Commun. Math. Phys. 108, 241–255 (1987)

Communications in
Mathematical
Physics

One Dimensional Spin Glasses with Potential Decay $1/r^{1+\varepsilon}$. Absence of Phase Transitions and Cluster Properties

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Abstract. One-dimensional Ising spin systems interacting via a two-body random potential are considered; a decay with the distance like $1/r^{1+\varepsilon}$ is assumed.

We consider only boundary conditions independent of the random realization of the interactions and prove uniqueness and cluster properties of Gibbs states with probability one.

1. Introduction

Spin glasses are at present one of the major areas of interest in Statistical Mechanics. Only few problems have so far been solved in a rigorous way. In particular the existence and the nature of phase transitions are still open problems even in the Sherrington-Kirkpatrick mean field theory, for which however a very precise heuristic theory exists (see [10]).

As far as rigorous results are concerned, we mention the proof of the existence of thermodynamics for interactions decaying like $r^{-\alpha d}$ with $\alpha > 1/2$ in d dimensions [4, 7, 9]. Khanin [8] proved the uniqueness of Gibbs distribution in one dimension for interactions decaying like $r^{-\alpha}$ with $\alpha > 3/2$. Cassandro et al. [2] proved under the same conditions the infinite differentiability of thermodynamic functions.

The one-dimensional case with $1 < \alpha \leqq 3/2$ appears qualitatively different from the former case, since here it is not true that the supremum of the interaction among two contiguous half-lines over all spin configurations is finite with probability one. This case has been considered in [5], where the authors deal with the problem of absence of symmetry breaking. They show essentially that the interaction among two contiguous half-lines is bounded if one excludes a subset of "bad" spin configurations of zero Gibbs measure. The situation is reminiscent of superstable unbounded spins (see [1]), but here the set of bad configurations depends on the random realization of the interaction.

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* Partially supported by GNAFA of CNR

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We remark that the proof of the main result of [5] is not correct even though we think that it is possible to get a weaker result (see Corollary 3.6 of the present paper) by using the ideas of [5] and some further considerations. For a family of Gibbs states $\mu^{(J)}$, obtained by J-independent boundary conditions, the authors can show that $\mu^{(J)}$ and the corresponding spin flipped state are mutually absolutely continuous. As they want to prove absence of symmetry breaking they are forced to consider extremal Gibbs states, which are a priori obtained by J dependent boundary conditions. However their proof which uses Fubini's theorem, can only be applied to J-independent boundary conditions. Similar considerations hold for [3]. It is thus not excluded that for a relevant set of interactions some "exotic" states, not necessarily symmetry preserving, could be obtained by imposing boundary conditions dependent on the realization of the interaction. We do not comment any further on this question that seems to us difficult to settle but not very relevant from the physical point of view.

In this paper we consider only interaction-independent boundary conditions and we prove that

i) the Gibbs expectation in a volume Λ of an observable localized far away from the boundary has a weak dependence on boundary conditions with large probability;

ii) for any given boundary condition the limiting Gibbs state exists, is a pure state and satisfies a suitable mixing property;

iii) any two boundary conditions give rise, with probability one, to the same Gibbs state.

The strategy of the proof is to consider the system as a nearest neighbour block model (which has exponential decay of correlations in the block distance) plus a small long-range perturbation. This can be shown to work with sufficiently high probability.

In Sect. 2 we give definitions and notation. In Sect. 3 we state our results and prove them by using two main Lemmas 3.1 and 3.2. In Sect. 4 we prove Lemmas 3.1 and 3.2. In the appendix we prove a proposition about factorization of products of transfer matrices.

2. Definitions and Notation

Given $\Lambda \subset \mathbb{Z}$ the configuration space in Λ is the set $\mathcal{S}_\Lambda = \{-1, 1\}^{|\Lambda|}$. Given $\Lambda_1 \subset \Lambda_2 \subset \mathbb{Z}$ and $s \in \mathcal{S}_{\Lambda_2}$, we denote by $s|_{\Lambda_1}$ the restriction of s to Λ_1. Given Λ_1, $\Lambda_2 \subset \mathbb{Z}$, with $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $s^{(1)} \in \mathcal{S}_{\Lambda_1}$, $s^{(2)} \in \mathcal{S}_{\Lambda_2}$, we denote by $s^{(1)} \vee s^{(2)}$ the configuration $s \in \mathcal{S}_{\Lambda_1} \cup \mathcal{S}_{\Lambda_2}$ such that $s|_{\Lambda_1} = s^{(1)}$, $s|_{\Lambda_2} = s^{(2)}$.

Given $\Lambda \subseteqq \mathbb{Z}$ we denote by $C(\Lambda)$ the space of all real valued functions on $\mathcal{S}_\Lambda$ continuous with respect to the usual product topology on $\mathcal{S}_\Lambda$, and we put for $f \in C(\Lambda): \|f\| = \sup_{s \in \mathcal{S}_\Lambda} f(s)$. An element f of $C(\Lambda)$ will be identified with that element $\tilde{f}$ of $C(\Lambda')$ with $\Lambda' \supset \Lambda$ such that $\tilde{f}(s) = f(s|_\Lambda)$.

We introduce a random variable J_{ij} for each unordered pair i, j, $i \neq j$, $i, j \in \mathbb{Z}$. The variables J_{ij} are assumed to be independent and identically distributed with common distribution $dF(x)$. We require

$$\int x dF(x) = 0, \quad \forall t \in \mathbb{R} : \int \exp(tx) \cdot dF(x) < \infty . \tag{2.1}$$

In particular for small $t: \int \exp(tx)dF(x) = \exp(\beta t^2) + O(t^3)$. We shall always assume that the realization of the variables J_{ij} is such that for every $i \in \mathbb{Z}$:

$$\sup_{j \neq i} \frac{|J_{ij}|}{\log|i-j|+1} < \infty . \tag{2.2}$$

This can be done as the set of such realizations has probability 1 due to the second hypothesis on the distribution $dF(x)$.

Given a real number $\alpha > 1$ (the interesting case will be $1 < \alpha \leqq 3/2$), we define the energy $H_\Lambda(s)$ [or $H(s)$ when no confusion can arise] of a configuration $s \in \mathscr{S}_\Lambda$ with Λ finite $\subset \mathbb{Z}$ by

$$H_\Lambda(s) = \sum_{\substack{i,j \in \Lambda \\ i \neq j}} \frac{J_{ij}}{|i-j|^\alpha} s_i s_j . \tag{2.3}$$

Given $\Lambda_1, \Lambda_2 \subset \mathbb{Z}$, $\Lambda_1 \cap \Lambda_2 = \emptyset$, with Λ_1 or Λ_2 finite, $s^{(1)} \in \mathscr{S}_{\Lambda_1}$, $s^{(2)} \in \mathscr{S}_{\Lambda_2}$, the interaction $W_{\Lambda_1,\Lambda_2}(s^{(1)}, s^{(2)})$ [or simply $W(s^{(1)}, s^{(2)})$] between $s^{(1)}$ and $s^{(2)}$ is defined by

$$W_{\Lambda_1,\Lambda_2}(s^{(1)}, s^{(2)}) = \sum_{\substack{i \in \Lambda_1 \\ j \in \Lambda_2}} \frac{J_{ij}}{|i-j|^\alpha} s_i^{(1)} s_j^{(2)} . \tag{2.4}$$

Inequality (2.2) implies that the series on the right-hand side (2.4) is absolutely convergent also when Λ_1 or Λ_2 (but not both) is infinite.

Let now $\Lambda \subset \mathbb{Z}$ be finite. Let $h \in C(\mathbb{Z})$ and ν be a finite, positive measure on $\mathscr{S}_{\Lambda^c}$. We define the probability measure $\mu_{\Lambda,\nu}^h$ on $\mathscr{S}_\mathbb{Z}$ by

$$\mu_{\Lambda,\nu}^h(f) = \frac{1}{Z_{\Lambda,\nu}^h} \sum_{s \in \mathscr{S}_\Lambda} \int f(s \vee \sigma) e^{h(s \vee \sigma)} d\nu(\sigma) \tag{2.5}$$

for $f \in C(\mathbb{Z})$, where

$$Z_{\Lambda,\nu}^h = \sum_{s \in \mathscr{S}_\Lambda} \int e^{h(s \vee \sigma)} d\nu(\sigma) . \tag{2.6}$$

Let a realization of the J_{ij}'s be given, let Λ be a finite subset of $\mathbb{Z}$, and let σ be a fixed configuration in $\mathscr{S}_\mathbb{Z}$. We define the Gibbs distribution $\bar{\mu}_{\Lambda,\sigma}$ in the volume Λ with boundary condition σ by

$$\bar{\mu}_{\Lambda,\sigma} = \mu_{\Lambda,\delta_{\sigma|\Lambda^c}}^{h_\Lambda} \tag{2.7}$$

where $h_\Lambda(s) = H_\Lambda(s|_\Lambda) + W_{\Lambda,\Lambda^c}(s|_\Lambda, s|_{\Lambda^c})$ for $s \in \mathscr{S}_\mathbb{Z}$ and $\delta_{\sigma|\Lambda^c}$ is the probability measure concentrated on the configuration $\sigma|\Lambda^c$.

3. Main Results

Our results follow from the following Lemmas 3.1 and 3.2 that will be proven in the next section.

Given an integer $L > 0$ we define $\Lambda_L = \{j \in \mathbb{Z} : -L \leqq j \leqq L\}$.

3.1. Lemma. *Let k be a positive integer. There exists $L_0(k)$ such that for $\Lambda = \Lambda_L$ with $L \geqq L_0$, every function $f \in C(\Lambda_k)$, $\|f\| \leqq 1$, and every two positive finite measures ν_1, ν_2 on $\mathscr{S}_{\Lambda^c}$:*

$$\mathbb{P}\left(|\mu_{\Lambda,\nu_1}^{h_\Lambda}(f) - \mu_{\Lambda,\nu_2}^{h_\Lambda}(f)| > \frac{1}{L^\varepsilon} \right) \leqq \exp(-(\log L)^{4/3}) , \tag{3.1}$$

where $\mathbb{P}$ *denotes the probability* w.r.t. *the* J_{ij}*'s,* h_Λ *is the function on* $\mathscr{S}_\mathbb{Z}$ *defined after* (2.7) *and* ε *is a positive constant that does not depend on* k.

3.2. Lemma. *Let* k *be a positive integer. There exists* $L_0(k)$ *such that for* $\Lambda=\Lambda_L$ *with* $L\geqq L_0$, $f\in C(\Lambda_k)$, $\psi\in C(\Lambda^c)$ *with* $\|f\|\leqq 1$, $\|\psi\|\leqq 1$ *and* ν *finite positive measure on* $\mathscr{S}_{\Lambda^c}$*:*

$$\mathbb{P}\left(|\mu^{h_\Lambda}_{\Lambda,\nu}(f\psi)-\mu^{h_\Lambda}_{\Lambda,\nu}(f)\mu^{h_\Lambda}_{\Lambda,\nu}(\psi)|>\frac{1}{L^\varepsilon}\right)\leqq\exp(-(\log L)^{4/3}). \tag{3.2}$$

3.3. Remark. Note that the events appearing on the left-hand side of (3.1) and (3.2) depend only on the J_{ij}'s contained in the definition of h_Λ. Therefore we are allowed to take for ν, ν_1, and ν_2 measures that depend on the remainder of the J_{ij}'s.

The following Theorems 3.4–3.6 follow easily from Lemmas 3.1 and 3.2.

3.4. Theorem. *Let* $\sigma\in\mathscr{S}_\mathbb{Z}$*. The limit*

$$\bar{\mu}_\sigma=\lim_{L\to\infty}\bar{\mu}_{\Lambda_L,\sigma} \tag{3.3}$$

exists in the weak sense with probability* 1.

Proof. Let $f\in C(\Lambda_k)$ for some positive k and let $k\leqq L_1\leqq L_2$. Then we have

$$\bar{\mu}_{\Lambda_{L_2},\sigma}=\mu^{h_{\Lambda_{L_1}}}_{\Lambda_{L_1},\lambda_\sigma}, \tag{3.4}$$

where λ_σ is the measure on $\mathscr{S}_{\Lambda^c_{L_1}}$ defined by

$$\lambda_\sigma(\varphi)=\sum_{s\in\mathscr{S}_{\Lambda_{L_2}\setminus\Lambda_{L_1}}}\varphi(\sigma|_{\Lambda^c_{L_2}}\vee s)\exp(H_{\Lambda_{L_2}\setminus\Lambda_{L_1}}(s)+W_{\Lambda_{L_2}\setminus\Lambda_{L_1},\Lambda^c_{L_2}}(s,\sigma|_{\Lambda^c_{L_2}})). \tag{3.5}$$

Let ϱ be a positive integer to be chosen later. By applying Lemma 3.1, we obtain from the representation (3.4) and the Borel-Cantelli Lemma that the sequence

$$\bar{\mu}_{\Lambda_{n^\varrho},\sigma}(f) \tag{3.6}$$

converges with probability 1 for ϱ sufficiently large. Using again Lemma 3.1 and the representation (3.4), we get that for n sufficiently large

$$\begin{aligned}\mathbb{P}\left(|\bar{\mu}_{\Lambda_{n^\varrho},\sigma}(f)-\bar{\mu}_{\Lambda_m,\sigma}(f)|>\frac{1}{n^{\varrho\varepsilon}}\text{ for some } m \text{ with } n^\varrho\leqq m\leqq(n+1)^\varrho\right)&\\ \leqq(n+1)^\varrho\exp(-(\log n^\varrho)^{4/3}),&\end{aligned} \tag{3.7}$$

so that we can apply again the Borel-Cantelli Lemma for ϱ sufficiently large and obtain that the whole sequence $\bar{\mu}_{\Lambda_m,\sigma}(f)$ converges with probability 1. The result is then obtained by considering a countable dense set of observables. □

3.5. Theorem. *Let* $\sigma,\tau\in\mathscr{S}_\mathbb{Z}$*, then there exists a set of interactions of probability one on which*

$$\bar{\mu}_\sigma=\bar{\mu}_\tau.$$

Proof. Let $f\in C(\Lambda_k)$ for some positive k. By applying Lemma 3.1 with ν_1 and ν_2 probability measures concentrated on $\sigma|_{\Lambda^c_L}$ and $\tau|_{\Lambda^c_L}$, we get that for L sufficiently large

$$\mathbb{P}\left(|\bar{\mu}_{\Lambda_L,\sigma}(f)-\bar{\mu}_{\Lambda_L,\tau}(f)|>\frac{1}{L^\varepsilon}\right)\leqq\exp(-(\log L)^{4/3}). \tag{3.8}$$

By Theorem 3.4 we can find a set of interactions of full probability on which both $\bar{\mu}_{\Lambda_L,\sigma}(f)$ and $\bar{\mu}_{\Lambda_L,\tau}(f)$ converge as L goes to infinity. Inequality (3.8) implies that on this set the limits are equal. The result is then obtained by considering a countable dense set of observables. □

From Theorem 3.5 we get in particular the absence of symmetry breaking. Let $\theta:\mathscr{S}_{\mathbb{Z}}\to\mathscr{S}_{\mathbb{Z}}$ be the spin flip transformation $(\theta s)_i=-s_i$ and θ^* be the induced map on the measures on $\mathscr{S}_{\mathbb{Z}}$. By the spin flip invariance of the interaction we get

$$\theta^*\bar{\mu}_\sigma=\bar{\mu}_{\theta\sigma}.$$

Therefore Theorem 3.5 implies the following:

3.6. Corollary. *For every $\sigma\in\mathscr{S}_{\mathbb{Z}}$ there is a set of interactions of full probability on which*

$$\theta^*\bar{\mu}_\sigma=\bar{\mu}_\sigma.$$

3.7. Theorem. *There exists a set of interactions of probability one on which the limit (3.3) exists and is a pure Gibbs state.*

Proof. Let $M>0$ be an integer. Given a finite volume Λ containing Λ_M, we define the Gibbs measure $\bar{\mu}^{(M)}_{\Lambda,\sigma}$ on $\mathscr{S}_{\Lambda\setminus\Lambda_M}$ by

$$\bar{\mu}^M_{\Lambda,\sigma}(f)=Z^{(M)^{-1}}_{\Lambda,\sigma}\sum_{s\in\mathscr{S}_{\Lambda\setminus\Lambda_M}}f(s)\exp(H_{\Lambda\setminus\Lambda_M}(s)+W_{\Lambda\setminus\Lambda_M,\Lambda^c}(s,\sigma|_{\Lambda^c}),\tag{3.9}$$

where $Z^{(M)}$ is the normalizing constant

$$Z^{(M)}_{\Lambda,\sigma}=\sum_{s\in\mathscr{S}_{\Lambda\setminus\Lambda_M}}\exp(H_{\Lambda\setminus\Lambda_M}(s)+W_{\Lambda\setminus\Lambda_M,\Lambda^c}(s,\sigma|_{\Lambda^c})).\tag{3.10}$$

For $f\in C(\Lambda\setminus\Lambda_M)$ the following relation holds:

$$\bar{\mu}^M_{\Lambda,\sigma}(f)=\frac{\bar{\mu}_{\Lambda,\sigma}(fe^{-W_{\Lambda_M,\Lambda\setminus\Lambda_M}})}{\bar{\mu}_{\Lambda,\sigma}(e^{-W_{\Lambda_M,\Lambda\setminus\Lambda_M}})}.\tag{3.11}$$

It is easy to check that on the set of interactions where the limit (3.3) exists, also the limit

$$\bar{\mu}^{(M)}_\sigma=\lim_{L\to\infty}\bar{\mu}^{(M)}_{\Lambda_L}\tag{3.12}$$

exists and $\bar{\mu}_\sigma$ and $\bar{\mu}^{(M)}_\sigma$ are connected by the relation

$$\bar{\mu}^M_\sigma(f)=\frac{\bar{\mu}_\sigma(fe^{-W_{\Lambda_M,\mathbb{Z}\setminus\Lambda_M}})}{\bar{\mu}_\sigma(e^{-W_{\Lambda_M,\mathbb{Z}\setminus\Lambda_M}})}.\tag{3.13}$$

This follows from the fact that due to our hypotheses [see (2.2)] the functions $\exp(-W_{\Lambda_M,\Lambda_L\setminus\Lambda_M})$ converge uniformly to $\exp(-W_{\Lambda_M,\mathbb{Z}\setminus\Lambda_M})$ as L tends to infinity. By a similar argument we get the relation

$$\bar{\mu}_\sigma=\mu^{h_{\Lambda_M}}_{\Lambda_M,\bar{\mu}^{(M)}_\sigma}.\tag{3.14}$$

On the other hand the measure $\bar{\mu}^{(M)}_\sigma$ does not depend on the J_{ij}'s in h_{Λ_M}, since it is the limit of the measures $\bar{\mu}^{(M)}_{\Lambda_L}$ which do not depend on these J_{ij}'s. Therefore we can apply to $\bar{\mu}_\sigma$ in the representation (3.14) the result of Lemma 3.2 and, in force of the characterization of pure Gibbs states given by Theorem 1.11 of [11], we obtain the result. □

4. Proofs of the Main Lemmas

Lemma 3.1 will follow from Propositions 4.1, 4.2, 4.3 below and from Proposition A.1. We give first some preliminary definitions.

Given a positive integer n and an odd integer N, consider the volume Λ, centered at the origin, with $|\Lambda|=(2N+1)n$. N and n will be suitable increasing functions of $|\Lambda|$ and they will be chosen later. As it will be clear from the proofs, we shall be able to treat the case of a general sufficiently large Λ; the particular case that we actually consider here is chosen only to simplify the notation.

We divide Λ into $2N+1$ intervals $\Lambda_{-N}, \ldots, \Lambda_N$, each containing n sites. We shall write $s^{(j)}$ for $s|_{\Lambda_j}$.

Let us now give an outline of the proof. In order to evaluate the quantities:

$$\mu^{h_\Lambda}_{\Lambda,\nu_1}(f)-\mu^{h_\Lambda}_{\Lambda,\nu_2}(f),$$

we introduce a sequence of approximations. At each step we make an error that we prove to be small with high probability.

The first step (see Proposition 4.1) consists in subtracting the interactions among non-contiguous blocks. In the second step (see Proposition 4.2) we cut off all the spin configurations that give rise to interactions among contiguous blocks that are bigger than a certain constant M. In the third step (see Proposition 4.3) we introduce another restriction on the spin configurations; this new restriction is local in the sense that it is not defined, like the previous one, in terms of pairs of blocks, but it involves configurations in single even blocks. If such a restriction is satisfied, then the effective interaction among even blocks is not too big. At this point we are able to apply the result of Proposition A.1 on the factorization properties of products of transfer matrices to the system relativized to even blocks with the restriction on the set of configurations in each block.

In Proposition A.1 we prove exponential loss of memory for such a system, so that for a suitable choice of the constants involved in the approximation we get the final result.

Before stating Proposition 4.1, let us establish some further definitions. For any given $s\in\mathscr{S}_{\mathbb{Z}}$, with the notation $s_\Lambda\equiv s|_\Lambda$, $\sigma=s|_{\Lambda^c}$, we set

$$H_\Lambda(s_\Lambda)+W_{\Lambda,\Lambda^c}(s_\Lambda,\sigma)=\tilde{H}(s_\Lambda,\sigma)+v(s_\Lambda,\sigma), \tag{4.1}$$

where, if σ_- and σ_+ are respectively the restrictions of σ to the left and the right part of Λ^c,

$$\begin{aligned}\tilde{H}(s_\Lambda,\sigma)&=W(\sigma_-,s^{(-N)})+\sum_{j=-N}^{N-1}[H(s^{(j)})+W(s^{(j)},s^{(j+1)})]\\&\quad+H(s^{N)})+W(s^{(N)},\sigma_+),\\ v(s_\Lambda,\sigma)&=\sum_{j=-N+1}^{N}W(\sigma_-,s^{(j)})+\sum_{j=-N}^{N-1}W(s^{(j)},\sigma_+)\\&\quad+\sum_{i=-N}^{N-2}\sum_{j=i+2}^{N}W(s^{(i)},s^{(j)}).\end{aligned} \tag{4.2}$$

$\tilde{H}$ describes a system with interaction of finite range n in Λ and boundary condition σ, whereas v contains all the interactions that jump at least one block of size n.

4.1. Proposition. *$\exists\lambda>0$, s.t., given any function $f\in C(\Delta)$ with $\Delta\subset\Lambda_0$, for any finite measure v on $\mathscr{S}_{\Lambda^c}$, if Λ is large enough, we have:*

$$\mathbb{P}(|\mu^{H}_{v,\Lambda}(f)-\mu^{\tilde{H}}_{v,\Lambda}(f)|>2\|f\|/|\Lambda|^{\lambda})\leqq\exp(-|\Lambda|^{4\lambda}). \tag{4.3}$$

Proof. It is immediate to check that

$$\mu^{H+W}_{v,\Lambda}(f)=\mu^{\tilde{H}}_{v,\Lambda}(f)+\mu^{H+W}_{v,\Lambda}(f)\,(1-\mu^{\tilde{H}}_{v,\Lambda}(e^{v}))+\mu^{\tilde{H}}_{v,\Lambda}(f(e^{v}-1)). \tag{4.4}$$

For any positive δ consider now the convex function g_δ defined by

$$g_\delta(x)=(1-\chi_\delta(x))\cdot 2\left(|x|-\frac{\delta}{2}\right), \tag{4.5}$$

where χ_δ is the characteristic function of the interval $\left[-\frac{\delta}{2},\frac{\delta}{2}\right]$. By Jensen's inequality we have

$$\mathbb{P}(\mu^{\tilde{H}}_{v,\Lambda}(|e^{v}-1|)>\delta)\equiv\mathbb{P}(g_\delta(\mu^{\tilde{H}}_{v,\Lambda}(|e^{v}-1|))>\delta)\leqq\mathbb{P}(\mu^{\tilde{H}}_{v,\Lambda}(g_\delta(|e^{v}-1|))>\delta). \tag{4.6}$$

The above probability can be evaluated by applying the Markov-Chebyshev inequality with the expectation taken with respect to the J_{ij}'s appearing in v. Since these J_{ij}'s do not appear in H, we can apply Fubini's theorem and obtain

$$\mathbb{E}\mu^{\tilde{H}}_{v,\Lambda}(g_\delta(|e^{v}-1|))=\mu^{\tilde{H}}_{v,\Lambda}(\mathbb{E}g_\delta(|e^{v}-1|)). \tag{4.7}$$

We shall get a bound for $\mathbb{E}g_\delta(|e^{v}-1|)$ which is uniform in the spin configuration. This bound allows us to estimate the expectation with respect to the remainder of the J_{ij}'s and to get the desired probability estimate by applying the Markov-Chebyshev inequality. By applying Schwarz inequality to Eq. (4.7), we get

$$\begin{aligned}\mu^{\tilde{H}}_{v,\Lambda}(\mathbb{E}g_\delta(|e^{v}-1|)\leqq\sup_{\sigma,s_\Lambda}&\left[\mathbb{P}\left(|e^{v(s_\Lambda,\sigma)}-1|>\frac{\delta}{2}\right)\right]^{1/2}\\ &\times 2[\mathbb{E}(e^{2v(s_\Lambda,\sigma)}+2e^{v(s_\Lambda,\sigma)}+1)]^{1/2}.\end{aligned}$$

If $\delta<1$, it will certainly be true that

$$\mathbb{P}\left(|e^{v}-1|>\frac{\delta}{2}\right)\leqq\mathbb{P}\left(|v|>\frac{\delta}{4}\right). \tag{4.8}$$

We can exploit now the hypotheses (2.1) made on the distribution of the J_{ij}'s. If we apply the Chebyshev exponential inequality to the right-hand side of Eq. (4.8) we get

$$\mathbb{P}\left(|v(s_\Lambda,\sigma)|>\frac{\delta}{4}\right)\leqq\exp\left[-c_1\frac{\delta^2}{\mathbb{E}v^2(s_\Lambda,\sigma)}\right]$$

for some positive constant c_1. Therefore

$$\begin{aligned}\mathbb{P}(\mu^{\tilde{H}}_{v,\Lambda}(|e^{v}-1|)>\delta)\leqq\sup_{\sigma,s_\Lambda}&\exp\left(-\frac{c_1\delta^2}{2\cdot\mathbb{E}v^2(s_\Lambda,\sigma)}\right)\\ &\times 2[\mathbb{E}(e^{2v(s_\Lambda\sigma)}+2e^{v(s_\Lambda\sigma)}+1)]^{1/2}\cdot\frac{1}{\delta}.\end{aligned} \tag{4.9}$$

From Eq. (4.3) it is easy to get that for some constant c_2,

$$\mathbb{E}(v^2(s_\Lambda, \sigma)) \leqq c_2 \frac{N}{n^2(\alpha-1)}. \tag{4.10}$$

We choose

$$\delta = \bar{\delta}\left(c_2 \frac{N}{n^{2(\alpha-1)}}\right)^{1/2} \tag{4.11}$$

and

$$N = [|\Lambda|^\gamma] \tag{4.12}$$

with γ such that

$$2(\alpha-1)(1-\gamma)-\gamma \equiv 10\lambda, \tag{4.13}$$

$$\bar{\delta} = |\Lambda|^{3\lambda}. \tag{4.14}$$

Therefore we finally get that for $|\Lambda|$ sufficiently large,

$$\mathbb{P}\left(\mu^{\tilde{H}}_{v,\Lambda}(|e^v - 1|) > \frac{1}{|\Lambda|^\lambda}\right) \leqq \exp(-|\Lambda|^{4\lambda}). \tag{4.15}$$

Equations (4.4) and (4.15) with the choices (4.12), (4.13), and (4.14) imply the result. □

4.2. Proposition. *In the same hypotheses of Proposition 4.1, let the characteristic functions $\chi_{i-1,i}$ be defined by (for $i=-N, \ldots, N+1$):*

$$\chi_{i-1,i}(s^{(i-1)}, s^{(i)}) = \begin{cases} 1 & \text{if } \; |W(s^{(i-1)}, s^{(i)}| \leqq M \\ 0 & \text{if } \; |W(s^{(i-1)}, s^{(i)}| > M, \end{cases} \tag{4.16}$$

where we have set

$$\begin{aligned} A_{-N-1} &\equiv \mathbb{Z}^- \cap \Lambda^c, \qquad A_{N+1} = \mathbb{Z}^+ \cap \Lambda^c, \\ s^{(-N-1)} &= \sigma_-, \qquad s^{(N+1)} = \sigma_+ . \end{aligned} \tag{4.17}$$

Then there exist positive constants c, c' such that

$$\mathbb{P}\left(\left|\mu^{\tilde{H}}_{v,\Lambda}(f) - \mu^{\tilde{H}}_{v,\Lambda}\left(f \prod_{i=-N}^{N+1} \chi_{i-1,i}\right)\right| > \|f\| N e^{-cM^2}\right) \leqq N e^{-c'M^2}. \tag{4.18}$$

Proof. We have

$$\begin{aligned} &\left|\mu^{H}_{v,\Lambda}(f) - \mu^{H}_{v,\Lambda}\left(f \prod_{i=-N}^{N+1} \chi_{i-1,i}\right)\right| \\ &\leqq \|f\| \sum_{i=-N}^{N+1} \mu^{H}_{v,\Lambda}(1-\chi_{i-1,i}) \\ &= \|f\| \sum_{i=-N}^{N+1} \mu^{\tilde{H}^i}_{v,\Lambda}((1-\chi_{i-1,i})e^{W_{i-1,i}}) \frac{1}{\mu^{\tilde{H}^i}_{v,\Lambda}(e^{W_{i-1,i}})}, \end{aligned} \tag{4.19}$$

where

$$\tilde{H}^i = \tilde{H} - W_{i-1,i}$$

and

$$W_{i-1,i} = W(s^{(i-1)}, s^i) \quad i=-N, \ldots, N-1 .$$

Now we evaluate separately the two factors appearing in each term of the right-hand side of Eq. (4.19).

By Jensen's inequality we have

$$\mu_{\nu,\Lambda}^{\tilde{H}_i}(e^{W_{i-1,i}}) \geqq \exp(\mu_{\nu,\Lambda}^{\tilde{H}_i}(W_{i-1,i})). \tag{4.20}$$

Moreover

$$\mathbb{P}(\exp \mu_{\nu,\Lambda}^{\tilde{H}_i}(W_{i-1,i}) < e^{-M}) \leqq \mathbb{P}(\mu_{\nu,\Lambda}^{\tilde{H}_i}(|W_{i-1,i}|) > M). \tag{4.21}$$

The method that we used to obtain Eq. (4.15) can be applied to estimate the right-hand side of Eq. (4.21). We get in this way

$$\mathbb{P}(\mu_{\nu,\Lambda}^{\tilde{H}_i}(|W_{i-1,i}|) > M) \leqq \exp(-c_3 M^2) \tag{4.22}$$

for some positive constant c_3.

In order to evaluate the first factor, we first compute the expectation w.r.t. the J's appearing in $W_{i-1,i}$. Since these J's do not appear in $\tilde{H}^i$, we can use Fubini's theorem to interchange the integrations. We get, for any positive γ,

$$\mathbb{P}(\mu_{\nu,\Lambda}^{\tilde{H}_i}((1-\chi_{i-1,i}) \exp W_{i-1,i}) > \gamma) \leqq \gamma^{-1} \mu_{\nu,\Lambda}^{\tilde{H}^i}(\mathbb{E}[(1-\chi_{i-1,i}) e^{W_{i-1,i}}]). \tag{4.23}$$

The right-hand side of Eq. (4.23) can be bounded by the Schwarz inequality by

$$\gamma^{-1} \mu_{\nu,\Lambda}^{\tilde{H}}([\mathbb{E}(1-\chi_{i-1,i})]^{1/2} [\mathbb{E}(e^{2W_{i-1,i}})]^{1/2}). \tag{4.24}$$

It is immediate to check that $\mathbb{E}(\exp(2W_{i-1,i})$ is bounded by a constant and that

$$\mathbb{E}(1-\chi_{i-1,i}) \leqq \sup_{\sigma, s_\Lambda} \mathbb{P}(|W_{i-1,i}| > M) \leqq e^{-c_4 M^2} \tag{4.25}$$

for a suitable positive constant c_4.

If we choose properly the constant γ in Eq. (4.23), we conclude the proof. □

We can write

$$\mu_{\nu,\Lambda}^{\tilde{H}}\left(f \prod_{i=-N}^{N+1} \chi_{i-1,i}\right) = \frac{\int d\nu(\sigma) \sum_{s^{(-N)} \dots s^{(N)}} \left(\prod_{i=-N}^{N} \mu_i(s^{i})\right) f \exp\left(\sum_{i=-N}^{N} W_{i-1,i}\right) \prod_{i=-N}^{N} \chi_{i-1,i}}{\int d\nu(\sigma) \sum_{s^{(-N)} \dots s^{(N)}} \left(\prod_{i=-N}^{N} \mu_i(s^{(i)})\right) \exp\left(\sum_{i=-N}^{N+1} W_{i-1,i}\right)}, \tag{4.26}$$

where we have used the notation introduced in Eqs. (4.16) and (4.17) and we have denoted by μ_i the normalized Gibbs measure on the isolated block $A_i : \mu_i(s^{(i)}) = (Z_{A_i}^{H_{A_i}})^{-1} \exp(H_{A_i}(s^{(i)}))$ for $i = -N, \dots, N$.

For any pair of consecutive even blocks consider the "effective interaction" $\tilde{W}_{2i, 2(i+1)}$ defined by

$$\begin{aligned} &\exp \tilde{W}_{2i, 2(i+1)}(s^{(2i)}, S^{(2i+2)}) \\ &= \sum_{s^{(2i+1)}} \mu_{2i+1}(s^{(2i+1)} \chi_{2i, 2i+1}(s^{(2i)}, s^{(2i+1)}) \chi_{2i+1, 2i+2}(s^{(2i+1)}, s^{(2i+2)}) \\ &\quad \times \exp[W_{2i, 2i+1}(s^{(2i)}, s^{(2i+1)}) + W_{2i+1, 2i+2}(s^{(2i+1)}, s^{(2i+2)})]. \end{aligned} \tag{4.27}$$

$\tilde{W}$ can take also the value $-\infty$.

4.3. Proposition. *In the same hypotheses of Proposition 4.1 there exist constants $\bar{c}$, $\bar{c}'$, and $\bar{c}''$ such that there exists a set of interactions J of probability bigger than $1-Ne^{-cM^2}$ on which the following is true:*

1) *for* $j=-(N-1)/2,\ldots,(N-1)/2$ *there exists* $\tilde{\Omega}_{2j}\in\mathscr{S}_{A_{2j}}$: $\forall s_{2j}\in\tilde{\Omega}_{2j}$, $s_{2j+2}\in\tilde{\Omega}_{2j+2}$:

$$(1-e^{-\bar{c}'M^2})e^{-2M}\leqq e^{\tilde{W}_{2j,2j+2}(s^{(2j)},s^{(2j+2)})}\leqq e^{2M},$$

2)

$$\left|\mu_{\nu,A}^{\tilde{H}}\left(f\prod_{i=-N}^{N+1}\chi_{i-1,i}\right)-\frac{\mu_{\nu,A}^{\tilde{H}}\left(f\prod_{i=-N}^{N}\chi_{i-1,i}\prod_{j=-N/2}^{N/2}\mathbb{1}_{\tilde{\Omega}_{2j}}\right)}{\mu_{\nu,A}^{\tilde{H}}\left(\prod_{i=-N}^{N}\chi_{i-1,i}\prod_{j=-N/2}^{N/2}\mathbb{1}_{\tilde{\Omega}_{2j}}\right)}\right|\leqq Ne^{-\bar{c}''M^2}.$$

Proof. Given the measures μ,ν on the spaces A,B we shall denote by $\mu\otimes\nu$ the product measure on the product space $A\otimes B$.

The following estimates can be obtained by the same methods that we used in the above situations:

$$\mathbb{P}(\mu_i\otimes\mu_{i+1}(1-\chi_{i-1,i})>e^{-c_5M^2})<e^{-c_6M^2}$$

for $i=-N,\ldots,N-1$ and

$$\begin{aligned}\mathbb{P}(\nu\otimes\mu_{-N}(1-\chi_{-N-1})>e^{-c_5M^2})<e^{-c_6M^2},\\ \mathbb{P}(\nu\otimes\mu_N(1-\chi_{N,N+1})>e^{-c_5M^2})<e^{-c_6M^2}.\end{aligned}\tag{4.28}$$

We call μ_{-N-1},μ_{N+1} the natural projections of ν to $\mathscr{S}_{A_{-N-1}},\mathscr{S}_{A_{N+1}}$ respectively. Inequality (4.28) implies that on a set of interactions of probability larger than $1-(2N+2)\exp(-c_6M^2)$, we have

$$\mu_j\otimes\mu_{j+1}(\chi_{j,j+1})>1-e^{-c_5M^2}\tag{4.29}$$

for $j=-N-1,\ldots,N$.

We assume that (4.29) are satisfied. Then it is easy to see that for $i=(-N-1)/2,\ldots,(N-1)/2$, there exist sets $\Omega_{2i}\in\mathscr{S}_{A_{2i}}$ such that

i) $$\mu_{2i}(\Omega_{2i})>1-e^{-c_7M^2},$$

ii) $$\forall s^{(2i)}\in\Omega^{(2i)}\exists\Omega_{2i+1}^{+}(s^{(2i)})\in\mathscr{S}_{A_{2i+1}}$$

with

$$\mu_{2i+1}(\Omega_{2i+1}^{+}(s^{(2i)}))>1-e^{-c_7M^2}$$

and

$$\chi_{2i,2i+1}(s^{(2i)},s^{(2i+1)})=1\quad\forall s^{(2i)}\in\Omega_{2i},\quad s^{(2i+1)}\in\Omega_{2i+1}^{+}(s^{(2i)})$$

for a suitable positive constant c_7.

In the same way we get that for $i=(-N+1)/2,\ldots,(N+1)/2$ there exist sets $\bar{\Omega}_{2i}\in\mathscr{S}_{A_{2i}}$ such that

i) $$\mu_{2i}(\bar{\Omega}_{2i})>1-e^{-c_7M^2},$$

ii) $$\forall s^{(2i)}\in\bar{\Omega}_{2i}\quad\exists\Omega_{2i-1}^{-}(s^{(2i)})$$

with

$$\mu_{2i-1}(\Omega_{2i-1}^{-}(s^{(2i)}))>1-e^{-c_7M^2}$$

and

$$\chi_{2i-1,2i}(s^{(2i-1)}, s^{(2i)}) = 1 \quad \forall s^{(2i)} \in \bar{\Omega}_{2i}, \qquad s^{(2i-1)} \in \Omega^{-}_{2i-1}(s^{(2i)}).$$

If in definition (4.27) we bound the sum from below by restricting it to the configurations in the set $\Omega^{+}_{2i+1}(s^{(2i)}) \cap \Omega^{-}_{2i+1}(s^{(2i+2)})$, we get that

$$\begin{aligned} &\exp \tilde{W}_{2i,2i+2}(s^{(2i)}, s^{(2i+2)}) \geqq (1 - e^{-c_8 M^2}) e^{-M}, \\ &\qquad \forall s^{(2i)}, s^{(2i+2)} \in \Omega_{2i} \otimes \bar{\Omega}_{2i+2}. \end{aligned} \tag{4.30}$$

On the other side we have of course $\tilde{W} < 2M$.

We set

$$\tilde{\Omega}_{2i} = \Omega_{2i} \cap \bar{\Omega}_{2i}. \tag{4.31}$$

On the set of realizations on which (4.29) are satisfied we have, using expression (4.26),

$$\begin{aligned} \mu^{\tilde{H}}_{\nu,\Lambda}\left(f \left(\prod_{i=-N}^{N+1} \chi_{i-1,i} \right) \cdot \mathbb{1}_{\tilde{\Omega}^c_{2j}} \right) &\leqq \|f\| e^{2M} \sum_{s^{(2j)} \in \tilde{\Omega}^c_{2j}} e^{2M} \mu_{2j}(s^{(2j)}) \\ &\leqq e^{4M} e^{-c_9 M^2} \|f\|, \end{aligned}$$

so that we conclude

$$\begin{aligned} \mathbb{P}\Bigg(\Bigg| &\mu^{\tilde{H}}_{\nu,\Lambda}\left(f \prod_{i=-N}^{N+1} \chi_{i-1,i} \prod_{j=-N/2}^{N/2} \mathbb{1}_{\tilde{\Omega}_{2j}} \right) \\ &- \mu^{\tilde{H}}_{\nu,\Lambda}\left(f \prod_{i=-N}^{N+1} \chi_{i-1,i} \right) \Bigg| > N e^{4M} e^{-c_9 M^2} \|f\| \Bigg) \\ &\leqq (2N+1) e^{-c_6 M^2}. \end{aligned} \tag{4.32}$$

From Eqs. (4.18), (4.32) with $f=1$ we immediately get

$$\begin{aligned} \mathbb{P}\Bigg(\Bigg| &\frac{\mu^{\tilde{H}}_{\nu,\Lambda}\left(f \prod_{i=-N}^{N+1} \chi_{i-1,i} \prod_{j=-N/2}^{N/2} \mathbb{1}_{\tilde{\Omega}_{2j}} \right)}{\mu^{\tilde{H}}_{\nu,\Lambda}\left(\prod_{i=-N}^{N} \chi_{i-1,i} \prod_{j=-N/2}^{N/2} \mathbb{1}_{\tilde{\Omega}_{2j}} \right)} - \mu^{\tilde{H}}_{\nu,\Lambda}\left(f \prod_{i=-N}^{N} \chi_{i-1,i} \prod_{j=-N/2}^{N/2} \mathbb{1}_{\tilde{\Omega}_{2j}} \right) \Bigg| \\ &> N \|f\| e^{4M} e^{-c_{10} M^2} \Bigg) \leqq N e^{-c_{11} M^2}. \end{aligned} \tag{4.33}$$

Equations (4.32) and (4.33) imply the result. □

Proof of Lemma 3.1.

From Propositions 4.1, 4.2, 4.3, we get

$$\begin{aligned} \mathbb{P}\Bigg(\Bigg| &\mu^{H}_{\nu,\Lambda}(f) - \frac{\mu^{\tilde{H}}_{\nu,\Lambda}\left(f \prod_{i=-N}^{N+1} \chi_{i-1,i} \prod_{j=-N/2}^{N/2} \mathbb{1}_{\tilde{\Omega}_{2j}} \right)}{\mu^{\tilde{H}}_{\nu,\Lambda}\left(\prod_{i=-N}^{N} \chi_{i-1,i} \prod_{j=-N/2}^{N/2} \mathbb{1}_{\tilde{\Omega}_{2j}} \right)} \Bigg| \\ &\geqq \|f\| \left(\frac{2}{|\Lambda|^{\lambda}} + N e^{-c_{12} M^2} \right) \Bigg) \leqq \exp(-|\Lambda|^{4\lambda} + N e^{-c_{13} M^2}). \end{aligned} \tag{4.34}$$

We remark that the sets $\tilde{\Omega}_i$ for $i=-N, N$ depend on the choice of the measure $\nu: \tilde{\Omega}_i = \tilde{\Omega}_i^{(\nu)}$. Now, given two measures ν, ν' we define $\tilde{\Omega}_i = \tilde{\Omega}_i^{(\nu)} \cap \tilde{\Omega}_i^{(\nu')}$ for $i=-N, N$.

For any positive δ such that

$$\left| \frac{\mu_{\nu,\Lambda}^{\tilde{H}}\left(f \prod_{i=-N}^{N+1} \chi_{i-1,i} \prod_{j=-N/2}^{N/2} \mathbb{1}_{\tilde{\Omega}_{2j}}\right)}{\mu_{\nu,\Lambda}^{\tilde{H}}\left(\prod_{i=-N}^{N} \chi_{i-1,i} \prod_{j=-N/2}^{N/2} \mathbb{1}_{\tilde{\Omega}_{2j}}\right)} - \frac{\mu_{\nu',\Lambda}^{\tilde{H}}\left(f \prod_{i=-N}^{N+1} \chi_{i-1,i} \prod_{j=-N/2}^{N/2} \mathbb{1}_{\tilde{\Omega}_{2j}}\right)}{\mu_{\nu',\Lambda}^{\tilde{H}}\left(\prod_{i=-N}^{N} \chi_{i-1,i} \prod_{j=-N/2}^{N/2} \mathbb{1}_{\tilde{\Omega}_{2j}}\right)} \right| < \delta, \tag{4.35}$$

we have

$$\begin{aligned} &\mathbb{P}(|\mu_{\nu,\Lambda}^{H}(f) - \mu_{\nu',\Lambda}^{H}(f)|) > \delta + 2\|f\|\left(\frac{2}{|\Lambda|^{\lambda}} + Ne^{-c_{12}M^2}\right) \\ &\leqq 2\exp(-|\Lambda|^{4\lambda} + 2Ne^{-c_{13}M^2}). \end{aligned} \tag{4.36}$$

Now we want to consider the quantity

$$\begin{aligned} &\frac{\mu_{\nu,\Lambda}^{\tilde{H}}\left(f \prod_{i=-N}^{N+1} \chi_{i-1,i} \prod_{j=-N/2}^{N/2} \mathbb{1}_{\tilde{\Omega}_{2j}}\right)}{\mu_{\nu,\Lambda}^{\tilde{H}}\left(\prod_{i=-N}^{N+1} \chi_{i-1,i} \prod_{j=-N/2}^{N/2} \mathbb{1}_{\tilde{\Omega}_{2j}}\right)} \\ &\qquad \equiv \frac{\int \nu(d\sigma) \sum_{s^{(-N)} \in \tilde{\Omega}_{-N}} \cdots \sum_{s^{(N)} \in \tilde{\Omega}_N} f(s^{(0)}) \cdot \prod_{i=-N/2}^{N/2} \mu_{2i}(s^{(2i)})}{\int \nu(d\sigma) \sum_{s^{(-N)} \in \tilde{\Omega}_N} \cdots \sum_{s^{(N)} \in \tilde{\Omega}_N} \prod_{i=-N/2}^{N/2} \mu_{2i}(s^{(2i)})} \\ &\qquad \times \frac{\exp \sum_{i=-N/2}^{N/2} \tilde{W}_{2i,2i+2}(s^{(2i)}, s^{(2i+2)})}{\exp \sum_{i=-N/2}^{N/2} \tilde{W}_{2i,2i+2}(s^{(2i)}, s^{(2i+2)})}, \end{aligned} \tag{4.37}$$

and the analogous quantity for ν'.

We can study the above quantities with the help of Proposition A.1 stated and proved in the Appendix. We must put there $q = \dfrac{N+1}{2}$, $S_i = \tilde{\Omega}_{2i}$, $T_{i,i+1} = e^{\tilde{W}_{2i,2i+2}}$, $\nu_i = \mu_{2i}$, $B = e^M$. If we take

$$\delta = 2e^{2M}(1 - e^{-4M})^{N/2},$$

and choose

$$M = (\log N)^{3/4}$$

the result of Lemma 3.1 follows. □

The proof of Lemma 3.2 is completely analogous to the previous one and we leave to the reader the task of making the obvious changes.

Appendix

For $i=-q-1,\ldots,q+1$ let S_i be a measurable space. Let ν_i be a probability measure for $-q\leqq i\leqq q$ and let $T_{i,i+1}(s,t)$ be a positive real function on $S_i\otimes S_{i+1}$ for $-q-1\leqq i\leqq q$ such that for every $s\in S_i$, $t\in S_{i+1}$,

$$B^{-1}\leqq T_{i,i+1}(s,t)\leqq B \tag{A1}$$

for some positive constant B.

Let the functions $T_{k,\ell}(s,t)$, $s\in S_k$, $t\in S_\ell$, be defined for $-q-1\leqq k<\ell\leqq q+1$ by the rule

$$T_{k,\ell}(s,t)=\int T_{k,\ell-1}(s,z)T_{\ell-1,\ell}(z,t)d\nu_{\ell-1}(z)\,. \tag{A2}$$

We have the following:

A.1. Proposition. *For every k with $-q-1\leqq k\leqq q-1$ there exist two positive functions $u(s)$, $v(t)$ defined respectively on S_k and S_{q+1} such that for every $s\in S_k$, $t\in S_{q+1}$,*

$$\left|\frac{T_{k,q+1}(s,t)}{u(s)v(t)}-1\right|\leqq 2B^2(1-B^{-4})^{q-k-1}\,. \tag{A3}$$

Similarly for $-q+1\leqq k\leqq q+1$ there exist positive functions $\bar u(s)$, $\bar v(t)$ on S_k, S_{-q-1} such that for every $s\in S_k$, $t\in S_{-q-1}$,

$$\left|\frac{T_{-q-1,k}(t,s)}{\bar v(t)\bar u(s)}-1\right|\leqq 2B^2(1-B^{-4})^{k+q-1}\,. \tag{A3'}$$

Proof. We shall prove (A3). (A3′) can be obtained in the same way with obvious changes.

We define the functions $P_{k,\ell}(s,t)$ for $-q-1\leqq k<\ell\leqq q$, $s\in S_k$, $t\in S_\ell$ by

$$P_{k,\ell}(s,t)=\frac{T_{k\ell}(s,t)\int T_{\ell,q}(t,z)d\nu_q(z)}{\int T_{k,q}(s,z)d\nu_q(z)}\,, \tag{A4}$$

where we agree that the integral in the numerator of the right-hand side of (A4) is absent for $\ell=q$. The functions $P_{k,\ell}(s,t)$ are the densities of the transition probabilities for a (inhomogeneous) Markov chain, i.e.

$$\begin{gathered}\int P_{k,\ell}(s,t)d\nu_\ell(t)=1\,,\\ P_{k,m}(s,t)=\int P_{k,\ell}(s,z)P_{\ell m}(z,t)d\nu_\ell(z)\,.\end{gathered} \tag{A5}$$

We first find upper and lower bounds for the values of $P_{k,\ell}(s,t)$. We have for $-q-1\leqq k<k+2\leqq\ell\leqq q+1$,

$$\frac{T_{k,\ell}(s,z)}{T_{k,\ell}(t,z)}=\frac{\int T_{k,k+1}(s,y)T_{k+1,\ell}(y,z)d\nu_{k+1}(y)}{\int T_{k,k+1}(t,y)T_{k+1,\ell}(y,z)d\nu_{k+1}(y)}\geqq B^{-2} \tag{A6}$$

and, similarly,

$$\frac{T_{k,\ell}(s,z)}{T_{k,\ell}(t,z)}\leqq B^2\,,\qquad B^{-2}\leqq\frac{T_{k,\ell}(z,s)}{T_{k,\ell}(z,t)}\leqq B^2 \tag{A7}$$

for every choice of s,t,z in the appropriate spaces.

It follows that

$$P_{k\ell}(s,t)=\frac{T_{k,\ell}(s,t)\int T_{\ell,q}(t,z)dv_q(z)}{\int T_{k\ell}(s,y)dv_\ell(y)\int T_{\ell,q}(y,z)dv_q(z)}\geqq B^{-4}. \tag{A8}$$

We can obtain from (A8) the following estimate for $-q-1\leqq k<\ell\leqq q$ and for every $s, s'\in S_k$

$$\int |P_{k,\ell}(s,t)-P_{k,\ell}(s',t)|dv_\ell(t)\leqq 2(1-B^{-4})^{\ell-k-1}. \tag{A9}$$

Indeed for $-q-1\leqq k\leqq k+2<\ell\leqq q$,

$$\begin{aligned}&\int |P_{k,\ell}(s,t)-P_{k,\ell}(s',t)dv_\ell(t)\\&\quad=\int |\int (P_{k,\ell-1}(s,z)-P_{k,\ell-1}(s',z))P_{\ell-1,\ell}(z,t)dv_{\ell-1}(z)|dv_\ell(t)\\&\quad=\int |\int B^{-4}(P_{k,\ell-1}(s,z)-P_{k,\ell-1}(s',z))P_{\ell-1,\ell}(z,t)dv_{\ell-1}(z)|dv_\ell(t)\\&\qquad+\int |\int (P_{k,\ell-1}(s,z)-P_{k,\ell-1}(s',z))\,(P_{\ell-1,\ell}(z,t)-B^{-4})dv_{\ell-1}(z)|dv_\ell(t).\end{aligned} \tag{A10}$$

The first term on the right-hand side of (A10) vanishes, whereas the second term can be bounded by

$$(1-B^{-4})\int P_{k,\ell-1}(s,z)-P_{k,\ell-1}(s',z)|dv_{\ell-1}(z), \tag{A11}$$

so that (A9) follows by induction.

Let now, for $-q-1\leqq k\leqq q-1$, $u(s)$ be the function on S_k defined by

$$u(s)=\int T_{k,q}(s,z)dv_q(z), \tag{A12}$$

and, given an arbitrary point s^* in S_k, let $v(t)$ be the function on S_{q+1} given by

$$v(t)=\int P_{k,q}(s^*,z)T_{q,q+1}(z,t)dv_q(z). \tag{A13}$$

By using (A1) and the fact that $P_{k,q}(s^*,z)dv_q(z)$ is a probability measure, we get

$$v(t)\geqq B^{-1} \tag{A14}$$

for every $t\in S_{q+1}$.

We have

$$\begin{aligned}\left|\frac{T_{k,q+1}(s,t)}{u(s)}-v(t)\right|&=|\int (P_{k,q}(s,z)-P_{k,q}(s^*,z))T_{q,q+1}(z,t)dv_q(z)|\\&\leqq 2(1-B^{-4})^{q-k-1}B\end{aligned} \tag{A15}$$

by (A9). Inequalities (A14) and (A15) imply (A3).

Acknowledgements. A. C. D. van E. wishes to express thanks to the Universities of Rome for their hospitality. Financial support from the Sonderforschungsbereich 123 of the Deutsche Forschungsgemeinschaft is gratefully acknowledged. We thank Donatella Capocaccia for helpful discussions.

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Communicated by T. Spencer

Received January 28, 1986; in revised form July 21, 1986