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# Nonlinear Parabolic Stochastic Differential Equations with Additive Colored Noise on $R^d \times R_+$: A Regulated Stochastic Quantization

Charles R. Doering*

Department of Physics, Center for Relativity and Department of Astronomy, The University of Texas at Austin, Austin, Texas 78712, USA

**Abstract.** We prove the existence of solutions to the nonlinear parabolic stochastic differential equation

$$(\partial/\partial t - \Delta)\varphi = -V'(\varphi) + \eta_c$$

for polynomials $V$ of even degree with positive leading coefficient and $\eta_c$ a gaussian colored noise process on $R^d \times R_+$. When $\eta_c$ is colored enough that the gaussian solution to the linear problem has Hölder continuous covariance, the nongaussian processes are almost surely realized by continuous functions. Uniqueness, regularity properties, asymptotic perturbation expansions and nonperturbative fluctuation bounds are obtained for the infinite volume processes. These equations are a cutoff version of the Parisi-Wu stochastic quantization procedure for $P(\varphi)_d$ models, and the results of this paper rigorously establish the nonperturbative nature of regularization via modification of the noise process. In the limit $\eta_c \to$ gaussian white noise we find that the asymptotic expansion and the rigorous bounds agree for processes corresponding to the (regulated) stochastic quantization of super-renormalizable and small coupling, strictly renormalizable scalar field theories and disagree for nonrenormalizable models.

## 1. Introduction and Overview

This work is motivated by the stochastic quantization procedure proposed by Parisi and Wu [1]. In this approach, the euclidean field measure for fields on $R^d$ with action functional $S$

$$d\mu(\varphi) = \exp(-S[\varphi])\prod d\varphi(x)/\int \exp(-S[\varphi])\prod d\varphi(x) \tag{1.1}$$

is considered as the formal stationary probability distribution of the random process defined by the stochastic differential equation (Langevin equation)

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* Current address: Center for Nonlinear Studies, MS-B258 Los Alamos National Laboratory Los Alamos, New Mexico 87545, USA

$$\partial\varphi(x,t)/\partial t = -\delta S/\delta\varphi(x,t) + \eta(x,t). \tag{1.2}$$

In this equation $t$ is an artificial time (called a "fictitious," "thermal" of "fifth" time) and $\eta(x,t)$ is a gaussian white noise on $R^d \times R_+$ defined by

$$E\{\eta(x,t)\} = 0, \quad E\{\eta(x,t)\eta(x',t')\} = 2\delta(t-t')\delta(x-x'). \tag{1.3}$$

The stochastic differential equation (1.2) gives rise to a functional Fokker–Planck equation for the time dependent probability density with the formally unique stationary solution (1.1).

This approach has generated considerable interest in recent years [2], especially in regards to its application to gauge theories [3–5]. If offers two new features over the functional integration formulation of these problems. First, the invariance problem becomes one of the relaxation of gauge variant quantities to a stationary state. No explicit gauge fixing terms are introduced and only gauge invariant quantities are expected to converge as $t \to \infty$. Second, it suggests new regularization procedures which are *not* action regularizations. Rather than introducing a lattice, forcing higher derivatives into the action, or considering non-integral space-time dimensions, the random driving noise force in the Langevin equation can be modified. This has the advantage of preserving the symmetries of the action while eliminating the UV divergences in dimensions $d > 1$. It has been suggested that the "time" part of the noise correlation function can be modified [6], and this has already been used in a statistical mechanical application of the stochastic differential equation [7]. Alternatively, we may modify the "space" part of the correlation function. In this case the stochastic differential equation and regularization can be given an interpretation in terms of a nonequilibrium chemical reaction-diffusion system [8]. These modifications of the gaussian white noise are known as "colored" noises. The spectrum of the colored fluctuations is not flat, but is of a finite bandwidth which we may refer to as a momentum cutoff.

In this paper, as a step toward the rigorous analysis of this regularized stochastic quantization procedure, we prove the existence and uniqueness of solutions to the nonlinear parabolic stochastic differential equation

$$(\partial/\partial t - \Delta)\varphi(x,t) = -V'(\varphi(x,t)) + \eta_c(x,t), \tag{1.4}$$

where $x \in R^d$, $\varphi$ is real, $V'$ is the derivative of a polynomial potential $V$ of even degree and positive leading coefficient, and $\eta_c$ is a properly regularized (colored) gaussian process on $R^d \times R_+$. This corresponds to the regulated stochastic quantization of a self-interacting scalar field theory with action

$$S[\varphi] = \int d^d x \{(1/2)(\nabla\varphi)^2 + V(\varphi)\}, \tag{1.5}$$

and establishes the nonperturbative nature of regularization via modification of the driving noise force. The existence proof consists of showing that, with probability one, there is a unique solution to the equivalent integral equation

$$\varphi(x,t) = -\int_0^t dt' \int d^d x' \, G(x-x', t-t') V'(\varphi(x',t')) + W_c(x,t)$$

$$+ \int d^d x' \, G(x-x',t)\varphi(x',0), \tag{1.6}$$

where $G$ is the kernel of the inverse of the parabolic operator $(\partial/\partial t - \Delta)$, $\varphi(x,0)$ is an initial condition, and $W_c(x,t)$ is a function-valued gaussian stochastic process defined by

$$(\partial/\partial t - \Delta) W_c(x,t) = \eta_c(x,t), \quad W_c(x,0) = 0. \tag{1.7}$$

This is not a standard problem in the theory of partial differential equations because

i) the force $(-V')$ is non-Lipschitz and not necessarily monotone,
ii) the domain $(R^d \times R_+)$ is not compact, and
iii) the random driving force $(W_c)$ is unbounded on the domain.

Partial results in this area have been obtained by Marcus who studied equations with Lipschitz forces in $d=1$ [9] and monotone forces in finite and infinite volume in $d=1$ [10, 11], and Faris and Jona-Lasinio who developed equations with non-monotone forces in finite volume in $d=1$ [12]. In this paper we present techniques which simultaneously treat all of these problems, are applicable to infinite volume equations with non-monotone forces, and are independent of dimension. This is important because previous work utilized regularity properties of the green's function $G$ in $d=1$ which do not survive the generalization to higher dimension.

In the next Sect. 2 we deal with the "free" process, i.e. the gaussian solution to (1.4) when $V$ is quadratic so that the equation is linear. This is the starting point for the construction of the solution to the full nonlinear problem and it introduces the need for a regularization in all dimensions $d > 1$. We show how to regularize these processes by changing the correlation function of the driving noise force and establish the properties required for the existence proof. Section 3 is the existence proof for a nonlinear process satisfying an equation in which the random driving force and the self-interaction are restricted to a compact domain—the "finite volume" solutions. The proof given here introduces, in a somewhat simplified setting, the techniques which are essential to the next Sect. 4 where we establish existence, uniqueness and regularity of solutions to the full nonlinear problem. The infinite volume limit involves the new identification of a family of Banach spaces which preserve the necessary positivity properties of the parabolic operator while eliminating the problems of the non-compact domain and unboundedness of the driving force.

In Sect. 5 we use the techniques developed for the existence proof to study the solutions to the nonlinear equations. We develop an asymptotic expansion for the nongaussian process corresponding to the coupling constant perturbation expansion. When the noise correlation function is euclidean invariant, this is a manifestly euclidean invariant expansion. Additionally, we obtain nonperturbative bounds on the moments that are uniform in the nonlinear coupling, the amplitude of the driving noise, and the momentum cutoff introduced by the regularization. Assuming spatial homogeneity and relaxation to a stationary state, we investigate the rigorous bounds on the quantity

$$\langle \varphi^N \rangle := \lim_{t \to \infty} E\{\varphi(x,t)^N\}, \tag{1.8}$$

corresponding to the vacuum expectation value of $\varphi^N$ for a regulated field theory model with the potential

$$V(\varphi)=(\lambda/N)\varphi^N. \tag{1.9}$$

An important new result is that the qualitative behavior of this quantity as a function of the coupling, cutoff and noise amplitude depend on degree of nonlinearity $N$ and the dimension $d$ in a way which exactly distinguishes between perturbatively renormalizable and nonrenormalizable interactions. In particular, for large values of the cutoff the perturbative result disagees with the true behavior for nonrenormalizable models even though no renormalization is attempted. We are able to show that for large values of the cutoff in nonrenormalizable models, the vacuum expectation value of the nonlinear potential dominates the vacuum expectation value of the kinetic term in the original action—even when the coupling is small.

This result not only gives considerable insight into the problems encountered in the perturbative renormalization of these theories, but it firmly establishes the usefulness of Parisi and Wu's stochastic quantization scheme in a nonperturbatively regularized form. The closing Sect. 6 is a brief conclusion and discussion of some of the open problems that remain in the rigorous study of nonlinear parabolic stochastic differential equations with colored noise.

## 2. Linear Equations and Regularization

The gaussian process $\eta$ on $R^{d+1}$ characterized by

$$E\{\eta(x,t)\}=0, \quad E\{\eta(x,t)\eta(x',t')\}=2\delta(t-t')\delta(x-x') \tag{2.1}$$

is a process indexed by the Schwartz space $S(R^{d+1})$. This means that given a function $f\in S(R^{d+1})$,

$$\eta(f)=\int dt \int dx\, f(x,t)\eta(x,t) \quad \text{(formally)} \tag{2.2}$$

is a gaussian random variable of mean zero and the covariance of any two random variables $\eta(f_1)$ and $\eta(f_2)$ is

$$E\{\eta(f_1)\eta(f_2)=\int dt \int dx\, f_1(x,t) f_2(x,t). \tag{2.3}$$

The measure on $S'(R^{d+1})$ (the space of tempered distributions) associated to $\eta$ is defined on the $\sigma$-algebra generated by the cylinder sets of the form

$$\{\eta\in S'(R^{d+1})|\eta(f_n)\in B_n,\dots,\eta(f_1)\in B_1\} \tag{2.4}$$

for any finite set of Schwartz functions $\{f_i\}$ and Borel sets $\{B_i\}$ in $R$. The free process of mass $m^2\geqq 0$ is defined here as the solution to

$$(\partial/\partial t-\Delta+m^2)W(x,t)=\eta(x,t), \quad W(x,0)=0. \tag{2.5}$$

Written as an integral equation,

$$W(x,t)=\int_0^t dt' \int dx'\, G(x-x',t-t')\eta(x,t), \tag{2.6}$$

where $G(x,t)$ is the kernel of the inverse of the operator $(\partial/\partial t-\Delta+m^2)$ given by

$$G(x,t)=\theta(t)(4\pi t)^{-d/2}\exp\{-m^2t-|x|^2/4t\}. \tag{2.7}$$

Equation (2.6) means that $W$ is a distribution valued gaussian process with mean zero and covariance,

$$\begin{aligned}
&E\{W(x,t)W(y,s)\}\\
&=\int_0^t dt'\int dx'\int_0^s ds'\int dy'\, G(x-x',t-t')G(y-y',s-s')E\{\eta(x',t')\eta(y',s')\}\\
&=\int\frac{d^dk}{(2\pi)^d}\frac{e^{ik(x-x')}}{k^2+m^2}\left[e^{-(k^2+m^2)|t-s|}-e^{-(k^2+m^2)(t+s)}\right].
\end{aligned} \tag{2.8}$$

In general, the measure on the $\sigma$-algebra generated by the cylinder sets of $S'(R^{d+1})$ defined by these processes are *not* supported on the set distributions which can be represented by functions. If we treat $W(x,t)$ as a gaussian random variable for each $(x,t)\in R^d\times R_+$, we find

$$E\{W(x,t)^2\}=\int\frac{d^dk}{(2\pi)^d}\frac{1-e^{-2(k^2+m^2)t}}{k^2+m^2}. \tag{2.9}$$

The expression above is not defined if $d\geqq 2$, i.e. $W(x,t)$ is certainly not a random variable pointwise in dimensions higher than 1. In order to perform the nonlinear operations necessary for the problem at hand, we must consider related processes which are almost surely given by functions.

The covariance of the driving process (2.1) can be replaced with something somewhat smoother. Consider the gaussian process $\eta_c$ of mean zero and covariance

$$E\{\eta_c(x,t)\eta_c(x',t')\}=2\delta(t-t')C(x-x'), \tag{2.10}$$

where $C(x)$ is a tempered distribution on $R^d$ with pointwise non-negative Fourier transform $C(k)$. Then by the equation

$$(\partial/\partial t-\Delta+m^2)W_c(x,t)=\eta_c(x,t),\quad W_c(x,0)=0, \tag{2.11}$$

we define the gaussian process $W_c$ of mean zero and covariance

$$E\{W_c(x,t)W_c(y,s)\}=\int\frac{d^dk}{(2\pi)^d}\frac{e^{ik(x-x')}}{k^2+m^2}C(k)[e^{-(k^2+m^2)|t-s|}-e^{-(k^2+m^2)(t-s)}]. \tag{2.12}$$

We require that for every $r>0$,

$$C(k)\cdot[k^2+m^2]^{-1}\cdot[1-\exp\{-(k^2+m^2)r\}]\in L^1(R^d). \tag{2.13}$$

Then $W_c(x,t)$ can be considered a random variable pointwise, and the covariance (2.12) is a continuous function of $x-y$, $t$ and $s$; as a function of $x-y$ the covariance is the Fourier transform of an $L^1$ function, and a simple application of the dominated convergence theorem ensures that it is continuous in $t$ and $s$. The same smoothing effect can also be achieved by changing the "time" part of the correlation function.

The continuity of the correlation function does not in general imply pathwise continuity of a process, and the almost sure continuity of the realizations of $W_c$ is convenient for the existence proof in the following sections. To ensure the almost sure continuity of the realizations we appeal to Kolmogorov's lemma [13]:

Let $\phi$ be a random process indexed by $R^n$. If there are constants $a > 0, b > 0$ and $c < \infty$ such that

$$E\{|\phi(x_1) - \phi(x_2)|^a\} \leqq c \|x_1 - x_2\|^{n+b}, \tag{2.14}$$

then almost surely the map

$$x \longrightarrow \phi(x) \tag{2.15}$$

is uniformly continuous on bounded sets.

For gaussian processes, Hölder continuity of the covariance with any exponent greater than zero is sufficient to guarantee the almost certain continuity of the sample paths. This is so because the difference between two jointly gaussian random variables is gaussian and any moment of this difference is just a power of the variance of the difference. For the case under consideration here,

$$\begin{aligned} &E\{[W_c(x,t) - W_c(y,s)]^2\} \\ &= \int \frac{d^dk}{(2\pi)^d} \frac{C(k)}{k^2+m^2}[2 - e^{-2(k^2+m^2)t} - e^{-2(k^2+m^2)s}] \\ &\quad - 2\int \frac{d^dk}{(2\pi)^d} \frac{e^{ik(x-y)}}{k^2+m^2} C(k)[e^{-(k^2+m^2)|t-s|} - e^{-(k^2+m^2)(t+s)}]. \end{aligned} \tag{2.16}$$

If, for instance, $C(k) \in L^1(R^d)$, then this is a differentiable function of the variables $x - y$ and $t$ and $s$ for $t \neq s$, with one sided derivatives at $t = s$. Then the criteria of Kolmogorov's lemma holds with $a = 2(d+1) + \varepsilon$ for any $\varepsilon > 0$. It is sufficient but not necessary that $C(k) \in L^1(R^d)$, and several examples of this are developed in [8]. *From this point on, any reference to $W_c$ will assume that an appropriate choice of $C$ has been made to ensure the almost certain continuity of the process.*

The global features of the sample paths are also of interest. Consider the almost surely positive random variable,

$$X = \int_0^\infty dt \int dx\, W_c(x,t)^{2n}(1 + |x|)^{-(d+\varepsilon_1)}(1+t)^{-(1+\varepsilon_2)}, \tag{2.17}$$

for $\varepsilon_1, \varepsilon_2 > 0$. Then

$$\begin{aligned} E\{X\} &= \int_0^\infty dt \int dx [(2n)!/(2^n n!)] E\{W_c(x,t)^2\}^n (1+|x|)^{-(d+\varepsilon_1)}(1+t)^{-(1+\varepsilon_2)} \\ &\leqq [(2n)!/(2^n n!)] \int \frac{d^dk}{(2\pi)^d} \frac{C(k)}{k^2+m^2} \int dt\, \theta(t)(1+t)^{-(1+\varepsilon_2)} \int dx (1+|x|)^{-(d+\varepsilon_1)}. \end{aligned} \tag{2.18}$$

Since this is finite, $X$ is almost surely finite. Thus the realizations are, with probability one in every $L^p(d\mu)$, $1 \leqq p < \infty$, for any algebraically decaying finite measure $\mu$ on $R^d \times R_+$.

Since $W_c$ takes values in $S'$ and has continuous realizations, the sample paths are almost surely polynomially bounded continuous functions on $R^d \times R_+$. With probability one the sample paths are *not* bounded as functions on $R^d \times R_+$. This follows from a straightforward argument as can be founded in [8], and is analogous to the law of the iterated logarithm applied to a finite dimensional Brownian motion [14].

To summarize, when the correlation function of the driving noise process is chosen so that the correlation function of the free process (2.12) is Hölder continuous with exponent greater than zero, the free process is given almost surely by a continuous function, unbounded on $R^d \times R_+$ with probability one. The realizations are, however, polynomially bounded and $p$-integrable with respect to any algebraically decaying measure.

## 3. Finite Volume Equations

We want to establish the existence and uniqueness of solutions to the integral equation

$$\varphi(x,t) = -\int_0^t dt' \int dx' G(x-x', t-t') V'(\varphi(x',t')) + F(x,t), \tag{3.1}$$

where $V'$ is a polynomial of odd degree $N-1$ with positive leading coefficient and $F$ is a random, almost surely polynomially bounded continuous function on $R^d \times R_+$. The initial condition is in $F$:

$$F(x,t) = W_c(x,t) + \int dx' \, G(x-x',t)\varphi(x',0), \tag{3.2}$$

where $W_c$ is as described in the previous section. By requiring $F$ to be polynomially bounded we stay in $S'(R^{d+1})$. Without loss of generality we may assume $V'(0)=0$. In the shorthand notation to be used here, equation (3.1) is

$$\varphi = -GV'(\varphi) + F. \tag{3.3}$$

In order to handle the fact that $F$ is not bounded, it is convenient to begin by considering the family of equations

$$\varphi_\Lambda = -G\Lambda \, V'(\varphi_\Lambda) + \Lambda \, F, \tag{3.4}$$

where $0 \leqq \Lambda(x) \leqq 1$ is a continuous function of compact support on $R^d$. We first prove the existence of these "finite volume" solutions and in the next section show that they converge, in a sense to be specified later, to a solution of the "infinite volume" equation (3.1). The global (i.e. for all times) existence of the solution to the finite volume equation follows from an extension of the argument in [12] which introduces some of the techniques necessary for the infinite volume problem in the next section. This is established in the following series of propositions and lemmas.

**Proposition 1.** *There exists $T_0 > 0$ so that there is a continuous bounded solution to*

$$\varphi_\Lambda = -G\Lambda \, V'(\varphi_\Lambda) + \Lambda \, F$$

*on $R^d \times [0, T_0]$.*

*Proof.* Note that

$$|V'(\varphi)| \leqq |\varphi| H(|\varphi|), \tag{3.5}$$

where $H$ is a polynomial of degree $N-2$ with positive coefficients. Since $\Lambda F$ is bounded on $R^d \times [0, T]$, there is a $T_a > 0$ so that

$$2^{N-1} T_0 H\left( \sup_{0 \leqq t \leqq T_0} \sup_x |\Lambda(x) F(x,t)| \right) \leqq 1. \tag{3.6}$$

Then it is straightforward to see that the map

$$\phi \longrightarrow -G\Lambda V'(\phi) + \Lambda F \tag{3.7}$$

takes the closed set $\{\phi \in C(R^d \times [0, T_0]) \colon \|\phi\|_\infty \leqq 2\|\Lambda F\|_\infty\}$ into itself. The function $V'$ is Lipschitz continuous on this set so the Picard iterates

$$\varphi_{\Lambda,0} = \Lambda F; \quad \varphi_{\Lambda,n+1} = -G\Lambda V'(\varphi_{\Lambda,n}) + \Lambda F \tag{3.8}$$

converge in $C(R^d \times [0, T_0])$ to the unique continuous solution [15]. ///

We now seek to establish the existence of this solution for all times. Let $T^*$ be the largest time such that there is a continuous solution to (3.4) on $R^d \times [0, T^*)$. Then either $T^* = \infty$ or not. If $T^* < \infty$, then it must be that

$$\sup_{0 \leqq t < T^*} \sup_x |\varphi_\Lambda(x,t)| = \infty, \tag{3.9}$$

for otherwise the map

$$\varphi \longrightarrow -G\Lambda\varphi[V'(\varphi_\Lambda)/\varphi_\Lambda] + \Lambda F \tag{3.10}$$

is Lipschitz from $C(R^d \times [0, T^*])$ into itself. The continuous fixed point of (3.10) is also a solution of the finite volume equation, so we could use $\varphi_\Lambda(x, T^*)$ as in initial condition to repeat the argument of Proposition 1 and extend the solution to a time greater that $T^*$ (the proof above only used the fact that $\Lambda F$ was bounded, not that it was of compact support in the spatial directions). Hence to prove that there is a global solution it is sufficient to show that if $T^* < \infty$ the solution is bounded on $R^d \times [0, T^*)$. This is established by the following six lemmas.

**Lemma 1.** *If $f(x,t)$ and $(\partial/\partial t - \Delta) f(x,t)$ are continuous on $R^d \times [0, T]$ and $|f|$, $|\nabla f| \in L^{2n+2}(R^d \times [0, T])$, then*

$$\int_0^T dt \int dx\, f^{2n+1}(\partial/\partial t - \Delta) f = (2n+2)^{-1}\left[\int dx\, f(x,T)^{2n+2} - \int dx\, f(x,0)^{2n+2}\right]$$
$$+ (2n+1)\int_0^T dt \int dx\, f^{2n}(\nabla f)^2. \tag{3.11}$$

*Proof.* Integration by parts. ///

**Lemma 2.** *If $\varphi_\Lambda$ is a continuous solution to*

$$\varphi_\Lambda = -G\Lambda V'(\varphi_\Lambda) + \Lambda F$$

*on $R^d \times [0, T]$, then $|\varphi_\Lambda|$ and $|\nabla\varphi_\Lambda|$ vanish exponentially as $|x| \to \infty$, uniformly in $t \leqq T$.*

*Proof.* (The need for this lemma is the reason for the $\Lambda$ in $G\Lambda V'$.) Suppose $x \notin \operatorname{supp}\Lambda$ ($\operatorname{supp}\Lambda$ is the support of the function $\Lambda(x)$). Then

$$|\varphi_\Lambda(x,t)| \leqq \int_0^t dt' \int dx'\, G(x-x', t-t')|\Lambda(x') V'(\varphi_\Lambda(x',t'))|$$

$$\leqq \sup_{t''\leqq T}\sup_{x''}|\Lambda(x'')V'(\varphi_\Lambda(x'',t''))|\int_0^t dt'\int dx'\chi_{\operatorname{supp}\Lambda}(x')G(x-x',t-t')$$

$$\leqq \sup_{t''\leqq T}\sup_{x''}|\Lambda(x'')V'(\varphi_\Lambda(x'',t''))|\operatorname{vol}(\operatorname{supp}\Lambda)$$

$$\times\int_0^t dt'[4\pi(t-t')]^{d/2}\exp\{-d(x,\operatorname{supp}\Lambda)^2/4(t-t')\}, \tag{3.12}$$

where vol(supp $\Lambda$) is the Lebesgue measure of the support of $\Lambda$ and $d(x, \operatorname{supp}\Lambda)$ is the distance from $x$ to the support. An obvious change of variables in the integral above implies

$$|\varphi_\Lambda(x,t)|\leqq\int_{1/(4T)}^{\infty} du\, u^{(d-4)/2}\exp\{-ud(x,\operatorname{supp}\Lambda)^2\}. \tag{3.13}$$

For $d\leqq 4$, this yields

$$|\varphi_\Lambda(x,t)|\leqq(\text{constant})\cdot T^{(4-d)/2}\cdot d(x,\operatorname{supp}\Lambda)^{-2}\cdot\exp\{-d(x,\operatorname{supp}\Lambda)^2/4T\}. \tag{3.14}$$

For $d>4$, simply integrate (3.13) by parts several times and note that there is an exponentially decaying expression in each term. The proof for $|\nabla\varphi_\Lambda|$ is almost exactly the same, noting that if $x\notin\operatorname{supp}\Lambda$,

$$|\nabla\varphi_\Lambda(x,t)|\leqq\int_0^t dt'\int dx'[2(t-t')]^{-1}|x-x'|G(x-x',t-t')|\Lambda(x')V'(\varphi_\Lambda(x',t'))|. \tag{3.15}$$

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**Lemma 3.** *If $\varphi_\Lambda$ is a continuous solution to*

$$\varphi_\Lambda=-G\Lambda V'(\varphi_\Lambda)+\Lambda F$$

*on $R^d\times[0,T]$, then $\varphi_\Lambda$ is in every $L^p(R^d\times[0,T],\Lambda(x)\,dx\,dt)$, $1\leqq p<\infty$.*

*Proof.* Denote by $\|\cdot\|_p^{T,\Lambda}$ the norm. Then note that

$$(\partial/\partial t-\Delta+m^2)(\varphi_\Lambda-\Lambda F)=-\Lambda V'(\varphi_\Lambda) \tag{3.16}$$

and

$$\varphi_\Lambda(x,0)-\Lambda(x)F(x,0)\equiv 0. \tag{3.17}$$

By Lemma 2 we may apply Lemma 1 to $(\varphi_\Lambda-\Lambda F)$ to find that

$$0\leqq\int_0^T dt\int dx(\varphi_\Lambda-\Lambda F)^{2n+1}(\partial/\partial t-\Delta+m^2)(\varphi_\Lambda-\Lambda F)$$

$$=-\int_0^T dt\int\Lambda(x)dx(\varphi_\Lambda-\Lambda F)^{2n+1}V'(\varphi_\Lambda). \tag{3.18}$$

The polynomial $V'$ is written

$$V'(\varphi_\Lambda)=\sum_{k=1}^{N-1}a_k\varphi_\Lambda^{\,k}, \tag{3.19}$$

where $a_{N-1} > 0$. Then (3.18) says

$$0 \leqq -\int_0^T dt \int \Lambda(x)\, dx \left[ \sum_{m=0}^{2n+1} \binom{2n+1}{m} \varphi_\Lambda^{2n+1-m} (-\Lambda F)^m \right] \times \left[ \sum_{k=1}^{N-1} a_k \varphi_\Lambda^k \right]. \tag{3.20}$$

Take the highest order term above to the left side to get the estimate

$$a_{N-1} \int_0^T dt \int \Lambda(x) dx\, \varphi_\Lambda^{2n+N} \leqq \int_0^T dt \int \Lambda(x)\, dx \left[ a_{N-1} \sum_{m=1}^{2n+1} \binom{2n+1}{m} \varphi_\Lambda^{2n+N-m} |\Lambda F|^m \right.$$
$$\left. + \sum_{k=1}^{N-2} |a_k| |\varphi_\Lambda|^{2n+k+1} + \sum_{k=1}^{N-2} |a_k| \sum_{m=1}^{2n+1} \binom{2n+1}{m} |\varphi_\Lambda|^{2n+k+1-m} |\Lambda F|^m \right]. \tag{3.21}$$

Using Hölder's inequality on each term on the right-hand above,

$$\begin{aligned} & a_{N-1} (\| \varphi_\Lambda \|_{2n+N}{}^{T,\Lambda})^{2n+N} \\ & \leqq \sum_{k=1}^{N-2} |a_k| (\| 1 \|_{2n+N}{}^{T,\Lambda})^{N-k-1} (\| \varphi_\Lambda \|_{2n+N}{}^{T,\Lambda})^{2n+k+1} \\ & \quad + a_{N-1} \sum_{m=1}^{2n+1} \binom{2n+1}{m} (\| \Lambda F \|_{2n+N}{}^{T,\Lambda})^m (\| \varphi_\Lambda \|_{2n+N}{}^{T,\Lambda})^{2n+N-m} \\ & \quad + \sum_{k=1}^{N-2} |a_k| \sum_{m=1}^{2n+1} \binom{2n+1}{m} (\| \Lambda F \|_{p(m,k)}{}^{T,\Lambda})^m (\| \varphi_\Lambda \|_{2n+N}{}^{T,\Lambda})^{2n+k+1-m}, \end{aligned} \tag{3.22}$$

where $p(m,k) = (2n+N)m/(N+m-k-1)$. A close inspection of the inequality above will reveal that the $(2n+N)^{\text{th}}$ power of $\| \varphi_\Lambda \|_{2n+N}{}^{T,\Lambda}$ is bounded by a polynomial $P$ of degree $2n+N-1$ in $\| \varphi_\Lambda \|_{2n+N}{}^{T,\Lambda}$. This means that there is some finite constant $C$ (depending on the coefficient of the polynomial $P$: the $a_k$'s, $N, n, T$, and $\Lambda F$) so that

$$\| \varphi_\Lambda \|_{2n+N}{}^{T,\Lambda} \leqq C < \infty. \tag{3.23}$$

Since this holds for any $n$, the lemma is proved. ///

The lemma above really follows trivially from the assumptions, but the usefulness of the estimates obtained will become apparent in the next proof.

**Lemma 4.** *If $\varphi_\Lambda$ is a continuous solution to*

$$\varphi_\Lambda = -G\Lambda V'(\varphi_\Lambda) + \Lambda F$$

*on $R^d \times [0, T^*)$, then $\varphi_\Lambda$ is in every $L^p(R^d \times [0, T^*), \Lambda(x)\, dx\, dt), p < \infty$.*

*Proof.* For each $T < T^*$, we have the inequality

$$(\| \varphi_\Lambda \|_{2n+N}{}^{T,\Lambda})^{2n+N} \leqq P(\| \varphi_\Lambda \|_{2n+N}{}^{T,\Lambda}), \tag{3.24}$$

where $P$ is the polynomial of degree $2n+N-1$ defined in the proof of Lemma 3. Note that the coefficients of $P$ are all bounded at $T \uparrow T^*$. Hence the inequality holds at $T^*$ and there is a finite constant $C^*$ (depending on the coefficients of $P$

at $T^*$) so that

$$\|\varphi_\Lambda\|_{2n+N}^{T^*,\Lambda} \leqq C^* < \infty. \tag{3.25}$$

///

**Lemma 5.** *The kernal $G(x,t)$ is in $L^p(R^d \times [0,T])$ for any $T < \infty$ and $p < 1 + 2/d$.*

*Proof.* Compute

$$\int_0^T dt \int dx\, G(x,t)^p = \int_0^T dt \int dx\, (4\pi t)^{-pd/2} \exp\{-pm^2 t - p|x|^2/4t\}$$

$$\leqq (\text{constant}) \cdot \int_0^T dt\, t^{d(1-p)/2}. \tag{3.26}$$

This is finite for $p < 1 + 2/d$. ///

**Lemma 6.** *If $\varphi_\Lambda$ is a continuous solution to*

$$\varphi_\Lambda = -G\Lambda V'(\varphi_\Lambda) + \Lambda F$$

*on $R^d \times [0, T^*)$, then it is bounded there.*

*Proof.* Let $p = 1 + 1/d$ and $q = d + 1$. Then

$$\begin{aligned} |\varphi_\Lambda(x,t)| &\leqq \left| \int_0^t dt' \int dx'\, G(x-x', t-t')\Lambda(x') V'(\varphi_\Lambda(x',t')) \right| + |\Lambda(x) F(x,t)| \\ &\leqq \int_{-\infty}^{\infty} dt' \int dx'\, \chi_{[0,T^*]}(t-t') G(x-x', t-t') \chi_{[0,T^*]}(t') |\Lambda(x') V'(\varphi_\Lambda(x',t'))| \\ &\quad + |\Lambda(x) F(x,t)|, \end{aligned} \tag{3.27}$$

where $\chi_{[0,T^*]}$ is the indicator function of the interval $[0, T^*]$. By Young's inequality,

$$\|\chi_{[0,T^*]}\varphi_\Lambda\|_\infty \leqq \|\chi_{[0,T^*]} G\|_p \|\chi_{[0,T^*]}\Lambda V'(\varphi_\Lambda)\|_q + \|\chi_{[0,T^*]}\Lambda F\|_\infty. \tag{3.28}$$

From Lemma 5, $\|\chi_{[0,T^*]} G\|_p$ is finite. Since $0 \leqq \Lambda \leqq 1$,

$$\|\chi_{[0,T^*]}\Lambda V'(\varphi_\Lambda)\|_q \leqq \|V'(\varphi_\Lambda)\|_q^{T^*,\Lambda}. \tag{3.29}$$

This is finite by Lemma 4. ///

We are now ready to state the following proposition.

**Proposition 2.** *Under the assumptions on $F$, there is a continuous solution to*

$$\varphi_\Lambda = -G\Lambda V'(\varphi_\Lambda) + \Lambda F$$

*on $R^d \times [0, \infty)$.*

*Proof.* This follows from Lemma 6 and the remarks made earlier. ///

The existence proof give in this section is also applicable to the case of the parabolic stochastic differential equation

$$(\partial/\partial t - \Delta)\varphi(x,t) = -V'(\varphi(x,t)) + \eta_c(x,t) \tag{3.30}$$

in a compact region of $R^d$ with Dirichlet boundary conditions. Then with probability one $W_c$ will satisfy the boundary conditions, and since the Green's function with Dirichlet boundary conditions is pointwise bounded by the infinite volume Green's function, all the estimates above are valid.

## 4. Infinite Volume Equations

The infinite volume of limit $\varphi_\Lambda$ will now be established in the Banach spaces $L^p(R^d \times [0, \infty), d\mu_\alpha)$, where $\mu_\alpha$ is the measure of finite total weight on $R^d \times R_+$ given by (for $\alpha > 0$)

$$d\mu_\alpha(x,t) = \exp\{-\alpha|x| - \alpha^2 t\}\, dx\, dt. \tag{4.1}$$

These spaces are really the natural ones in which to consider the problem at hand because, as will be shown, the essential positivity properties of the parabolic operator $(\partial/\partial t - \Delta)$ are preserved while the problems of the noncompactness of the domain and the unboundedness of the driving force function are circumvented by the damping power of the measure. The fundamental lemma is

**Lemma 7.** *If $f(x,t)$ and $(\partial/\partial t - \Delta)\, f$ are continuous on $R^d \times [0, T]$ and $|f|^{2n+2}$ and $|\nabla f|^{2n+2}$ are in $L^1(R^d \times [0, T], d\mu_\alpha)$, then*

$$\begin{aligned}
&\int d\mu_\alpha \chi_{[0,T]} f^{2n+1} (\partial/\partial t - \Delta) f \\
&= (2n+2)^{-1} \int dx\, e^{-\alpha|x|} [\exp\{-\alpha^2 T\}\, f(x,T)^{2n+2} - f(x,0)^{2n+2}] \\
&\quad + (2n+1) \int d\mu_\alpha \chi_{[0,T]} f^{2n} |\nabla f|^2 \\
&\quad + \begin{cases} (2n+2)^{-1} \alpha(d-1) \int d\mu_\alpha(x,t) \chi_{[0,T]}(t) |x|^{-1} f(x,t)^{2n+2}; & d \geqq 2 \\ (2n+2)^{-1} 2\alpha \int dt\, \chi_{[0,T]}(t) f(0,t)^{2n+2} & ; \quad d = 1. \end{cases}
\end{aligned} \tag{4.2}$$

*Proof.* Integration by parts, noting that

$$\Delta e^{-\alpha|x|} = \alpha^2 e^{-\alpha|x|} + \begin{cases} -\alpha(d-1)|x|^{-1} e^{-\alpha|x|}; & d \geqq 2 \\ -2\alpha\delta(x) & ; \quad d = 1. \end{cases} \tag{4.3}$$

///

It should be mentioned that Marcus [11] worked with a measure quite close to the one used here for the one dimensional problem: he allowed exponential decay in the spatial direction but none in the $t$-direction. This forced him to require that $m > 0$ and $\alpha < m/2$, restrictions which are not necessary with the measures $\mu_\alpha$ considered here. The point now is that with the aid of this lemma, we may find bounds on the $L^p(d\mu_\alpha)$ norms of $\varphi_\Lambda$ that are *uniform* in $\Lambda$. We will denote by $\|\cdot\|_p^{\Lambda,T,\alpha}$ and $\|\cdot\|_p^\alpha$ the norms on $L^p(R^d \times [0, T], \Lambda\, d\mu_\alpha)$ and $L^p(R^d \times [0, \infty), d\mu_\alpha)$ respectively.

**Lemma 8.** *If $\varphi_\Lambda$ satisfies*

$$\varphi_\Lambda = -G\Lambda\, V'(\varphi_\Lambda) + \Lambda F,$$

*then $\varphi_\Lambda$ is bounded in every $L^p(R^d \times [0, T], \Lambda\, d\mu_\alpha)$ and $\Lambda\varphi_\Lambda$ is bounded in every $L^p(R^d \times [0, \infty), d\mu_\alpha)$ uniformly in $\Lambda\, (p < \infty)$.*

*Proof.* Note that

$$(\partial/\partial t - \Delta + m^2)(\varphi_\Lambda - \Lambda F) = -\Lambda\, V'(\varphi_\Lambda), \quad \varphi_\Lambda(x,0) - \Lambda(x) F(x,0) \equiv 0, \tag{4.4}$$

and $|\varphi_\Lambda|$ and $|\nabla\varphi_\Lambda|$ decay exponentially as $|x|\to\infty$ for finite times. Thus for any $T<\infty$ we have by Lemma 7,

$$\begin{aligned}0 &\leqq \int d\mu_\alpha \chi_{[0,T]}(\varphi_\Lambda - \Lambda F)^{2n+1}(\partial/\partial t - \Delta + m^2)(\varphi_\Lambda - \Lambda F)\\ &= -\int d\mu_\alpha \chi_{[0,T]}\Lambda(\varphi_\Lambda - \Lambda F)^{2n+1} V'(\varphi_\Lambda).\end{aligned} \tag{4.5}$$

Then, by a calculation exactly like that the in Lemma 3,

$$\begin{aligned}&a_{N-1}(\|\varphi_\Lambda\|_{2n+N}{}^{\Lambda,T,\alpha})^{2n+N}\\ &\leqq \sum_{k=1}^{N-2}|a_k|(\|1\|_{2n+N}{}^{\Lambda,T,\alpha})^{N-k-1}(\|\varphi_\Lambda\|_{2n+N}{}^{\Lambda,T,\alpha})^{2n+k+1}\\ &\quad + a_{N-1}\sum_{m=1}^{2n+1}\binom{2n+1}{m}(\|\Lambda F\|_{2n+N}{}^{\Lambda,T,\alpha})^m(\|\varphi_\Lambda\|_{2n+N}{}^{\Lambda,T,\alpha})^{2n+N-m}\\ &\quad + \sum_{k=1}^{N-2}|a_k|\sum_{m=1}^{2n+1}\binom{2n+1}{m}(\|\Lambda F\|_{p(m,k)}{}^{\Lambda,T,\alpha})^m(\|\varphi_\Lambda\|_{2n+N}{}^{\Lambda,T,\alpha})^{2n+N-m},\end{aligned} \tag{4.6}$$

where $p(m,k)=(2n+N)m/(N+m-k-1)$. The coefficients of the polynomial of degree $2n+N-1$ above are bounded as both $T\to\infty$ and $\Lambda\uparrow 1$:

$$\|1\|_p{}^{\Lambda,T,\alpha}\uparrow\|1\|_p{}^\alpha<\infty,\quad \|\Lambda F\|_p{}^{\Lambda,T,\alpha}\uparrow\|F\|_p{}^\alpha<\infty. \tag{4.7}$$

Hence there is again a finite constant $C$ which depends only on $p,\alpha$, the $a_k$'s and $F$ such that

$$\|\varphi_\Lambda\|_p{}^{\Lambda,T,\alpha}\leqq C<\infty. \tag{4.8}$$

Since $\Lambda^p\leqq\Lambda$ for any $p\geqq 1$,

$$\|\Lambda\varphi_\Lambda\|_p{}^\alpha\leqq\lim_{T\to\infty}\|\varphi_\Lambda\|_p{}^{\Lambda,T,\alpha}<\infty. \tag{4.9}$$

///

**Lemma 9.** *$G$ is a bounded map from*

a) *$L^p(d\mu_\alpha)$ to $L^p(d\mu_\alpha)$ if $p>1$,*
b) *$L^1(\chi_{[0,T]}d\mu_\alpha)$ to $L^1(\chi_{[0,T]}d\mu_\alpha)$ if $T<\infty$,*
c) *$L^1(d\mu_\alpha)$ to $L^1(d\mu_\alpha)$ if $m^2>0$.*

*Proof.* Let $g\in L^p(d\mu_\alpha)$ and $f=Gg$. Then

$$\chi_{[0,T]}(t)|f(x,t)|\leqq\int_{-\infty}^{\infty}dt'\int dx'\chi_{[0,T]}(t-t')G(x-x',t-t')\chi_{[0,T]}(t')|g(x',t')|. \tag{4.10}$$

Since

$$e^{|a|-|b|}\leqq e^{|a-b|}, \tag{4.11}$$

we have

$$\begin{aligned}&\chi_{[0,T]}(t)\exp\{-\alpha^2t/p-\alpha|x|/p\}|f(x,t)|\\ &\leqq\int dt'\int dx'\chi_{[0,T]}(t-t')G(x-x',t-t')\\ &\quad\times\exp\{-\alpha^2(t-t')/p+\alpha|x-x'|/p\}\\ &\quad\times\chi_{[0,T]}(t')|g(x',t')|\exp\{-\alpha^2t'/p-\alpha|x'|/p\}.\end{aligned} \tag{4.12}$$

Then Young's inequality says

$$\|\chi_{[0,T]}f\|_p^{\alpha} \leqq \int dt \int dx\, \chi_{[0,T]}(t)G(x,t)\exp\{-\alpha^2 t/p + \alpha|x|/p\}\, \|\chi_{[0,T]}g\|_p^{\alpha}. \tag{4.13}$$

It is straightforward to check that the operator norm $\|G\|_{p\to p}^{T,\alpha}$ satisfies

$$\begin{aligned}\|G\|_{p\to p}^{T,\alpha} &\leqq \int dt \int dx\, \chi_{[0,T]}(t)G(x,t)\exp\{-\alpha^2 t/p + \alpha|x|/p\}\\ &\leqq (\text{constant})\cdot \int dt\, \chi_{[0,T]}(t)\exp\{-(m^2+(1-1/p)\alpha^2/p)t\}\\ &\quad \times \int_{-\infty}^{\infty} du(|u| + \alpha t^{1/2}/p)^{d-1}\exp\{-u^2\},\end{aligned} \tag{4.14}$$

which is finite for all $T<\infty$, and finite for $T=\infty$ if either $p>1$ or $m^2>0$. ///

**Lemma 10.** *If $\varphi_\Lambda$ satisfies*

$$\varphi_\Lambda = -G\Lambda V'(\varphi_\Lambda) + \Lambda F,$$

*then it is bounded uniformly in $\Lambda$ in every $L^p(d\mu_\alpha)$ $(1 \leqq p < \infty)$.*

*Proof.* Without loss of generality we may take $p>1$. Then, by the preceding lemma,

$$\|\varphi_\Lambda\|_p^{\alpha} \leqq \|G\|_{p\to p}^{T,\alpha}\cdot \|\Lambda V'(\varphi_\Lambda)\|_p^{\alpha} + \|F\|_p^{\alpha}. \tag{4.15}$$

By Lemma 8, $\|\Lambda V'(\varphi_\Lambda)\|_p^{\alpha}$ is bounded uniformly in $\Lambda$. ///

With this machinery in hand we are now ready to establish the convergence of the finite volume solutions to the infinite volume solutions in all $L^p(d\mu_\alpha)$, $p<\infty$. It is first convenient to consider the case of a monostable potential $V$. This means that $V$ has a unique minimum and $V'$ is monotonically increasing. For such potentials there is a constant $k>0$ so that

$$[V'(\varphi_1) - V'(\varphi_2)](\varphi_1 - \varphi_2) \geqq k(\varphi_1 - \varphi_2)^N. \tag{4.16}$$

For example, if

$$V'(\varphi) = \varphi^3 + \varphi^2 + \varphi, \tag{4.17}$$

then

$$\begin{aligned}&[V'(\varphi_1) - V'(\varphi_2)](\varphi_1 - \varphi_2)\\ &\quad = (1/4)(\varphi_1 - \varphi_2)^4 + [(3/4)(\varphi_1+\varphi_2)^2 + (\varphi_1+\varphi_2) + 1]\cdot(\varphi_1 - \varphi_2)^2\\ &\quad \geqq (1/4)(\varphi_1 - \varphi_2)^4.\end{aligned} \tag{4.18}$$

since

$$\inf_{u\in R}[(3/4)u^2 + u + 1] = 4/3 \geqq 0. \tag{4.19}$$

The procedure we follow is to show that the family $\Lambda\varphi_\Lambda$ is Cauchy in each $L^p(d\mu_\alpha)$ and that the limit is a continuous solution of the infinite volume equation.

**Lemma 11.** *If $V$ is monostable, then $\Lambda\varphi_\Lambda$ is Cauchy in each $L^p(d\mu_\alpha)$, $p<\infty$, as $\Lambda\uparrow 1$.*

*Proof.* Let $\varphi_\Lambda$ and $\varphi_{\Lambda'}$ satisfy

$$\varphi_\Lambda = -G\Lambda V'(\varphi_\Lambda) + \Lambda F, \quad \varphi_{\Lambda'} = -G\Lambda' V'(\varphi_{\Lambda'}) + \Lambda' F. \tag{4.20}$$

Then

$$(\partial/\partial t - \Delta + m^2)[(\varphi_{\Lambda'} - \varphi_\Lambda) - (\Lambda' - \Lambda)F] = -[\Lambda' V'(\varphi_{\Lambda'}) - \Lambda V'(\varphi_\Lambda)] \quad (4.21)$$

and, by Lemma 7,

$$\begin{aligned} 0 &\leqq \int d\mu_\alpha [(\varphi_{\Lambda'} - \varphi_\Lambda) - (\Lambda' - \Lambda)F]^{2n+1}(\partial/\partial t - \Delta + m^2)[(\varphi_{\Lambda'} - \varphi_\Lambda) - (\Lambda' - \Lambda)F] \\ &= -\int d\mu_\alpha [(\varphi_{\Lambda'} - \varphi_\Lambda) - (\Lambda' - \Lambda)F]^{2n+1}[\Lambda' V'(\varphi_{\Lambda'}) - \Lambda V'(\varphi_\Lambda)] \\ &= -\int d\mu_\alpha [\Lambda' \varphi_{\Lambda'} - \Lambda \varphi_\Lambda]^{2n+1}[V'(\varphi_{\Lambda'}) - V'(\varphi_\Lambda)] + R, \end{aligned} \quad (4.22)$$

where $R$ consists of terms of the form

$$\int d\mu_\alpha (1 - \Lambda')^{n_1}(1 - \Lambda)^{n_2}(\Lambda - \Lambda')^{n_3}\Lambda'^{n_4}\Lambda^{n_5}\varphi_{\Lambda'}{}^{n_6}\varphi_\Lambda{}^{n_7}F^{n_8}, \quad (4.23)$$

with at least one of $n_1, n_2, n_3 > 0$ (and, of course, all $n_i \geqq 0$). Since $\varphi_\Lambda$ and $F$ are uniformly bounded as $\Lambda \uparrow 1$ in every $L^p(d\mu_\alpha)$, $R \to 0$ as $\Lambda, \Lambda' \uparrow 1$. Thus

$$\int d\mu_\alpha [\Lambda' \varphi_{\Lambda'} - \Lambda \varphi_\Lambda]^{2n+N} \leqq R/k \longrightarrow 0. \quad (4.24)$$

///

**Proposition 3.** *If $V$ is monostable, then there is a unique solution to*

$$\varphi = -GV'(\varphi) + F$$

*that is in every $L^p(d\mu_\alpha), p < \infty$.*

*Proof.* (Existence) The family $\Lambda \varphi_\Lambda$ has a limit in $L^p(d\mu_\alpha)$, call it $\varphi$. We need to check that $\varphi$ satisfies the integral equation, but this is trivial since $\Lambda V'(\varphi_\Lambda)$ converges to $V'(\varphi)$ in $L^p(d\mu_\alpha)$ and $G$ is closed.

(Uniqueness) Let $\varphi_1$ and $\varphi_2$ be two solutions. Then by Lemma 7, for any $n$ we have

$$\begin{aligned} 0 &\leqq \int d\mu_\alpha (\varphi_1 - \varphi_2)^{2n+1}(\partial/\partial t - \Delta + m^2)(\varphi_1 - \varphi_2) \\ &= -\int d\mu_\alpha (\varphi_1 - \varphi_2)^{2n+1}[V'(\varphi_1) - V'(\varphi_2)] \\ &\leqq -k\int d\mu_\alpha (\varphi_1 - \varphi_2)^{2n+N}. \end{aligned} \quad (4.25)$$

Now choose $n > (p - N)/2$. Then $\|\varphi_1 - \varphi_2\|_p{}^\alpha = 0$. ///

Everything but the continuity of the solution has been established at this point. The finite volume solutions also converge, however, in the Banach space of continuous functions $C^\alpha$ with norm

$$\|f\|_\infty{}^\alpha = \sup_{t \geqq 0} \sup_x \exp\{-\alpha^2 t - \alpha|x|\}|f(x,t)|. \quad (4.26)$$

(Note that this is *not* the norm on $L^\infty(d\mu_\alpha)$: since $\mu_\alpha$ and Lebesgue measure have the same negligible sets, $L^\infty(d\mu_\alpha) = L^\infty$ (Lebesgue).) This is established by the following lemma:

**Lemma 12.** *$G$ is a bounded map from $L^p(d\mu_\alpha)$ into $C^\alpha$ for all $p > 1 + d/2$.*

*Proof.* Without loss of generality we may take $m^2 > 0$. Let $g \in L^p(d\mu_\alpha)$, $f = Gg$ and $1/p + 1/q = 1$. Then

$$\begin{aligned}
&\exp\{-\alpha^2 t-\alpha|x|\}|f(x,t)| \\
&\quad\leqq \int dt' \int dx' \chi_{[0,\infty]}(t-t')G(x-x',t-t')\exp\{-\alpha^2(t-t')+\alpha|x-x'|\} \\
&\qquad\times \chi_{[0,\infty]}(t')|g(x',t')|\exp\{-\alpha^2 t'-\alpha|x'|\} \\
&\quad\leqq \{\int dt \int dx\, \chi_{[0,\infty]}(t)G(x,t)^q \exp\{-q\alpha^2 t+q\alpha|x|\}\}^{1/q} \\
&\qquad\times \{\int dt \int dx\, \chi_{[0,\infty]}(t)|g(x',t')|^p \exp\{-p\alpha^2 t'-p\alpha|x'|\}\}^{1/p}
\end{aligned} \tag{4.27}$$

by Young's inequality. Since $p>1$, the second factor above is less than $\|g\|_p{}^\alpha$. Straightforward estimates show that

$$\begin{aligned}
&\int dt \int dx\, \chi_{[0,\infty]}(t)G(x,t)^q \exp\{-q\alpha^2 t+q\alpha|x|\} \\
&\quad\leqq (\text{constant})\cdot \int_0^\infty dt\, t^{d(1-q)/2}\exp\{-qm^2 t\}\int_{-\infty}^{\infty} du[|u| \\
&\qquad + q^{1/2}t^{1/2}\alpha]^{d-1}\exp\{-u^2\},
\end{aligned} \tag{4.28}$$

which is finite for any $q<1+2/d$. This means $p>1+d/2$. ///

Since $\Lambda F\to F$ in $C^\alpha$ and $\Lambda V'(\varphi_\Lambda)\to V'(\varphi)$ in every $L^p(d\mu_\alpha)$, we have

$$\|\Lambda\varphi_\Lambda-\varphi\|_\infty{}^\alpha \longrightarrow 0 \quad \text{as} \quad \Lambda\uparrow 1. \tag{4.29}$$

As convergence in $C^\alpha$ implies uniform convergence on compact sets and each $\Lambda\varphi_\Lambda$ is continuous, $\varphi$ is continuous. Thus we have proven.

**Theorem.** *If $V$ is a monostable polynomial potential and $F$ is in $C^\alpha$ and $L^p(d\mu_\alpha)$, $p<\infty$, then the equation*

$$\varphi = -GV'(\varphi)+F$$

*has a unique continuous solution in $C^\alpha$ and $L^p(d\mu_\alpha)$, $p<\infty$.*

The procedure used above for $V$ monostable will not work directly for $V$ multistable. Consider, for example,

$$V'(\varphi)=\varphi^3-\varphi. \tag{4.30}$$

Then the best estimate we can make along the lines of (4.18) is

$$[V'(\varphi_1)-V'(\varphi_2)](\varphi_1-\varphi_2)\geqq(1/4)(\varphi_1-\varphi_2)^4-(\varphi_1-\varphi_2)^2. \tag{4.31}$$

Performing the same calculation as in Lemma 11 with two finite volume solutions $\varphi_\Lambda$ and $\varphi_{\Lambda'}$, we find

$$\int d\mu_\alpha[\Lambda'\varphi_{\Lambda'}-\Lambda\varphi_\Lambda]^{2n+4}\leqq\int d\mu_\alpha[\Lambda'\varphi_{\Lambda'}-\Lambda\varphi_\Lambda]^{2n+2}+R, \tag{4.32}$$

where, as before, $R$ consists of terms which vanish as $\Lambda,\Lambda'\uparrow 1$. We are not able to conclude that the difference vanishes, but only that it remains bounded in every $L^p(d\mu_\alpha)$. It is clear that this is the case for any multistable potential. We noticed no difference between mono- and multi-stable potentials in finite volume, but it seems we may have a nonexistence or nonuniqueness problem in infinite volume when a phase transition is possible. In fact, this is *not* the case. The infinite volume limit here just requires a more general argument than that above.

For any globally stable polynomial potential $V(\varphi)$ with an extremum at $\varphi=0$, the addition of a large enough quadratic term will change it into a monostable potential. Equivalently, for any polynomial $V'(\varphi)$ that is eventually monotone increasing, the addition of a large enough linear term will render the sum monotone increasing everywhere. We may effectively accomplish this for the problem at hand by multiplying the finite volume equation by a factor $e^{-\beta t}$. In general,

$$e^{-\beta t}\partial f/\partial t = \partial(e^{-\beta t}f)/\partial t + \beta e^{-\beta t}f. \tag{4.33}$$

The finite volume solutions satisfy

$$(\partial/\partial t - \Delta + m^2)(\varphi_\Lambda - \Lambda F) = -\Lambda V'(\varphi_\Lambda), \tag{4.34}$$

so that, multiplying by $e^{-\beta t}$,

$$(\partial/\partial t - \Delta + m^2)[e^{-\beta t}(\varphi_\Lambda - \Lambda F)] + \beta e^{-\beta t}(\varphi_\Lambda - \Lambda F) = -e^{-\beta t}\Lambda V'(\varphi_\Lambda). \tag{4.35}$$

Rearranging,

$$\begin{aligned}(\partial/\partial t &- \Delta + m^2)[e^{-\beta t}(\varphi_\Lambda - \Lambda F)]\\ &= -e^{-\beta t}[\Lambda V'(\varphi_\Lambda) + \beta\varphi_\Lambda] + \beta e^{-\beta t}\Lambda F\\ &= -e^{-\beta t}\Lambda[V'(\varphi_\Lambda) + \beta\varphi_\Lambda] - \beta e^{-\beta t}(1-\Lambda)\varphi_\Lambda + \beta e^{-\beta t}\Lambda F.\end{aligned} \tag{4.36}$$

By choosing $\beta$ large enough, the term $V'(\varphi_\Lambda) + \beta\varphi_\Lambda$ becomes monotone increasing in $\varphi_\Lambda$. We are now ready to state

**Lemma 13.** *For $\beta$ large enough, $e^{-\beta t}\Lambda\varphi_\Lambda$ is Cauchy in each $L^p(d\mu_\alpha)$, $p<\infty$, as $\Lambda \uparrow 1$.*

*Proof.* Let $\varphi_\Lambda$ and $\varphi_{\Lambda'}$ satisfy

$$\varphi_\Lambda = -G\Lambda V'(\varphi_\Lambda) + \Lambda F, \quad \varphi_{\Lambda'} = -G\Lambda' V'(\varphi_{\Lambda'}) + \Lambda' F. \tag{4.37}$$

Then

$$\begin{aligned}(\partial/\partial t &- \Delta + m^2)[e^{-\beta t}(\varphi_{\Lambda'} - \varphi_\Lambda) - e^{-\beta t}(\Lambda' - \Lambda)F)]\\ &= -e^{-\beta t}[\Lambda(V'(\varphi_\Lambda) + \beta\varphi_\Lambda) - \Lambda'(V'(\varphi_{\Lambda'}) + \beta\varphi_{\Lambda'})]\\ &\quad - \beta e^{-\beta t}[(1-\Lambda')\varphi_{\Lambda'} - (1-\Lambda)\varphi_\Lambda] + \beta e^{-\beta t}(\Lambda' - \Lambda)F.\end{aligned} \tag{4.38}$$

By Lemma 7,

$$\begin{aligned}0 &\leqq \int d\mu_\alpha[e^{-\beta t}(\varphi_{\Lambda'} - \varphi_\Lambda) - e^{-\beta t}(\Lambda' - \Lambda)F]^{2n+1}\\ &\quad\times(\partial/\partial t - \Delta + m^2)[e^{-\beta t}(\varphi_{\Lambda'} - \varphi_\Lambda) - e^{-\beta t}(\Lambda' - \Lambda)F]\\ &= -\int d\mu_\alpha e^{-(2n+2)\beta t}[(\varphi_{\Lambda'} - \varphi_\Lambda) - (\Lambda' - \Lambda)F]^{2n+1}\\ &\quad\times[\Lambda(V'(\varphi_\Lambda) + \beta\varphi_\Lambda) - \Lambda'(V'(\varphi_{\Lambda'}) + \beta\varphi_{\Lambda'})] + R_1,\end{aligned} \tag{4.39}$$

where $R_1 \to 0$ as $\Lambda, \Lambda' \uparrow 1$, due to the boundedness of $\varphi_{\Lambda'}$, $\varphi_\Lambda$, and $F$ in every $L^p(d\mu_\alpha)$. Choose $\beta$ large enough so that there is a $k>0$ satisfying

$$[(V'(\varphi_\Lambda) + \beta\varphi_\Lambda) - (V'(\varphi_{\Lambda'}) + \beta\varphi_{\Lambda'})](\varphi_{\Lambda'} - \varphi_\Lambda) \geqq k(\varphi_{\Lambda'} - \varphi_\Lambda)^N, \tag{4.40}$$

so that (note that without loss of generality $N>2$)

$$\int d\mu_\alpha e^{-(2n+N)\beta t}(\Lambda'\varphi_{\Lambda'} - \Lambda\varphi_\Lambda)^{2n+N} \leqq R_2/k, \tag{4.41}$$

where $R_2 \to 0$ as $\Lambda, \Lambda' \uparrow 1$. ///

We may now prove the following

**Theorem.** *If $V$ is a polynomial of even degree with positive leading coefficient, and $F$ is in $C^\alpha$ and $L^p(d\mu_\alpha)$, $p<\infty$, then the equation*

$$\varphi = -GV'(\varphi)+F$$

*has a unique continuous solution in $C^\alpha$ and $L^p(d\mu_\alpha)$, $p<\infty$.*

*Proof.* (Existence) By Lemma 13, there is a $\beta$ so that $e^{-\beta t}\Lambda\varphi_\Lambda$ has a limit in every $L^p(d\mu_\alpha)$: call it $e^{-\beta t}\varphi$. Since

$$e^{-\beta t}\varphi_\Lambda = e^{-\beta t}\Lambda\varphi_\Lambda + e^{-\beta t}(1-\Lambda)\varphi_\Lambda, \tag{4.42}$$

and $\varphi_\Lambda$ is uniformly bounded in every $L^p(d\mu_\alpha)$, $e^{-\beta t}\varphi_\Lambda$ converges to $e^{-\beta t}\varphi$ in each $L^p(d\mu_\alpha)$. Choose $\alpha^2>\beta$. Then for $1/p=1/q+1/r$, $1<q<\alpha^2/\beta$,

$$\begin{aligned}\|\varphi_\Lambda-\varphi\|_p^\alpha &= \|e^{\beta t}(e^{-\beta t}\varphi_\Lambda - e^{-\beta t}\varphi)\|_p^\alpha\\ &\leqq \|e^{\beta t}\|_q^\alpha\|(e^{-\beta t}\varphi_\Lambda - e^{-\beta t}\varphi)\|_r^\alpha \longrightarrow 0 \text{ as } \Lambda\uparrow 1,\end{aligned} \tag{4.43}$$

so that $\varphi_\Lambda\to\varphi$ in every $L^p(d\mu_\alpha)$, $\alpha^2>\beta$. Hence $\varphi$ satisfies

$$\varphi = -GV'(\varphi)+F, \tag{4.44}$$

and is continuous by Lemma 12. For $\alpha^2\leqq\beta$, Lemma 10 ensures that the $\varphi_\Lambda$'s are bounded in each $L^p(d\mu_\alpha)$ uniformly in $\Lambda$, so there is a weakly convergent sequence $\varphi_{\Lambda,n}\to\varphi'\in L^p(d\mu_\alpha)$. But $e^{-\beta t}\varphi_{\Lambda,n}\to e^{-\beta t}\varphi$ strongly in $L^p(d\mu_\alpha)$, so $\varphi=\varphi'$ almost everywhere in $R^d\times R_+$ and $\varphi\in L^p(d\mu_\alpha)$. Lemma 12 guarantees that $\varphi\in C^\alpha$ for every $\alpha>0$.

(Uniqueness) Let $\varphi_1$ and $\varphi_2$ be two solutions. Then by multiplying by $e^{-\beta t}$ for a large enough $\beta$, it is easy to see that $e^{-\beta t}\varphi_1 = e^{-\beta t}\varphi_2$ almost everywhere in $R^d\times R_+$. Hence $\varphi_1=\varphi_2$. ///

The theorem above gives exponential bounds on the finite volume solutions. We may also establish polynomial bounds. The measures $\mu_\alpha$ are convenient due to the validity of Lemma 7, but other measures may be utilized with the same result. The analog to Lemma 7 is

**Lemma 14.** *Let $p(x,t)\in L^1(R^d\times[0,T])$ with*

$$\|(\partial p/\partial t+\Delta p)/p\|_\infty<\infty.$$

*Let $f$ and $(\partial/\partial t-\Delta)f$ be continuous with $pf^{2n+2}$, $p|\nabla f|^{2n+2}\in L^1(R^d\times[0,T])$. Then*

$$\begin{aligned}&\int dt\,dx\,\chi_{[0,T]}pf^{2n+1}(\partial/\partial t-\Delta)f\\ &\quad=(2n+2)^{-1}\int dx\,[p(x,T)f(x,T)^{2n+2}-p(x,0)f(x,0)^{2n+2}]\\ &\qquad+(2n+1)\int dt\,dx\,\chi_{[0,T]}pf^{2n}|\nabla f|^2\\ &\qquad-(2n+2)^{-1}\int dt\,dx\,\chi_{[0,T]}(\partial p/\partial t+\Delta p)f^{2n+2}\\ &\quad\geqq-(2n+2)^{-1}\int dx\,p(x,0)f(x,0)^{2n+2}\\ &\qquad-(2n+2)^{-1}\|(\partial p/\partial t+\Delta p)/p\|_\infty\int dt\,dx\,\chi_{[0,T]}pf^{2n+2}.\end{aligned} \tag{4.45}$$

*Proof.* Integration by parts, as in Lemma 7. ///

Choose $p(x,t)$ to be a positive function such as

$$p(x,t)=(1+t)^{-(1+n_1)}(1+|x|)^{-(1+n_2)};\quad n_1,n_2>0. \tag{4.46}$$

Then by calculations like those performed in Lemmas 8 and 10, the finite volume solutions are in every $L^p(p\,dx\,dt)$, $p<\infty$, uniformly in $\Lambda$ as long as the driving force $F$ is. This is the case if the initial condition is in every $L^p(p\,dx\,dt)$, and the rest of the force is given by a sample path of the gaussian process as described in Sect. 2. By mimicking the argument in the proof of the previous theorem, the infinite volume solutions are then seen to be in every $L^p(p\,dx\,dt)$. This establishes the final result summarized as a

**Theorem.** *Let $\eta_c$ be a gaussian process such that $W_c$, defined by*

$$(\partial/\partial t-\Delta+m^2)W_c=\eta_c, \quad W_c(x,0)\equiv 0,$$

*has a Hölder continuous covariance. Let $p(x,t)$ on $R^d\times R_+$ satisfy*

$$\|(\partial p/\partial t+\Delta p)/p\|_\infty<\infty.$$

*Let $\varphi_0(x)\in L^p(p(x,0)\,dx)$, $\forall p<\infty$. Then the equation*

$$(\partial/\partial t-\Delta+m^2)\varphi=-V'(\varphi)+\eta_c, \quad \varphi(x,0)=\varphi_0(x),$$

*has, with probability one, a unique continuous solution which is in every $L^p(p(x,t)\,dx\,dt)$.*

## 5. Asymptotic Expansions and Nonperturbative Bounds

In this section we study the solution to

$$(\partial/\partial t-\Delta+m^2)\varphi=-\lambda V'(\varphi)+\eta_c, \tag{5.1}$$

and show that the expansions of the process and its moments in terms of the nonlinear coupling $\lambda$ are asymptotic as $\lambda\to 0$. The expansion about $\lambda=0$ is the usual coupling constant perturbation expansion around the free process. The ideas behind these expansions are very simple, but the results of the last chapter—specifically the *a priori* bounds on the exact solution—are necessary to bound the remainder terms.

For the coupling constant expansion, Eq. (5.1) is written

$$\varphi=\varphi_0-\lambda GV'(\varphi), \tag{5.2}$$

where $\varphi_0$ is the free process and the term from the initial condition. The expansion is generated by iterating Eq. (5.2),

$$\begin{aligned}\varphi&=\varphi_0-\lambda GV'(\varphi_0-\lambda GV'(\varphi))\\&=\varphi_0-\lambda GV'(\varphi_0-\lambda GV'(\varphi_0-\lambda GV'(\varphi)))=\cdots\text{etc.}\end{aligned} \tag{5.3}$$

Since $V'$ is a polynomial, the iterates above are polynomials in $\lambda$ with coefficients depending on $\varphi_0$ and $\varphi$. The coefficients of $\lambda^0,\ldots,\lambda^n$ in the $n^{\text{th}}$ iteration depend only on $\varphi_0$. For example, if

$$V(\varphi)=(1/4)\varphi^4, \tag{5.4}$$

then the first iteration gives

$$\begin{aligned}\varphi&=\varphi_0-\lambda G\varphi_0{}^3+3\lambda^2G[\varphi_0{}^2G\varphi^3]-3\lambda^3G[\varphi_0(G\varphi^3)^2]+\lambda^4G[(G\varphi^3)^3]\\&=\varphi_0-\lambda G\varphi_0{}^3+\lambda^2R_1.\end{aligned} \tag{5.5}$$

Then remainder term is bounded in every $L^p(d\mu_\alpha)$ almost surely and in mean, uniformly in as $\lambda \to 0$. This is a result of Lemma 9 and a calculation like that in Lemma 8. It is clear that this procedure can be carried on indefinitely, so that for any $n$,

$$\varphi = \sum_{k=0}^{n} \lambda^k \phi_n[\varphi_0] + \lambda^{n+1} R_n[\varphi_0, \varphi], \tag{5.6}$$

and for every $p < \infty$,

$$\lim_{\lambda \to 0} \lambda [\int d\lambda_\alpha E\{|R_n|^p\}]^{1/p} = 0. \tag{5.7}$$

This is sufficient to ensure the asymptotic nature of the perturbative expansion for the correlation functions when the errors are bounded in the spaces $L^p(d\mu_\alpha)$.

Although we have included a mass in the equations above, the result is still valid if $m^2 = 0$. At first this might seem surprising because of the well known "infrared" divergences in the usual functional integral perturbation expansions about massless free fields, but the distinction between that situation and the one here is two-fold. First, we are bounding our error terms in $L^p(d\mu_\alpha)$ which places *no* weight on the equilibrium ($t \to \infty$) configurations of the fields. Second, the extra time variable acts as an infrared regulator before the equilibrium limit is taken. This is obvious from the one- and two-dimensional cases where $\Delta$ has no inverse on an unbounded domain. This nonexistent inverse is the desired covariance for the free massless field in the functional integral formulation. The massless free *process*, however, has a perfectly well defined covariance—it just diverges pointwise as $t \to \infty$. The fact that our measure $d\mu_\alpha$ puts no weight on the interesting (for some applications) equilibrium process is not a problem. Below we will see how to recover equilibrium quantities from $L^p(d\mu_\alpha)$ norms.

Now we turn to calculate nonperturbative bounds on

$$\langle \varphi^N \rangle \equiv \lim_{t \to \infty} E\{\varphi(x,t)^N\} \tag{5.8}$$

under the assumptions of spatial translation invariance and the existence of a stationary equilibrium for the potential

$$V(\varphi) = N^{-1} \lambda \varphi^N. \tag{5.9}$$

For convenience we will regulate the process with a simple momentum cutoff $\kappa$, i.e.

$$C(k) = \theta(\kappa - |k|). \tag{5.10}$$

Also for convenience, we will consider the case $d > 2$ although a similar analysis can be carried out for $m^2 > 0$, $d = 2$ and $d = 1$ (where no regulator is necessary). The stochastic differential equation is then

$$(\partial/\partial t - \Delta)\varphi = -\lambda \varphi^{N-1} + \sigma \eta_c, \tag{5.11}$$

where we have included the amplitude $\sigma$ of the driving noise. We introduce an auxiliary gaussian process $\psi$ defined by

$$(\partial/\partial t - \Delta)\psi = -\beta\psi + \sigma\eta_c, \tag{5.12}$$

where $\beta$ is defined implicitly by

$$\beta = \gamma\lambda(\Sigma^2)^{(N/2)-1}, \quad \Sigma^2 = (2\pi)^{-d}\sigma^2 \int d^dk\, C(k)[k^2+\beta^2]^{-1}. \tag{5.13}$$

The parameter $\gamma$ is a constant, with the choice

$$\gamma = N!/[2^{N/2}(N/2)!] \tag{5.14}$$

Corresponding to the "equivalent-linearization" of the nonlinear process as described by Ito [16]. The equivalent-linearization of the problem is the gaussian process "closest" to the nonlinear process in the sense that the expectation of the logarithm of the Radon–Nikodym derivative of the nonlinear process with respect to the gaussian process is extremized by this choice. For $N=4$ this choice corresponds to the "self-consistent" Wick ordering of the quartic potential.

Subtract Eqs. (5.11) and (5.12) to find

$$(\partial/\partial t - \Delta)(\varphi - \psi) = -\lambda\varphi^{N-1} + \beta\psi, \tag{5.15}$$

where (again for convenience) we take the initial condition of both processes to be the stationary state of $\psi$, independent of the future evolution. Multiply this equation by $(\varphi - \psi)$ and integrate with respect to $d\mu_\alpha$. In light of the inequality (from Lemma 7)

$$0 \leqq \int d\mu_\alpha(\varphi-\psi)(\partial/\partial t - \Delta)(\varphi-\psi), \tag{5.16}$$

we have

$$0 \leqq -\lambda\int d\mu_\alpha\varphi^N + \beta\int d\mu_\alpha\varphi\psi + \lambda\int d\mu_\alpha\varphi^{N-1}\psi - \beta\int d\mu_\alpha\psi^2. \tag{5.17}$$

Evaluate the expectation of this quantity and apply Hölder's inequality to the two middle terms:

$$\begin{aligned} 0 \leqq {} & -\lambda\int d\mu_\alpha E\{\varphi^N\} + \beta\left[\int d\mu_\alpha E\{\varphi^N\}\right]^{1/N}\left[\int d\mu_\alpha E\{|\psi|^{N/(N-1)}\}\right]^{(N-1)/N} \\ & + \lambda\left[\int d\mu_\alpha E\{\varphi^N\}\right]^{(N-1)/N}\left[\int d\mu_\alpha E\{\psi^N\}\right]^{1/N} - \beta\int d\mu_\alpha E\{\psi^2\}. \end{aligned} \tag{5.18}$$

Noting that $\psi$ is gaussian and

$$E\{\psi^2\} = \Sigma^2, \quad E\{|\psi|^{N/(N-1)}\} = C_1\Sigma^{N/(N-1)}, \quad E\{\psi^N\} = C_2\Sigma^N, \tag{5.19}$$

we have, by inserting (5.13) and (5.19) into (5.18)

$$\begin{aligned} 0 \leqq {} & -\int d\mu_\alpha E\{\varphi^N\} + \gamma C_1{}^{(N-1)/N}\Sigma^{(N-1)}\left[\int d\mu_\alpha\right]^{(N-1)/N}\left[\int d\mu_\alpha E\{\varphi^N\}\right]^{1/N} \\ & + C_2{}^{1/N}\Sigma\left[\int d\mu_\alpha\right]^{1/N}\left[\int d\mu_\alpha E\{\varphi^N\}\right]^{(N-1)/N} - \gamma\Sigma^N\int d\mu_\alpha. \end{aligned} \tag{5.20}$$

Let us denote

$$\langle\varphi^N\rangle_\alpha \equiv \left[\int d\mu_\alpha\right]^{-1}\left[\int d\mu_\alpha E\{\varphi^N\}\right], \quad a \equiv \gamma C_1{}^{(N-1)/N}, \quad b \equiv C_2{}^{1/N}. \tag{5.21}$$

Then Eq. (5.20) reads

$$0 \leqq -\langle\varphi^N\rangle_\alpha + a\Sigma^{(N-1)}\langle\varphi^N\rangle_\alpha{}^{1/N} + b\Sigma\langle\varphi^N\rangle_\alpha{}^{(N-1)/N} - \gamma\Sigma^N. \tag{5.22}$$

From this it follows that there are constant $r_2 > r_1 > 0$ so that

$$r_1\Sigma^N \leqq \langle\varphi^N\rangle_\alpha \leqq r_2\Sigma^N. \tag{5.23}$$

These constants depend only on $\gamma$ and $N$.

Note that (5.23) holds uniformly in $\alpha$. If $E\{\varphi(x,t)^N\}$ is independent of $x \in R^d$ and approaches a limit as $t \to \infty$, then

$$\lim_{\alpha \to 0} \langle \varphi^N \rangle_\alpha = \lim_{t \to \infty} E\{\varphi(x,t)^N\} = \langle \varphi \rangle. \tag{5.24}$$

Thus,

$$r_1 \Sigma^N \leqq \langle \varphi^N \rangle \leqq r_2 \Sigma^N. \tag{5.25}$$

A similar calculation, utilizing Lemma 7 with $n > 0$, will yield bounds on higher moments of the nonlinear process.

It is straightforward to compute $\Sigma^N$ from (5.13) in the limits of large or small coupling, noise amplitude, or cutoff. From this the behavior of $\langle \varphi^N \rangle$ is determined as given below.

1) $\kappa$ fixed, $\lambda \to 0$: $\quad \langle \varphi^N \rangle \approx \sigma^{2N} \kappa^{N(d-2)/2}$ (5.26)

2) $\kappa$ fixed, $\lambda\sigma \to \infty$: $\quad \langle \varphi^N \rangle \approx \sigma^2 \kappa^d / \lambda$ (5.27)

3) $\kappa \to 0, \lambda, \sigma$ fixed:

$$\begin{aligned} &\langle \varphi^N \rangle \approx \sigma^2 \kappa^d/\lambda; && (N-2)\cdot(d-2) < 4 \\ &\langle \varphi^N \rangle \approx \sigma^2 \kappa^d/\lambda; && (N-2)\cdot(d-2) = 4, \lambda\sigma \gg 1 \\ &\langle \varphi^N \rangle \approx \sigma^{2N} \kappa^{N(d-2)/2}; && (N-2)\cdot(d-2) = 4, \lambda\sigma \ll 1 \\ &\langle \varphi^N \rangle \approx \sigma^{2N} \kappa^{N(d-2)/2}; && (N-2)\cdot(d-2) > 4 \end{aligned} \tag{5.28}$$

4) $\kappa \to \infty, \lambda, \sigma$ fixed:

$$\begin{aligned} &\langle \varphi^N \rangle \approx \sigma^{2N} \kappa^{N(d-2)/2}; && (N-2)\cdot(d-2) < 4 \\ &\langle \varphi^N \rangle \approx \sigma^{2N} \kappa^{N(d-2)/2}; && (N-2)\cdot(d-2) = 4, \lambda\sigma \ll 1 \\ &\langle \varphi^N \rangle \approx \sigma^2 \kappa^d/\lambda; && (N-2)\cdot(d-2) = 4, \lambda\sigma \gg 1 \\ &\langle \varphi^N \rangle \approx \sigma^2 \kappa^d/\lambda; && (N-2)\cdot(d-2) > 4. \end{aligned} \tag{5.29}$$

Note than in cases 3 and 4 the cutoff dependence is really the same when $(N-2)\cdot(d-2) = 4$ whether $\lambda\sigma \ll 1$ or $\lambda\sigma \gg 1$ (i.e., $N(d-2) = 2d$ in this case). We express them as above for easy comparison with their neighbors.

In the limits of infinite or vanishing cutoff ($\kappa \to \infty$ or $\kappa \to 0$), the nonperturbative bounds on $\langle \varphi^N \rangle$ obtained in the last section depend on the dimension $d$ of the underlying space and the degree of nonlinearity $N$. The distinction between the cases is exactly that determining whether a self-interacting scalar quantum field theory is super-renormalizable (SR$\Leftrightarrow(N-2)\cdot(d-2) < 4$), strictly renormalizable (R$\Leftrightarrow$ $(N-2)\cdot(d-2) = 4$), or nonrenormalizable (NR$\Leftrightarrow(N-2)\cdot(d-2) > 4$) according to perturbation theory. The bounds change in the R case, going from the SR result for weak coupling and small noise amplitude to the NR result for strong coupling and large noise amplitude. We first remark that the distinction between SR, R and NR models exists outside of renormalization theory: no attempt is made here to subtract divergence appearing as $\kappa \to \infty$.

Focus on the bounds in the limit $\kappa \to \infty$ and suppress the noise amplitude $\sigma$. The zeroth order perturbative result for $\langle \varphi^N \rangle$ in these models is

$$\langle \varphi^N \rangle_0 = \frac{N}{2^{N/2}(N/2)!} \left\{ \int \frac{d^d k}{(2\pi)^d} \frac{\theta(\kappa - |k|)}{k^2} \right\}^{N/2} \approx \kappa^{N(d-2)/2} \tag{5.30}$$

in all dimensions $d > 2$, in agreement only with the super-renormalizable and strictly renormalizable, small coupling and noise amplitude case. The condition for nonrenormalizability is just that which ensures $2d < N(d-2)$ so the true divergence of $\langle \varphi^N \rangle$ for the NR models is strictly less than that predicted by the approximation in perturbation theory. For SR and R models the limits $\kappa \to \infty$ and $\lambda \to 0$ may be exchanged (compare (5.26) and (5.29)) but this is not the case for NR models. Holding $\kappa$ fixed and taking $\lambda \to 0$ we recover the perturbative result.

The vacuum expectation value of the potential,

$$\langle V(\varphi) \rangle = N^{-1} \lambda \langle \varphi^N \rangle, \tag{5.31}$$

becomes independent of the coupling in the large cutoff limit for the NR theories; $\langle V(\varphi) \rangle$ is not necessarily small even if the coupling constant is small. This result foreshadows the problems encountered in the perturbative treatment of these models in this limit. In the derivation of (5.25) we made use of only part of the full power of Lemma 7. The $L^2(d\mu_\alpha)$ norms of $\nabla\varphi$ and $\nabla\psi$ can be easily taken into account, and when combined with the result (5.25), we find that

$$|\langle (\nabla\varphi)^2 \rangle - \langle (\nabla\psi)^2 \rangle| \leqq O(\langle V(\varphi) \rangle). \tag{5.32}$$

This bounds the kinetic part of the action

$$S[\varphi] = \int d^d x \{(1/2)(\nabla\varphi)^2 + (\lambda/N)\varphi^N\} \tag{5.33}$$

for the nonlinear process. The quantity $\langle (\nabla\psi)^2 \rangle$ is easily evaluated in the limit $\kappa \to \infty$. In the SR case, $\langle (\nabla\varphi)^2 \rangle$ is proportional to a power of $\kappa$ greater than that controlling $\langle V(\varphi) \rangle$, so that the potential can be considered a perturbation of the free field action. The strictly renormalizable models are a borderline situation, but $\langle V(\varphi) \rangle$ can be made as small as desired by reducing the coupling (in the context of operators on Sobolev spaces for the non-stochastic partial differential equation, this fact is noted in [17]). This is not possible for nonrenormalizable models; $\langle (\nabla\varphi) \rangle$ and $\langle V(\varphi) \rangle$ are controlled by the same power of $\kappa$ and $\langle V(\varphi) \rangle$ is independent of the coupling $\lambda$. The nonlinearity is *not* small compared to the free field action, so the self-interaction in NR models precludes those models' approximation by a free theory.

## 6. Summary and Discussion

The central result of this paper is the proof of the existence, uniqueness and regularity of solutions to a large class of nonlinear parabolic stochastic differential equations with additive noise. New techniques are developed which permit the simultaneous generalization of the previous work to infinite volume, arbitrary dimension and non-monotone nonlinearities. In particular, the method of obtaining *a priori* bounds together with the utilization of bounded, positively preserving measures have not been previously exploited—even in the context of deterministic nonlinear parabolic equations [18]. These techniques are not only applicable to the study of many problems in partial differential equations, but have proved to be a valuable computational tool as well.

The asymptotic expansions and uniform nonperturbative bounds on the nongaussian processes obtained here highlight both the value of these tools and the usefulness of the dynamic, Langevin equation approach to the study of nongaussian functional measures. Because the infinite volume process satisfies an equation, we have a new way to probe the equilibrium measure. Rigorous calculations can be made with these equations akin to rigorous integrations by parts in the path integral [17], but with access to different information. A calculation like that in the last section has no counterpart in the functional integral formulation, where one considers only *one* stochastic process with *various* measures, as opposed to the stochastic differential equation viewpoint where there are *various* processes with *one* underlying measure.

Several important questions remain in the study of these regularized processes defined by parabolic equations. For nonlinear equations regularization via modification of the driving noise correlation function destroys the formal equilibration to a Gibbs measure such as (1.1). Although the terms in the unregulated perturbation expansion are equivalent to those in other approaches [19], and this paper has established the nonperturbative nature of this kind of regularization, the approach to equilibrium and the existence of a stationary state are central to the applications. In the novel approach to gauge theories presented by Parisi–Wu stochastic quantization, the approach to equilibrium is crucial, for it is here that the invariance of the action comes into play. The possibility of equilibrium phase transitions is intimately related to the degeneracy of a stationary state for these stochastic dynamic systems, and a nonequilibrium approach to these problems raises the possibility of new effects [20].

The "space" regularization developed here, as opposed to the "time" regularization proposed in [6], is the appropriate one for generalization to models with invariances. Smearing the noise in the fictitious time parameter destroys the Markov character of the evolution so that a formal analysis via Fokker–Planck equations (as in [4]) is prohibited. In fact, it has been shown [21] that this approach is incompatible with a gauge-invariant perturbation expansion proposed in [4]. Recent application of a spatial regularization to gauge theories [22] involves a field-dependent regulator where the instantaneous effect of the noise depends on the state of the system. A finite, gauge-covariant perturbation expansion can then be developed, but the appearance of a *multiplicative* noise in the stochastic differential equations presents a new challenge for the rigorous, nonperturbative analysis of these theories.

Nonlinear partial differential equations with colored noise for scalar variables arise in a variety of nonequilibrium applications [23] and often the noise is coupled multiplicatively to the state variables. These systems are a generalization of ordinary stochastic differential equations with a nontrivial diffusion term, and their analysis requires the full utilization of probabilistic concepts (i.e., Markov and martingale properties and stochastic integration) which are not necessary for many equations with purely additive noise. The mathematical analysis and precise determination of the physical content of these models remains a major open problem.

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