p/q,r}}$ would contradict the hypothesis. Similarly, there is $T_2 \in \mathscr{F}_\omega$ satisfying $T_2 \cap I_{p/q,r-1} = \emptyset$. So $T = T_1 \cap T_2$ being in $\mathscr{F}_\omega$ must be nonempty and $T \cap I_{p/q,r} = T \cap I_{p/q,r-1} = \emptyset$. Since $\left(\frac{p}{q}, \frac{r}{q}\right) \in \overline{T}$, T is included in a finite union of sectors, otherwise $S(\omega)$ would contain other points than $\left\{\left(\frac{p}{q}, \frac{r}{q}\right)\right\}$. These sectors being separated by lines in $\mathscr{D}$, one of them is in the filter $\mathscr{F}_\omega$. The intersection of two of these sectors being empty, only one, say $S_{p,q,r,j,+}$, is in $\mathscr{F}_\omega$.

Define arbitrarily small (for large integers n) triangles inside the sectors

$$T_{p,q,r,j,+,n} = S_{p,q,r,j,+} \cap (A_{nq,1+np})^c,$$
$$T_{p,q,r,j,-,n} = S_{p,q,r,j,-} \cap (A_{nq,np}).$$

The $T_{p,q,r,j,s,n}$'s are all in $\mathscr{T}$ and actually in $\mathscr{F}_\omega$: If not, the intersection of the complement of a triangle and its sector would be in $\mathscr{F}_\omega$, but this is impossible because then $\left(\frac{p}{q}, \frac{r}{q}\right)$ would not be in the closure of the intersection.

Let us now show that if $T \in \mathscr{F}_\omega$, then there exists $n > 0$ with the inclusion $T \supset T_{p,q,r,j,+,n}$: $T \cap T_{p,q,r,j,s,n}$ is in $\mathscr{F}_\omega$, so is not empty and the assertion is proved for n large enough.

For $T \in \mathscr{F}_\omega$, we get by this result

$$\omega(\chi_T) = 1 = \lim_{0<\varepsilon\to 0} \chi_T\left(\left(\frac{p}{q} + \varepsilon, \frac{r}{q} + (rp' + jq + \delta)\varepsilon\right)\right) \qquad \forall \delta \in [0, q[$$

and $\omega = \left(\frac{p}{q}, \frac{r}{q}, j, +\right) \in \Omega^2_{p/q,+}$.

(iv) $S(\omega) = \left\{\frac{p}{q}\right\} \times \left[\frac{r}{q}, \frac{r+1}{q}\right]$: We assert that $\omega \in \Omega^3_{p/q}$.

Let T in $\mathscr{F}_\omega$: thus, $\overline{T} \supset S(\omega) \supset I_{p/q,r}$. If $T \cap I_{p/q,r} = \emptyset$, then the boundary of T contains the vertical segment $I_{p/q,r}$, which is impossible because its boundary must be in $\mathscr{D}$. It follows that $T \supset I_{p/q,r}$ for $T \in \mathscr{F}_\omega$ and if $\delta \in [0, 1[$ then

$$\omega(\chi_T) = 1 = \chi_T\left(\frac{p}{q'}, \frac{r+\delta}{q}\right).$$

Hence $\omega = \left(\frac{p}{q'}, \frac{r}{q}\right) \in \Omega^3_{p/q}$.

This concludes the proof of Theorem 5.

Remarks 1. (i) $\underline{p}$ is not open on Ω:

Let ω be in Ω and let $T \in \mathscr{F}_\omega$. If $i_T = \inf(pr_1(\bar{T}))$ and $s_T = \sup(pr_1(\bar{T}))$, then $i_T = \frac{p'}{q'}$ and $s_T = \frac{p''}{q''}$ by construction. There exist ω' in $\Omega^2_{p'/q',+}$ and ω'' in $\Omega^2_{p''/q'',-}$ with $T \in \mathscr{F}_{\omega'} \cap \mathscr{F}_{\omega''}$. This implies $\underline{p}(V_T) = \left[\frac{p'}{q'}, \frac{p''}{q''}\right]$.

(ii) Suppose that p is a continuous open map from Ω onto $\Gamma = p(\Omega)$. Then, Γ is a topological quotient of Ω for an equivalence relation the classes of which are saturated by p. Then no fiber $p^{-1}(\gamma)$ has a p-isolated point in Ω.

(iii) The map ϕ has been defined by extension on $\Omega(\mathscr{B})$. Actually, one can check that $\phi : \Omega \to \Omega$ is explicitly defined on each component of Ω by

$$\phi(\alpha, x) = (\alpha, x - \alpha),$$
$$\phi(\alpha, k\alpha, s) = (\alpha, (k-1)\alpha, s),$$
$$\phi\left(\frac{p}{q'}, \frac{r}{q}, j, s\right) = \left(\frac{p}{q'}, \frac{r-p}{q} \,(\mathrm{mod}\, 1), j + q', s\right), \qquad s \in \{+, -\},$$
$$\phi\left(\frac{p}{q'}, \frac{r}{q}\right) = \left(\frac{p}{q'}, \frac{r-p}{q} \,(\mathrm{mod}\, 1)\right).$$

We now introduce the space Γ of Sect. 2:

Let p be the map from Ω onto the disjoint union

$$\Gamma = [0, 1[\cup (([0, 1[\cap \mathbb{Q}) \times \{+, -\})$$

defined by

$$p(\omega) = \underline{p}(\omega) \quad \text{if} \quad \omega \in \Lambda \cup \Omega^1 \cup \Omega^3,$$
$$p(\omega) = (\underline{p}\omega), s) \quad \text{if} \quad \omega \in \Omega^2_s,\ s \in \{+, -\}.$$

For $p(\omega)$ in Γ, let $\mathscr{B}(p(\omega))$ be the set of neighborhoods $V_{p(\omega)}(\varepsilon)$ of $p(\omega)$ given for $\varepsilon > 0$ by

$$\{p(\omega') \mid \omega' \in \Omega, \underline{p}(\omega') \in]\underline{p}(\omega) - \varepsilon, \underline{p}(\omega) + \varepsilon[\} \quad \text{if} \quad \omega \in \Lambda \cup \Omega^1 \cup \Omega^3,$$
$$\{p(\omega') \mid \omega' \in \Omega, \underline{p}(\omega') \in [\underline{p}(\omega), \underline{p}(\omega) + \varepsilon[\} \backslash \{(\underline{p}(\omega), -)\} \quad \text{if} \quad \omega \in \Omega^2_+,$$
$$\{p(\omega') \mid \omega' \in \Omega, \underline{p}(\omega') \in]\underline{p}(\omega) - \varepsilon, \underline{p}(\omega)]\} \backslash \{(\underline{p}(\omega), +)\} \quad \text{if} \quad \omega \in \Omega^2_-.$$

Naturally we do not take into account $(0, -)$ and $(1, +)$.

$\mathscr{B}(p(\omega))$ defines a fundamental basis of neighborhoods of $p(\omega)$ giving a Hausdorff topology $\mathscr{T}_p$ on Γ. Then, p and Γ satisfy the hypothesis of Theorem 3:

Lemma 11. *p is a continuous open map from Ω onto Γ such that $p \circ \phi = p$.*

Proof. $p \circ \phi = p$ follows from the definitions.

To prove the continuity of p, it is sufficient, by the metrizability of Ω, to verify that $p(\omega)=\lim\limits_{n\to\infty} p(\omega_n)$ for any ω in Ω and any sequence $\{\omega_n\}_n$ $\mathscr{T}$-converging to ω.

Thanks to Lemma 9, p is a continuous surjection from $\Omega(\mathscr{B})$ onto $\mathbb{T}$. Thus p is clearly continuous at each $\omega\in\Lambda\cup\Omega^1\cup\Omega^3$.

Suppose that $\omega=\left(\frac{p}{q},\frac{r}{q},j,+\right)\in\Omega^2$. The triangles

$$T_m=S_{p,q,r,j,+}\cap(A_{mq,1+mp})^c$$

are in $\mathscr{F}_\omega$ for $m>0$. Thus $p(\omega_n)$ is in

$$pr_1(\overline{T_m})=\left[\frac{p}{q},\frac{p}{q}+\frac{1}{mq}\right]$$

for n large enough, so p is continuous on Ω^2_+.
Similar arguments give the continuity on Ω^2_-.
Before proving that p is open, let us observe a useful fact:

Let $\omega\in\Omega$, $T\in\mathscr{F}_\omega$ and let $\gamma\in\Gamma$ be such that its spatial part $\underline{\gamma}$ ($\gamma=(\underline{\gamma},s)$), where $\underline{\gamma}\in\mathbb{T}$ and $s\in\{+,-\}$) is in the interval $]i_T,s_T[$ with

$$i_T=\inf(pr_1(\overline{T}))\quad\text{and}\quad s_T=\sup(pr_1(\overline{T}))\,.$$

There exists $\omega'\in\Omega$ such that $S(\omega')=(\gamma,x)$ is in the interior of T (thus $\omega'\in V_T$) and $p(\omega')=\gamma$.
In particular for convex polygons T in $\mathscr{F}_\omega$,

$$p(V_T)\supset\,]i_T,s_T[\,\cup\,]i_T,s_T[\,\times\{+\}\cup\,]i_T,s_T[\,\times\{-\}\,.$$

To prove that p is open, we need only show that for $\omega\in\Omega$ and T a convex polygon in $\mathscr{F}_\omega$, $p(V_T)$ is open in Γ:

Let $\omega'\in V_T$. We assert that there is $\varepsilon>0$ such that $V_{p(\omega')}(\varepsilon)$ is included in $p(V_T)$.

When $p(\omega')\in\,]i_T,s_T[$, the remark gives the assertion.

When $p(\omega')=i_T$, $\omega'\in\Omega^2$ and i_T and s_T are in $\mathbb{Q}$. Write $i_T=\frac{p}{q}$ and let $r, 0\leqq r\leqq q-1$ and $j\in\mathbb{Z}$ be such that $\omega'=\left(\frac{p}{q},\frac{r}{q},j,+\right)$ (note that T is not in $\mathscr{F}_{(p/q,r/q,j,-)}$ for any r and j). If ε satisfies $\varepsilon<s_T-i_T$, then $p(V_T)\supset V_{p(\omega')}(\varepsilon)$.

Proof of Theorem 2. We assert that $\sigma^\pm_{p/q}=\sigma(\eta_{(p/q,\pm)}(h))$ for h as in Proposition 3.

The map: $\gamma\in\Gamma\to\sigma(\eta_\gamma(h))$ is continuous (Lemma 11, Theorems 3 and 5). Let α_n be a sequence in $\mathbb{T}$ such that $\frac{p}{q}<\alpha_n$ and α_n tends to $\frac{p}{q}$. Since $\mathbb{T}$ is included in Γ, for each ε, α_n is in $V_{(p/q,+)}(\varepsilon)$ when n is large enough and

$$\sigma(\eta_{(p/q,+)}(h))=\lim_{\alpha_n\to p/q}\sigma(\eta_{\alpha_n}(h))\,.$$

By extracting a subsequence, we may assume that the sequence $\{\alpha_n\}_n$ is in $\mathbb{T}\backslash\mathbb{Q}$ or in $\mathbb{Q}$.

In the former case, $\sigma(\eta_{\alpha_n}(h))=\sigma_{\alpha_n}$ (Proposition 3) and the assertion is proved. Note that this argument is also valid for hamiltonians of type (4).

In the latter case, $\alpha_n=p_n/q_n$ and

$$p^{-1}(\alpha_n)=\Omega^3_{\alpha_n}=\bigcup_{r\in\{0,\ldots,q_n-1\}}\left(\alpha_n,\frac{r}{q_n}\right).$$

Again by (6), (7) and Proposition 1, we get

$$\sigma(\eta_{\alpha_n}(h)) = \overline{\bigcup_{r\in\{0,\ldots,q_n-1\}} \sigma\left(H\left(\alpha_n, \frac{r}{q_n}\right)\right)}.$$
$$= \sigma_{\alpha_n}.$$

6. The Spectrum Around a Rational Number

In the previous sections, we showed that the limits $\sigma^+_{p/q}, \sigma^-_{p/q}$ exist for the spectrum of the hamiltonian $H\left(\frac{p}{q}, x\right)$. Here we give more details on a general operator in the algebra $\mathscr{A} = C(\Omega) \times_\phi \mathbb{Z}$ associated to a "limit" character in $\Omega^2_{p/q,\pm}$.

The situation we want to describe now is, typically, the effect of an arbitrarily large impurity placed somewhere inside a periodic partition of $\mathbb{Z}$. So we introduce the

Definition 4. Let A be a bounded operator on $l^2(\mathbb{Z})$. Then A is called *eventually q-periodic* if there exists an impurity domain $I = \{n\in\mathbb{Z} \mid a<n<b\}$ with $b-a=c$ modulo q, satisfying for $\psi \in l^2(\mathbb{Z})$,

$$T^{-q}AT^{+q}\psi = A\psi \quad \text{when} \quad [b,\infty[\supset \text{support}(\psi),$$
$$T^{+q}AT^{-q}\psi = A\psi \quad \text{when} \quad]-\infty,a]\supset \text{support}(\psi),$$
$$\exists k_0 \geqq 0 \text{ such that } \forall k>k_0,\ T^{kq+c}AT^{-kq-c}\psi(n) = A\psi(n) \text{ when } n>b.$$

Note that the strong limit of $T^{kq}AT^{-kq}$ when k goes to $\pm\infty$ is a q-periodic operator, which we shall denote by A_{per}.

Similarly, a bounded function on $\mathbb{Z}$ is said to be eventually q-periodic if the multiplication by this function is eventually q-periodic on $l^2(\mathbb{Z})$.

Given $\left(\frac{p}{q}, \frac{r}{q}\right)$, we define two elements of Ω by

$$\omega_{j,\pm} = \left(\frac{p}{q}, \frac{r}{q}, j, \pm\right),$$
$$\omega_\pm = \left(\frac{p}{q}, 0, 0, \pm\right).$$

Lemma 12. (i) *Let*

$$d_{j,\pm}(n) = \Pi_{\omega_{j,\pm}}(\chi_0)(n) - \Pi_{(p/q,r/q)}(\chi_0)(n).$$

Then,

$$d_{j,\pm}(n) = \lambda \sum_{m=1}^{\infty} (\delta_{-(\pm mq+jq+rq')} - \delta_{-(\pm mq+jq+rq'+1)})(n).$$

(ii) $$T^{-(jq+rp')}\Pi_{\omega_\pm}(\chi_0)T^{jq+rp'} = \Pi_{\omega_{j,\pm}}(\chi_0).$$

(iii) $$\operatorname*{strong-lim}_{j\to\infty} \Pi_{\omega_{j,\pm}}(\chi_0) = \Pi_{(p/q,r/q)}(\chi_0).$$

Proof. (i) By definition, $d_{j,\pm}(n)$ is

$$\lim_{\pm\varepsilon\to 0} \chi_{[1-(p/q\pm\varepsilon),1[}\left(n\frac{p}{q} + \frac{r}{q} \pm \varepsilon(rq'+jq+n)\right) - \chi_{[1-p/q,1[}\left(n\frac{p}{q} + \frac{r}{q}\right).$$

We treat only the case $+$.

Let us assume first that $n\frac{p}{q}+\frac{r}{q}\in\mathbb{Z}$. Then $n=-rq'+kq$ for some $k\in\mathbb{Z}$. Thus,

$$\begin{aligned} d_{j,+}(n) &= \lim_{0<\varepsilon\to 0} \chi_{[1-p/q-\varepsilon,1[}(\varepsilon q(k+j))-0 \\ &= \begin{cases} 0 & \text{if } k\geqq -j \\ 1 & \text{if } k<-j\,. \end{cases} \end{aligned}$$

If now $n\frac{p}{q}+\frac{r}{q}\in\left(\mathbb{Z}-\frac{p}{q}\right)$, then $n=-rq'+kq-1$ for some $k\in\mathbb{Z}$, and

$$d_{j,+}(n)=\begin{cases} 0 & \text{if } k\geqq -j \\ -1 & \text{if } k<-j\,. \end{cases}$$

Finally, suppose that $n\frac{p}{q}+\frac{r}{q}\notin\mathbb{Z}\cup\left(\mathbb{Z}-\frac{p}{q}\right)$.

Let $t=1-\frac{p}{q}-\left(n\frac{p}{q}+\frac{r}{q}\right)$ (mod 1). Then $t\neq 0$ by hypothesis. Assume that $t>0$ and choose a sufficiently small ε that

$$0<\varepsilon<\left|\frac{t}{rp'+jq+n+1}\right|.$$

Then,

$$\begin{aligned} 0<n\frac{p}{q}+\frac{r}{q}+\varepsilon(rq'+jq+n) &\leqq n\frac{p}{q}+\frac{r}{q}+\varepsilon|rq'+jq+n+1|-\varepsilon \\ &< n\frac{p}{q}+\frac{r}{q}+t-\varepsilon \\ &=1-\frac{p}{q}-\varepsilon\,. \end{aligned}$$

Thus $d_{j,+}(n)=0$. In the same way, $d_{j,+}(n)=0$ when $t<0$.
(ii) is immediate and (iii) follows from (ii).

Remark 2. Note that the difference between $\Pi_{\omega_{j,\pm}}(\chi_0)$ and $\Pi_{(p/q,r/q)}(\chi_0)$ is not a compact operator, so the proof of Theorem 4 is not direct.

Let $\mathscr{B}_0(\omega_\pm)$ denote the algebra of bounded functions on $\mathbb{Z}$ generated by $\Pi_{\omega_\pm}(\chi_n)$, $n\in\mathbb{Z}$.

Lemma 13. *Any f in $\mathscr{B}_0(\omega_\pm)$ is eventually q-periodic.*

Proof. The previous lemma shows that, when j tends to $\pm$ infinity, the strong limit of $T^{-j}\Pi_{\omega_\pm}(\chi_0)T^{+j}$ coincides with some q-periodic function. Thus, $\Pi_{\omega_\pm}(\chi_0)$ is eventually q-periodic. By translation, this is also true for $\Pi_{\omega_\pm}(\chi_n)=\Pi_{\phi^{-n}(\omega_\pm)}(\chi_0)$. Since the eventually q-periodic functions form an algebra, the lemma is proved.

Definition 5. A bounded operator A on $l^2(\mathbb{Z})$ has a finite interaction range when there exists $r>0$ such that $\langle e_n, Ae_m\rangle=0$ if $|n-m|>r$, where $\{e_n\}_n$ is the canonical basis of $l^2(\mathbb{Z})$.

Theorem 6. *Let A be a self-adjoint bounded operator on $l^2(\mathbb{Z})$ with a finite interaction range and which is eventually q-periodic. Let A_{per} be its q-periodic part.*

Then,

$$\sigma_{\mathrm{ess}}(A)=\sigma_{\mathrm{ess}}(A_{\mathrm{per}}) \quad and \quad \sigma_{\mathrm{ac}}(A)=\sigma_{\mathrm{ac}}(A_{\mathrm{per}}).$$

Proof. Let $I=]a,b[$ denote the impurity domain of A and P_- (respectively P_0, P_+) the projection onto $l^2(]-\infty,a[)$ (respectively $l^2(]a,b[)$, $l^2([b,\infty[)$).

If $\approx$ means equality modulo a finite rank operator, we have:

$$P_-AP_+\approx 0,$$

$$P_+AP_-\approx 0,$$

$$A\approx P_+AP_+\oplus P_-AP_-.$$

Consider the partial isometry S_I on $l^2(\mathbb{Z})$ associated to the impurity domain I:

$$S_I\psi(n)=\begin{cases}\psi(n+b) & n\geqq 0\\ \psi(n+a+1) & n<0.\end{cases}$$

Then $S_I^*S_I=\mathbb{1}$, $S_IS_I^*=\mathbb{1}-P_0$. Clearly, the operator

$$B=S_I(P_+AP_+\oplus P_-AP_-)S_I^*$$

consists in chopping off the impurity and gluing together the periodic parts on the left and on the right. B is not yet a periodic operator because the P_+ and P_- parts are disconnected, but $S_I^*BS_I\approx A_{\mathrm{per}}$. The theorem is now a consequence of the classical Weyl and Birman-Kato-Rosenblum theorems [K.].

Note that if A is an eventually periodic self-adjoint operator, the strong limit A_{per} of T^jAT^{-j} when j goes to $\pm$ infinity exists and satisfies $\sigma(A)\supset\sigma(A_{\mathrm{per}})$. Using this result, together with Lemmas 12 and 13, we obtain the

Corollary 2. *Let $h\in\mathscr{A}_0$. Then $\Pi_{\omega_j,\pm}(h)\approx\Pi_{(p/q,r/q)}(h)$ up to a partial isometry S_I associated to the impurity domain of $\Pi_{\omega_j,\pm}(h)$ and*

$$\sigma_{\mathrm{ess}}(\Pi_{\omega_j,\pm}(h))=\sigma_{\mathrm{ac}}(\Pi_{\omega_j,\pm}(h))=\sigma(\Pi_{(p/q,r/q)}(h)).$$

Now Corollary 1 is immediate.

Proof of Theorem 4. Let $h\in\mathscr{A}$ and $\varepsilon>0$. There exists $h_\varepsilon\in\mathscr{A}_0$ such that $\|h-h_\varepsilon\|<\varepsilon$. So

$$\|\Pi_{\omega_j,\pm}(h)-\Pi_{\omega_j,\pm}(h_\varepsilon)\|<\varepsilon.$$

(i) The strong limit of $\Pi_{\omega_j,\pm}(h_\varepsilon)=T^{jq}\Pi_{(p/q,r/q,0,\pm)}(h_\varepsilon)T^{-jq}$ when $j\to\pm\infty$ is equal to $\Pi_{(p/q,r/q)}(h_\varepsilon)$ (see proof of Lemma 13). Thus,

$$s-\lim_{j\to\pm\infty}\Pi_{\omega_j,\pm}(h)=\Pi_{(p/q,r/q)}(h)$$

and

$$\sigma(\Pi_{\omega_j,\pm}(h))=\sigma(\Pi_{\omega_{j'},\pm}(h)) \quad \text{for} \quad j\neq j'.$$

Thus, $\sigma(\Pi_{\omega_j,\pm}(h))$ contains $\sigma(\Pi_{(p/q,r/q)}(h))$.

(ii) Let G be an open gap in $\sigma(\Pi_{(p/q,r/q)}(h))$, d be the length of G and choose $0<\varepsilon_0<\frac{d}{10}$.

By perturbation theory, $\sigma(\Pi_{(p/q,r/q)}(h_\varepsilon))$ does not intersect G^{ε_0} for $\varepsilon<\varepsilon_0$, where $X^\varepsilon=[\inf(X)+\varepsilon,\ \sup(X)-\varepsilon]$. Corollary 1 implies that

$$\sigma_{\mathrm{ess}}(\Pi_{\omega j,\pm}(h_\varepsilon))=\sigma_{\mathrm{ac}}(\Pi_{\omega j,\pm}(h_\varepsilon))=\sigma(\Pi_{(p/q,r/q)}(h_\varepsilon)).$$

We avoid the accumulation points of $\sigma_{\mathrm{ess}}(\Pi_{\omega_j,\pm}(h_\varepsilon))$ if we restrict to $G^{2\varepsilon_0}$. Now the number of eigenvalues (with multiplicity) of $\Pi_{\omega_j,\pm}(h_\varepsilon)$ contained in $G^{4\varepsilon_0}$ is uniformly bounded by the number of eigenvalues (with multiplicity) of $\Pi_{\omega_j,\pm}(h_{\varepsilon_0})$ contained in $G^{2\varepsilon_0}$ for

$$\|\Pi_{\omega j,\pm}(h_\varepsilon)-\Pi_{\omega j,\pm}(h_{\varepsilon_0})\|<2\varepsilon_0.$$

Thus, $\sigma(\Pi_{\omega_j,\pm}(h))\cap G^{5\varepsilon_0}$ is a set of isolated eigenvalues with finite multiplicity. Taking $\varepsilon_0\to 0$, one concludes that $\sigma(\Pi_{\omega_j,\pm}(h))\cap G$ is in $\sigma_{\mathrm{discrete}}(\Pi_{\omega_j,\pm}(h))$.

In the case of the Kohmoto model $H(\alpha,x)$ of type (1), one can give a more precise result. The associated operator h in $C(\Omega)\times{}_\phi\mathbb{Z}$ has the form

$$h=U+U^*+\lambda\chi_0. \tag{9}$$

We suppose in what follows the coupling constant λ to be non-zero:

$$\begin{aligned}\sigma^{\pm}_{p/q}&=\sigma(\eta_{(p/q,\pm)}(h)) && \text{(proof of Theorem 2)}\\ &=\overline{\bigcup_{\omega\in\Omega^2_{p/q,\pm}}\sigma(\Pi_\omega(h))} && \text{[cf. (6)]}\\ &=\sigma(\Pi_{(p/q,0,0,\pm)}(h)) && \text{(Lemma 12)}.\end{aligned}$$

The impurity domains are of different sizes for $\Pi_{(p/q,0,0,+)}(h)$ and $\Pi_{(p/q,0,0,-)}(h)$: if the continued fraction expansion of p/q contains an even (respectively odd) number of quotients, the first (respectively second) one has size q' and the other $q-q'$.

Let us first recall notations and the transfer matrix technique (for more details, see [B.I.S.T.]). In order to find a generalized (not normalized a priori) eigenvector $\psi_n(n\in\mathbb{N})$ of the hamiltonian (1), corresponding to the eigenvalue E, it is equivalent to solve, up to a common scalar multiple, for $(\Phi_n(n\in\mathbb{N}))\neq 0$ $(\Phi_n\in\mathbb{C}^2)$, the following set of equations:

$$\Phi_{n+1}=T_n(E)\Phi_n,$$

where

$$\Phi_n=\begin{pmatrix}\psi(n+1)\\ \psi(n)\end{pmatrix}$$

and

$$T_n(E)=\begin{pmatrix}E-\lambda v_{\alpha,x}(n) & -1\\ 1 & 0\end{pmatrix}.$$

We also introduce the "resonant transfer matrices":

$$M_q(E)=T_q(E)T_{q-1}(E)\dots T_1(E)T_0(E),$$
$$M_{q'}(E)=T_{q'}(E)T_{q'-1}(E)\dots T_1(E)T_0(E).$$

Considering the case that the continued fraction expansion of p/q contains an even number of quotients (the other case can be treated similarly), we consider now the problem of finding a generalized eigenvalue for the hamiltonian $\Pi_{(p/q,0,0,-)}(h)$

[respectively $\Pi_{(p/q,0,0,+)}(h)$]. On writing $\Phi=\Phi_0$ and $\Phi'=\Phi_{q'}$ (respectively $\Phi'=\Phi_{q-q'}$), it is immediate to verify that this problem is equivalent to solving, up to a common scalar multiple, the following set of equations for a non-zero ξ_n ($\xi_n\in\mathbb{C}^2$, $n\in\mathbb{N}$):

$$\begin{gathered}\xi_0=\Phi\\ \xi_1=\Phi'\\ M_q(E)\xi_n=\xi_{n+1}\qquad(n\geqq 1\text{ or }n\leqq -1)\\ M_{q'}(E)\xi_0=\xi_1\qquad(\text{respectively }M_{q'}(E)^{-1}M_q(E)\xi_0=\xi_1).\end{gathered}\tag{10}$$

We can now state the following:

Lemma 14. *The following conditions are equivalent:*

(i) $$E\in\sigma_{\text{discrete}}(\Pi_{(p/q,0,0,-)}(h))\cup\sigma_{\text{discrete}}(\Pi_{(p/q,0,0,+)}(h)).$$

(ii) *The spectrum of $M_q(E)$ has multiplicity 1 and Φ and Φ' are eigenvectors of $M_q(E)$ with different eigenvalues. Moreover, $M_{q'}(E)\Phi$ is a multiple of Φ' or $M_{q'}(E)\Phi'$ is a multiple of Φ.*

(iii) $$\operatorname{Tr}(M_q(E))=\pm\sqrt{4+\lambda^2}.$$

Moreover, if $E\in\sigma_{\text{discrete}}(\Pi_{(p/q,0,0,-)}(h))$, then the operator $(\Pi_{(p/q,0,0,+)}(h))$ admits a generalized eigenvector with eigenvalue E which increases at both $n\to\pm\infty$. The same statement holds on replacing $+$ by $-$.

Before going to the proof of this lemma, let us show that it implies Theorem 7: First, we remark that any (possibly complex) solution E of the equations (iii) appears as an element of the spectrum of a selfadjoint operator and consequently is real. So, the equations (iii) have $2q$ real solutions. By the remark at the end of Lemma 14, the discrete spectra of the operators $\Pi_{(p/q,0,0,-)}(h)$ and $\Pi_{(p/q,0,0,+)}(h)$ are disjoint and the result follows.

Proof of Lemma 14. A general solution of Eqs. (10) is a superposition of solutions for which Φ and Φ' are eigenvectors of $M_q(E)$. Assume (i) and suppose that $M_q(E)$ has its spectrum contained in the unit complex circle. Then ξ_n cannot tend to 0 at infinity. It follows that $\sigma(M_q(E))=\left\{\left(\beta,\frac{1}{\beta}\right)\right\}$ with $|\beta|<1$, since $\det(M_q(E))=1$. For the same reason, Φ (respectively Φ') must be an eigenvector of $M_q(E)$ corresponding to the eigenvalue $\frac{1}{\beta}$ (respectively β) and (ii) follows. The converse is trivial. The remark at the end of the statement follows easily from the same argument: an exchange of Φ and Φ' allows to transform the set of equations for the operator $\Pi_{(p/q,0,0,+)}(h)$ into those for $\Pi_{(p/q,0,0,-)}(h)$.

Using Lagrange polynomials in order to express the eigenprojections of $M_q(E)$, we get that (ii) is equivalent to

$$(M_q(E)-\beta)M_{q'}(E)(M_q(E)-\beta)=0$$

or

$$\left(M_q(E)-\frac{1}{\beta}\right)M_{q'}(E)\quad\left(M_q(E)-\frac{1}{\beta}\right)=0.$$

Since both operators on the left-hand side have one dimensional kernel and range, one has equivalently:

$$\mathrm{Tr}(M_q(E)M_{q'}(E)) = \beta\,\mathrm{Tr}(M_{q'}(E))$$

or

$$\mathrm{Tr}(M_q(E)M_{q'}(E)) = \frac{1}{\beta}\,\mathrm{Tr}(M_{q'}(E)).$$

Using the fundamental invariant (see [B.I.S.T.])

$$(\mathrm{Tr}(M_q(E)))^2 + (\mathrm{Tr}(M_{q'}(E)))^2 + (\mathrm{Tr}(M_q(E)M_{q'}(E)))^2 \\ - \mathrm{Tr}(M_q(E))\,\mathrm{Tr}(M_{q'}(E))\,\mathrm{Tr}(M_q(E)M_{q'}(E)) = 4 + \lambda^2,$$

one gets the equivalence of (ii) and (iii) easily and this ends the proof.

Remark 3. The method of proof does not allow to decide simply, amongst the whole set of solutions of Lemma 14 (iii), what are the eigenvalues of the operators $\Pi_{(p/q,0,0,-)}(h)$ or $\Pi_{(p/q,0,0,+)}(h)$. Numerically, it seems that all but one of the eigenvalues appear in one gap of their common periodic part, and each gap of each operator contains one eigenvalue, the last eigenvalue can appear as either the least upper bound or the greatest lower bound of the spectrum, these two situations being mutually exclusive with respect to the exchange of $+$ and $-$.

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