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The Topology of Asymptotically Euclidean Static Perfect Fluid Space-Time

A.K.M. Masood-ul-Alam

Department of Mathematics, Australian National University, GPO Box 4, Canberra, A.C.T. 2601, Australia

Abstract. It is shown that a geodesically complete, asymptotically Euclidean, static perfect fluid space-time satisfying the time-like convergence condition and having a connected fluid region is diffeomorphic to $\mathbb{R}^3 \times \mathbb{R}$.

Introduction

In this paper we prove that a geodesically complete, asymptotically Euclidean, static perfect fluid space-time with connected fluid region and satisfying the time-like convergence condition is diffeomorphic to $\mathbb{R}^3 \times \mathbb{R}$. It is believed that such a space-time would be spherically symmetric at least for physically reasonable conditions on the density function ρ and the pressure function p.

The above assertion (that the space-time is diffeomorphic to $\mathbb{R}^3 \times \mathbb{R}$) has been claimed in [LB] provided the Poincaré conjecture is valid. In fact a theorem due to Gannon [G] says that such a space-time is diffeomorphic to $N \times \mathbb{R}$, where N is a simply connected complete 3-manifold. The asymptotic conditions then imply that N has the same homotopy as $\mathbb{R}^3$ ([LB]; results in [G] and [LB] do not require that space-time be static). Thus Gannon's result reduced the question to proving the non-existence of fake 3-cells in N (in particular it would give the full result if the 3 dimensional Poincaré conjecture were known to be true). Proving the non-existence of fake 3-cells in an appropriate class of asymptotically Euclidean Riemannian 3-manifolds is thus the main point of this paper. Here we note that a theorem due to Schoen and Yau (Theorem 3 in [SY]) says that a complete, noncompact 3-manifold with positive Ricci curvature is diffeomorphic to $\mathbb{R}^3$.

The main arguments needed are given in Sect. 2 and Sect. 3 of the paper. In Sect. 2 we point out that the theorem of Meeks, Simon and Yau (Theorem 1 in [MSY]) applied in the present context, gives (in case there are fake 3-cells) a totally geodesic embedded sphere not intersecting the "fluid region," where $\rho + p > 0$. The argument to show this is similar to an argument of Frankel and Galloway [FG] (who also used the existence theorem of [MSY]), but here we need an additional approximation argument to make the appropriate stability statement.

In Sect. 3 we use an argument (inspired partly by an argument of Robinson [R] from his proof of spherical symmetry of the static vacuum solution) to prove that there cannot exist a totally geodesic embedded sphere which does not intersect the fluid region. Combined with the result of Sect. 2, this completes the proof of the main theorem for Riemannian 3-manifolds.

In Sect. 4 we complete the proof of the main result for static perfect fluid space-time. The result differs from the similar result in the paper of Frankel and Galloway in that, in our case we may allow $\rho+p$ to vanish outside a connected region.

In Sect. 5 we consider various generalizations of the main theorem. In particular, the topology of suitable space-like hypersurfaces in a certain class of space-times, not necessarily static or perfect fluid, is investigated.

1. Notation and Main Theorem

With regards to tensor notation we use the following conventions.

Italic capital indices $A, B, C, \ldots$ run from 0 to 3, Greek indices $\alpha, \beta, \gamma, \ldots$ run from 1 to 3 and Italic indices $a, b, c, \ldots$ run from 2 to 3.

In local co-ordinates, for a metric $g_{\alpha\beta}$ the Ricci curvature is $\mathrm{Ric}(g)_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$, where $R_{\alpha\beta\mu\nu}$ is the Riemann curvature tensor. For a vector field Z^{β} we have $R^{\alpha}{}_{\beta\mu\nu}Z^{\beta} = Z^{\alpha}{}_{;\nu\mu} - Z^{\alpha}{}_{;\mu\nu}$. Here ; denotes covariant differentiation in the g metric.

Let Σ be a smooth surface embedded in the smooth three dimensional Riemannian manifold (N, g). (By smooth we shall mean C^{∞}.) In local coordinates $\|$ indicates covariant differentiation with respect to the induced connection on Σ. For a tensor T belonging to the tensor bundle of Σ we have

$$T^{\alpha\ldots}{}_{\gamma\ldots\|\varepsilon} = \bar{T}^{\beta\ldots}{}_{\delta\ldots;\sigma} h^{\alpha}{}_{\beta} h^{\delta}{}_{\gamma} \cdots h^{\sigma}{}_{\varepsilon} \quad \text{on } \Sigma,$$

where $\bar{T}$ is a local extension of T in a neighbourhood of Σ in N, and where $h_{\alpha\beta} = g_{\alpha\beta} - n_{\alpha}n_{\beta}$, $h^{\alpha}{}_{\beta} = g^{\alpha\sigma}h_{\sigma\beta}$ on Σ, n_{α} being the unit normal form on Σ. Clearly $h^{\alpha\beta}n_{\alpha} = 0$, $h^{\alpha\beta}{}_{\|\varepsilon} = 0$.

The second fundamental form A of Σ is the tensor $A_{\alpha\beta} = h^{\nu}{}_{\alpha}h^{\mu}{}_{\beta}\bar{n}_{\nu;\mu}$, where $\bar{n}_{\alpha}$ is any local extension of n_{α} in N. Mean curvature $H = A_{\alpha\beta}g^{\alpha\beta}$. $|A|$ denotes the length of the second fundamental form: $|A|^2 = A_{\alpha\beta}A^{\alpha\beta}$.

The Riemann curvature of the induced metric on Σ is denoted by ${}^{h}R_{\alpha\beta\gamma\delta}$, where

$${}^{h}R^{\alpha}{}_{\beta\gamma\delta}Y^{\beta} = Y^{\alpha}{}_{\|\delta\gamma} - Y^{\alpha}{}_{\|\gamma\delta}$$

for all vector fields Y on Σ. The indices are raised using g. We include the embedding equations for easy reference. Gauss' equation: ${}^{h}R^{\alpha}{}_{\beta\gamma\delta} = R^{\varepsilon}{}_{\rho\mu\nu}h^{\alpha}{}_{\varepsilon}h^{\rho}{}_{\beta}h^{\mu}{}_{\gamma}h^{\nu}{}_{\delta} + A^{\alpha}{}_{\gamma}A_{\beta\delta} - A^{\alpha}{}_{\delta}A_{\beta\gamma}$. Contracted Gauss' equation: ${}^{h}R = R - 2\,\mathrm{Ric}(g)_{\alpha\beta}n^{\alpha}n^{\beta} + H^2 - |A|^2$, ${}^{h}R$ and R being respectively the scalar curvatures of Σ and N. Codazzi's equation: $A^{\alpha}{}_{\beta\|\alpha} - A^{\alpha}{}_{\alpha\|\beta} = \mathrm{Ric}(g)_{\varepsilon\sigma}n^{\sigma}h^{\varepsilon}{}_{\beta}$.

For a vector field Y on N, $Y|_p$ denotes its value at a point $p \in N$. $g(,)$ denotes the inner product on the tangent space of N and $\langle , \rangle$ denotes the corresponding induced inner product on Σ. In general the induced inner product on the tensor bundles of N and Σ will be denoted by $g(,)$ and by $\langle , \rangle$ respectively. $|T|^2 \equiv g(T, T)$. For a tensor field T defined on $\zeta_1 \subset N$ and for $\zeta_2 \subset \zeta_1$, $T|\zeta_2$ denotes the restriction of T on ζ_2.

$|\Sigma|$ denotes the area (that is the usual two dimensional Hausdroff measure $\mathscr{H}^2$ in N) of Σ considered as a submanifold of the Riemannian manifold (N, g). L^n denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$.

∇ and Δ will respectively denote the gradient operators and the Laplacian on (N, g). ∇_Σ and Δ_Σ are the corresponding operators on Σ with respect to the induced metric.

$B_R(x)$ is the open ball in $\mathbb{R}^3$ with radius R and centre at $x \in \mathbb{R}^3$.

For a bounded, open, connected subset Ω of $\mathbb{R}^n$, $C^{k,\lambda}(\Omega)$, $k \geqq 0$, $\lambda \in (0,1)$, denotes the space of C^k functions on Ω whose k^{th} order partial derivatives are locally Hölder continuous in Ω with exponent λ. $C^{k,1}(\Omega)$ denotes the space of C^k functions whose k^{th} order partial derivatives are locally Lipschitz continuous in Ω. For $1 \leqq p < \infty$, $W^{k,p}(\Omega)$ denotes the Sobolev space; that is, $W^{k,p}(\Omega)$ is the set of all k times weakly differentiable functions u in Ω such that $D^\alpha u \in L^p(\Omega)$ for all $|\alpha| \leqq k$.

The main theorem we prove is the following (to compare with physics literature, see 1.10 and Sect. 4):

Theorem 1. *Assumptions.* $(N, \{\mathscr{U}_a, \varphi_a\}_{a \in \mathscr{A}})$ *is a* C^∞ *simply connected three-dimensional differentiable manifold with a complete Riemannian metric* g *satisfying the following conditions:*

(a) *In local co-ordinates given by* φ_a, $g_{\alpha\beta} \in C^{1,1}(\varphi_\alpha(\mathscr{U}_a))$ *and* g *satisfies*

$$\mathrm{Ric}(g)_{\alpha\beta} = V^{-1} V_{;\alpha\beta} + \Phi_1 g_{\alpha\beta} \tag{1.1}$$

and

$$\Delta V = V \Phi_2, \tag{1.2}$$

where ; *denotes the covariant differentiation for* g, Δ *is Laplacian for* g, V *is a locally* $C^{1,1}$ *positive function on* N, Φ_1, Φ_2 *are bounded measurable functions on* N, *and* $\Phi_2 \geqq 0$. *(The above equations are assumed to hold almost everywhere in* N.)

(b) *There exists an open connected set* $\mathscr{Q} \subset N$ *such that* $\operatorname*{ess\,inf}_K (\Phi_1 + \Phi_2) > 0$ *for all compact* $K \subset \mathscr{Q}$, *and* $\Phi_1, \Phi_2 = 0$ *in* $N \sim \bar{\mathscr{Q}}$.

(c) *There exists an open connected set* $N_0 \subset N$ *such that* $\bar{N}_0$ *is compact and* $N \sim \bar{N}_0$ *is diffeomorphic to* $\mathbb{R}^3 \sim \bar{B}_1$, *where* $\bar{B}_1$ *is the closed unit ball centred at the origin and, with respect to the standard co-ordinate system in* $\mathbb{R}^3$, *we have, on* $N \sim \bar{N}_0$,

$$g_{\alpha\beta} = \delta_{\alpha\beta} + O(|x|^{-\lambda}) \quad \textit{and} \quad \frac{\partial g_{\alpha\beta}}{\partial x^\sigma} = O(|x|^{-1-\lambda}), \tag{1.3}$$

for some $\lambda \in (0,1)$, *where* $|x| \equiv \left(\sum_{\alpha=1}^{3} (x^\alpha)^2 \right)^{1/2} \to \infty$.

Assertion. N *is topologically* $\mathbb{R}^3$ *and hence diffeomorphic to* $\mathbb{R}^3$.

1.4. *Remark.* Henceforth the co-ordinate system in (c) above will be referred to as the asymptotic co-ordinate system.

1.5. *Remark.* On $N \sim \bar{\mathscr{Q}}$ we have $\Delta V = 0$. Hence on $N \sim \bar{\mathscr{Q}}$, V is locally $C^{2,\mu}$, $\mu \in (0,1)$ (See the paragraph following 3.7 in Sect. 3).

It can be easily shown (see [MA]) that when $\bar{\mathscr{Q}}$ is compact, by virtue of 1.3 we have in the asymptotic co-ordinate system,

$$V = C_1 - \frac{C_2}{|x|} + \eta, \quad \text{where} \quad \eta = O(|x|^{-1-\beta}), \quad \frac{\partial \eta}{\partial x^\alpha} = O(|x|^{-2-\beta}), \tag{1.6}$$

and

$$\left(R^{-3}\int_{N_{3R}\sim N_R}\left|\frac{\partial^2\eta}{\partial x^\alpha \partial x^\tau}\right|^2\right)^{1/2}=O(R^{-3-\beta})\quad\text{for some}\quad \beta\in(0,1),$$

constants $C_1>0$ and $C_2\geqq 0$. Here $N_R\subset N$ is such that in the asymptotic coordinate system, $N\sim N_R=\mathbb{R}^3\sim B_R(0)$. Without loss of generality we write (1.6) as

$$V=1-\frac{m}{|x|}+\eta,\quad\text{where}\quad m\geqq 0\quad\text{and}\quad \eta\text{ is as above.}\tag{1.7}$$

It is well-known that when $m=0$ and Φ_1 tends to 0 at infinity, then g is flat. To see this we note that in this case (1.2) and (1.7) imply $V\equiv$ constant. Hence by (1.1) we have $\mathrm{Ric}(g)_{\alpha\beta}=\Phi_1 g_{\alpha\beta}$. The contracted Bianchi identity for g (see [MA]) then implies that $\Phi_1\equiv$ constant. Thus $\Phi_1\equiv 0$ and g is Ricci flat and hence flat.

1.8. *Remark.* The contracted Bianchi identity for g (in weak sense, see Eq. (A2) in [MA] where the third derivatives of the metric are avoided by integration by parts after contracting the usual expression for the contracted Bianchi identity with a smooth vector field having compact support) implies that $\Phi_2-\Phi_1$ is a Lipschitz function on N and

$$(\Phi_2-\Phi_1)_{;\beta}=-2V^{-1}(\Phi_1+\Phi_2)V_{;\beta}.\tag{1.9}$$

For simplicity we put

$$\begin{cases}\Phi_2-\Phi_1=16\pi p,\\ 3\Phi_1+\Phi_2=16\pi\rho,\end{cases}\tag{1.10}$$

where the factor 16π has been used in accordance with the usual convention of physics literature.

It is shown in [KS] that if $\bar{\mathscr{D}}$ is compact and p is non-negative then ρ cannot be a Lipschitz function of p unless $p\equiv 0$. In [KS] it is assumed that V has one critical point. We give an alternative argument which does not need the assumptions on the critical points of V. If ρ is a Lipschitz function of p then (1.9) and (1.2) give

$$\Delta p+\left(2+\frac{d\rho}{dp}\right)V^{-1}V^{;\beta}p_{;\beta}=-4\pi(\rho+p)(\rho+3p)\leqq 0.$$

Now p vanishes outside $\bar{\mathscr{D}}$ and $N\sim\bar{\mathscr{D}}\neq\emptyset$. Hence by the (generalized) strong maximum principle (Theorem 8.19 in [GT]), $p\equiv 0$.

1.11. *Remark.* The assumption $\Phi_2\geqq 0$ in (a) is not necessary in case $\mathscr{D}$ is unbounded. This is because in this case, if a sphere $S\subset N$ with $S\cap\mathscr{D}=\emptyset$ bounds the compact manifold-with-boundary $N_1\subset N$ then $N_1\cap\mathscr{D}=\emptyset$ (since, by hypothesis $\mathscr{D}$ is connected; for details see Remark 3.1 in Sect. 3). Hence Case II in the proof of Theorem 5 cannot occur automatically. The proof of Case II uses (1.7) and that $V<1$ on $N\sim\bar{\mathscr{D}}$. The condition $\Phi_2\geqq 0$ enables us to use the maximum principle to deduce that $V<1$ on N and also to determine the sign of C_2 in (1.6).

1.12. *Remark.* A physical example where g and V are piecewise C^2 and ρ is discontinuous across $\partial\mathscr{D}$ is the static stellar model described in [KS] and [L]. Our asymptotic conditions are slightly more general than those given in [KS] or [K].

1.13. *Remark.* Further generalizations of Theorem 1 are given in Sect. 5. In particular alternative definitions of "$\mathscr{Q}$" are considered.

2. Stable Minimal Surfaces

In this section we use a theorem of Meeks, Simon and Yau (Theorem 1 in [MSY]) regarding the existence of minimal surfaces in 3-manifolds to prove that either N is topologically Euclidean or there exists an embedded totally geodesic sphere in $N \sim \mathscr{Q}$. The proof is essentially a straightforward modification of a result due to Frankel and Galloway (Corollary to Theorem 1 in [FG]). However, since the metric is only $C^{1,1}$ in our case, we have to deduce a suitable form of the "stability inequality" involving the Ricci curvature which, in our case, is only defined almost everywhere and hence does not, in general, make sense on the minimal surface.

Let $\mathscr{C}$ denote the collection of all connected compact C^2 2-dimensional surfaces-without-boundary embedded in N, and let $\mathscr{C}_1$ denote the collection of compact embedded surfaces Σ such that each component of Σ is an element of $\mathscr{C}$. Given $\Sigma \in \mathscr{C}$ we let $I(\Sigma)$ be the collection of all $\hat{\Sigma}$ such that $\hat{\Sigma}$ is isotopic to Σ via a smooth isotopy: $\psi: [0,1] \times N \to N$, where

(i) $\psi_0(x) \equiv \psi(0,x) = 1_N(x)$, 1_N being the identity map on N;
(ii) ψ_t, defined by $\psi_t(x) = \psi(t,x)$, $(t,x) \in [0,1] \times N$, is a diffeomorphism of N onto N;
(iii) $\psi_t | N \sim K = 1_{N \sim K}$ for $t \in [0,1]$ for some compact set $K \subset N$ independent of t.

We shall say that a two-sided surface $S \in \mathscr{C}$ is area minimizing if there exists $d > 0$ such that

$$|S| \leqq \inf_{\substack{\hat{\Sigma} \in I(S) \\ \hat{\Sigma} \subset \{x \in N | \operatorname{dist}(x,S) < d\}}} |\hat{\Sigma}|. \tag{2.1}$$

Now from the asymptotic conditions (c) in the hypothesis of the main Theorem 1 it follows that there exists a smooth sphere S_r given by $|x| = r$ in the asymptotic co-ordinate system such that the mean curvature of S_r in (N, g) with respect to the outward normal is strictly positive. Let $\bar{\mathscr{N}}_1$ be the simply connected compact submanifold of N with boundary $\partial \bar{\mathscr{N}}_1 = S_r$ (that is, in asymptotic co-ordinate system $N \sim \bar{\mathscr{N}}_1 = \mathbb{R}^3 \sim \bar{B}_r(0)$). Then

Theorem 2. [MSY]. *Either $\bar{\mathscr{N}}_1$ is diffeomorphic to a closed unit ball in $\mathbb{R}^3$ or there exists a $C^{2,\alpha}$, $\alpha \in (0,1)$, embedded area minimizing minimal sphere S in the interior $\mathscr{N}_1$ of $\bar{\mathscr{N}}_1$.*

Proof of 2. It is a particular case of the more general fundamental existence theorem (Theorem 1) in [MSY]. We note that the only places in the proof of the latter theorem where the smoothness of the metric is used are in the definition of homogeneous regularity and in the arguments relating to homothetic expansion on p. 639. (As an alternative method we can avoid the use of the above theorem for $C^{1,1}$ metric altogether and can directly apply the smooth metric version of this theorem (which is proved in [MSY]) to a sequence of smooth metrics approximating the $C^{1,1}$

metric to prove our final theorem (3) of this section. This alternative approach is based on convergence of sequences $\{\Sigma_k\}$ of smooth compact surfaces in a Riemannian 3-manifold (N, g) where Σ_k is stable minimal in the smooth metrics ${}^k g$ approximating the $C^{1,1}$ metric g. For details see [MA]). On p. 639 in [MSY] it is assumed that N can be locally isometrically embedded in the Euclidean space. This can be avoided by considering the local co-ordinates representation for N in the neighbourhood of a point x_0 with the $C^{1,1}$ metric $g_{\alpha\beta}$ satisfying $g_{\alpha\beta}(0)=\delta_{\alpha\beta}$ and $(\partial g_{\alpha\beta}/\partial x^\tau)(0)=0$, where 0 is the image of x_0. The definition of homogeneous regularity can be modified by having a countable atlas with uniformly bounded $C^{1,1}$ norm of the metric in each chart. By hypothesis N has such an atlas. Finally, we note that since N is simply connected there cannot be any embedded projective space in N (by Theorem 4.7 on p. 108 in [H]). $\square$

We shall now deduce a suitable form of the "stability inequality" from 2.1 and use this inequality to prove the existence of a totally geodesic embedded sphere $S \subset N \sim \mathscr{Q}$ in case N is not topologically Euclidean. In particular, we prove the following theorem:

Theorem 3. *Either N is diffeomorphic to $\mathbb{R}^3$ or there exists a $C^{2,\alpha}$, $\alpha\in(0,1)$, embedded totally geodesic sphere S in $N \sim \mathscr{Q}$.*

Proof of 3. We take an 1-parameter family $\{\varphi_t\}_{0\leqq t\leqq 1}$ of diffeomorphisms $\mathscr{N}_2 \to \mathscr{N}_2$ of some neighbourhood $\mathscr{N}_2$ of S in N such that

(i) $\varphi(t,x) \equiv \varphi_t(x)$ is a C^2 map: $(-1,1)\times\mathscr{N}_2 \to \mathscr{N}_2$;
(ii) $\varphi_0(x) \equiv x, x\in\mathscr{N}_2$;
(iii) $\partial\varphi(t,x)/\partial t|_{0,S} = \xi\hat{n}$,

where ξ is a smooth positive function and $\hat{n}$ is a smooth vector field on S such that for some given $\varepsilon>0$, the C^1 norm of $(\xi - V)$ in $\mathscr{N}_2$ is less than $\varepsilon, |\hat{n}-n_0|^2<\varepsilon$ and $|\nabla_Y(\hat{n}-n_0)|^2 < \varepsilon|Y|^2$ for all vectors $Y\in T_xN, x\in S, n_0$ being a unit normal vector field on S.

Now let $f(t)=|S_t|$, where $S_t=\varphi_t(S)$. Then $df/dt|_0=0$, since S is area minimizing. Also given any interval $I_j=(-1/j, 1/j)$ (however small) about $t=0$ and given any $\bar{t}\in I_j$, we have

$$0 \leqq f(\bar{t}) - f(0) = \int_0^{\bar{t}} \frac{df}{dt}dt = \int_0^{\bar{t}}\int_0 \frac{d^2f}{ds^2}dsdt.$$

Thus $d^2f/ds^2 \geqq 0$ on a subset $A_j \subset I_j$ with A_j having positive measure.

Now there exist $C_j \subset A_j$ with C_j having positive measure such that, for all $t\in C_j$, $\partial^2 g_{\alpha\beta}/\partial x^\sigma\partial x^\tau$ and $(\partial^2 V/\partial x^\sigma \partial x^\tau)$ exist almost everywhere on S_t. To see this we consider the local chart $(\mathscr{U}_i, \varphi_i)$. Let B be the set $\{x\in\mathscr{U}_i | x\in\varphi_t(S), t\in[-(1/j),(1/j)]$ for sufficiently large j and some $(\partial^2 g_{\alpha\beta}/\partial x^\sigma\partial x^\tau)$ is not defined at $x\}$.

Then

$$0 = \int_{\varphi_i(B)} |\nabla t|\, dL^3 = \int_{-\bar{t}}^{\bar{t}} \mathscr{H}^2(\varphi_i(B\cap S_t))dt$$

by the co-area formula. Thus for almost all $t\in[-\bar{t},\bar{t}]$, $\mathscr{H}^2(B\cap S_t)=0$.

For any $t \in C_j$ we have the following second variation formula on S_t:

$$\left(\frac{d^2}{ds^2}|S_s|\right)_{s=t} = \int_{S_t}\left(\operatorname{div}_{S_t} Z + (\operatorname{div}_{S_t} X)^2 + \sum_{a=1}^{2} |(\nabla_{\tau_a} X)^{\perp}|^2 - \sum_{b,a=1}^{2} g(\tau_a, \nabla_{\tau_b} X) g(\tau_b, \nabla_{\tau_a} X) - \sum_{a=1}^{2} g(R(\tau_a, X)X, \tau_a)\right),$$

where τ_a is an orthonormal vector basis on $S_t, X = (d/ds)(\varphi_s(x))|_{s=t}, x \in S_t$; $Z = (d^2/ds)(\varphi_s(x))|_{s=t}, x \in S_t$; $(\nabla_{\tau_a} X)^{\perp}$ is the part of $\nabla_{\tau_a} X$ normal to S_t and $(R(\cdot,\cdot)\cdot)$ is the Riemann tensor.

Restricting t to a subset D_j of C_j with D_j having positive measure, we have $X = \xi(n + \eta)$, where n is the unit normal vector to S_t and where $|n|^2 < \varepsilon, |\nabla_Y \eta|^2 < \varepsilon |Y|^2$ for $Y \in T_x N, x \in S_t$. Hence we have, for $t \in D_j$,

$$\begin{aligned} 0 &\leqslant \left(\frac{d^2}{ds^2}|S_s|\right)_{s=t} \\ &= \int_{S_t} (g(Z,n)H_t + \xi^2 H_t^2 + |\nabla_{S_t} \xi|^2 - |A_t|^2 \xi^2 - \xi^2 \operatorname{Ric}(g)(n,n)) + c\varepsilon, \end{aligned} \tag{2.2}$$

where A_t is the second fundamental form of S_t and H_t is the mean curvature of S_t. c is a constant depending only on C^1 norm of ξ. This implies

$$\int_{S_t} (\xi^2(|A_t|^2 + \operatorname{Ric}(g)(n,n)) - |\nabla_{S_t} \xi|^2) \leqslant \int_{S_t} (g(Z,n)H_t + \xi^2 H_t^2) + c\varepsilon.$$

Now there exists $E_j \subset D_j$ also with positive measure such that (1.1) and (1.2) hold almost everywhere on S_t for all $t \in E_j$. Hence using the fact that the C^1 norm of $(\xi - V)$ in $\mathcal{N}_2$ is less than ε we get, for all $t \in E_j$,

$$\int_{S_t} (V^2(|A_t|^2 + \operatorname{Ric}(g)(n,n)) - |\nabla_{S_t} V|^2) \leqq \int_{S_t} (g(Z,n)H_t + V^2 H_t^2) + c\varepsilon. \tag{2.3}$$

This implies, since $V|S_t$ is $W^{2,p}(S_t)$,

$$\int_{S_t} (V^2(|A_t|^2 + \operatorname{Ric}(g)(n,n)) + V\Delta_{S_t} V) \leqq \int_{S_t} (g(Z,n)H_t + V^2 H_t^2) + c\varepsilon. \tag{2.4}$$

Now using the following well known formula

$$\Delta \psi = \Delta_{S_t} \psi + \psi_{;\alpha\beta} n^\alpha n^\beta + H_t \nabla \psi(n) \quad \text{for any } C^2\ \psi, \tag{2.5}$$

which also holds for V on S_t, $t \in E_j$, we have

$$\begin{aligned} &\int_{S_t} (V^2(|A_t|^2 + \operatorname{Ric}(g)(n,n)) + V\Delta V - VV_{;\alpha\beta} n^\alpha n^\beta) \\ &\quad \leqq \int_{S_t} (g(Z,n)H_t + V^2 H_t^2 + VH_t \nabla V(n)) + c\varepsilon. \end{aligned} \tag{2.6}$$

But $V^2 \operatorname{Ric}(g)(n,n) + V\Delta V - VV_{;\alpha\beta} n^\alpha n^\beta = V^2(\Phi_1 + \Phi_2) \geqq V^2 h$, where h is a continuous function in N with $h > 0$ in $\mathcal{Q}$. Hence

$$\int_{S_t} V^2(|A_t|^2 + h) \leqq \int_{S_t} (g(Z,n)H_t + V^2 H_t^2 + VH_t \nabla V(n)) + c\varepsilon$$

for all $t \in E_j$.

But $E_j \subset (-1/j, 1/j)$. Hence letting $j \to \infty$ and noting that $S = S_0$ is a C^2 minimal surface, we have

$$\int_S V^2(|A|^2 + h) \leqq 0,$$

where A is the second fundamental form of S. Since $h > 0$ in $\mathscr{D}$ we get that $|A| = 0$ and $|S \cap \mathscr{D}| = 0$. Since S is C^2 we have proved Theorem 3. □

3. Proof of the Main Theorem

In this section we shall prove that no embedded totally geodesic sphere exists in $N \sim \mathscr{D}$, and thereby we shall complete the proof of the main theorem.

3.1. *Remark.* Since N is simply connected and S is compact co-dimension 1 embedded submanifold of N, S separates N and $N \sim S$ has exactly two closed components, say N_1 and N_2 having boundary S (see Lemma 4.4 and Theorem 4.6 on p. 107 in [H]). It follows from the asymptotic condition (c) that exactly one of the components, say N_1, is bounded. Thus $\bar{N}_1$ is a compact manifold with boundary S so that Stoke's formula holds. Similar is the case with the set obtained from N_2 by deleting the exterior of any asymptotic sphere S_R given by $|x| = R$ in the asymptotic coordinate system for sufficiently large R.

In the following lemma we deduce some formulae we shall need later.

Lemma 4. *Let S be a (C^2) totally geodesic embedded sphere in (N, g) such that $S \subset N \sim \mathscr{D}$. We suppose n is a continuous unit normal form on S. Then*

(i) $g(n, \nabla V) = m'$, *a constant on S*; (3.2)

(ii) $\int_S \frac{|\nabla_S V|^2}{V^2} = 4\pi$, *where ∇_S is the gradient operator on S with respect to the metric induced from g; and provided $V < 1$,* (3.3)

(iii) *for a sequence T_l of smooth spheres in $N \sim \bar{\mathscr{D}}$ converging to S in the C^2 sense*

$$\lim_{l \to \infty} \int_{T_l} \frac{cV^2 + a}{V(1 - V^2)^3} g(\tilde{n}, \nabla w) = \int_S 2m' \left\langle \nabla_S \left(\frac{cV^2 + a}{V(1 - V^2)^3} \right), \nabla_S V \right\rangle, \tag{3.4}$$

where $w = |\nabla V|^2$, $\tilde{n} = \tilde{n}(l)$ is the smooth unit normal form on T_l consistent in direction with n, $\langle , \rangle$ denotes induced inner product on T_l and c, a are arbitrary constants to be specified later.

3.5. *Remark.* If $S \subset N \sim \bar{\mathscr{D}}$, we may replace (3.4) by the pointwise relation $g(n, \nabla w) = -m'RV$ on S, R being the scalar curvature of S. However S may touch $\partial \mathscr{D}$, where the metric is not C^2. The extra term $(cV^2 + a)/(V(1 - V^2)^3)$ has been introduced for later application. In (3.2) the sign of the constant m' depends on which normal direction is considered.

Proof of 4. We first note that there exists a sequence T_l of smooth spheres in $N \sim \bar{\mathscr{D}}$ converging to S in the C^2 sense. This is because S separates N (see Remark 3.1) and by hypothesis $\mathscr{D}$ is connected.

Now in $N \sim \bar{\mathscr{D}}$, (1.1) and (1.2) become

$$\mathrm{Ric}(g)_{\alpha\beta} = V^{-1} V_{;\alpha\beta} \tag{3.6}$$

and

$$\Delta V = 0 \tag{3.7}$$

respectively.

Standard PDE regularity theorems (for example, Theorem 9.19 in [GT]) applied to 3.7 gives V to be locally $W^{3,p}$, $1 < p < \infty$, on $N \sim \bar{\mathcal{Q}}$. From 3.6 it follows that $\mathrm{Ric}(g)$ is locally $W^{1,p}$ on $N \sim \bar{\mathcal{Q}}$.

Now for the smooth compact embedded surface T_l in $N \sim \bar{\mathcal{Q}}$ we can use the following weak form of Codazzi's equation:

$$\int_{T_l} b^\mu \,\mathrm{Ric}(g)_{\alpha\beta}(\delta^\alpha_\mu - \tilde{n}^\alpha \tilde{n}_\mu)\tilde{n}^\beta = \int_{T_l} (b^\mu{}_{\|\mu} H - b^\mu{}_{\|\beta} A^\beta{}_\mu) \tag{3.8}$$

for any C^1 vector field b^μ on T_l. Here $A_{\alpha\beta} = A_{\alpha\beta}(l)$ is the second fundamental form of T_l and $H = H(l)$ is the mean curvature of T_l.

Now let f be any C^2 function on S and let $\bar{f}$ be a C^2 extension of f in a fixed neighbourhood of S such that $\bar{f}$ restricted to T_l is a C^2 function on T_l for all sufficiently large l.

On any such T_l in local co-ordinates we have

$$\bar{f}^{\|\sigma}(V_{;\alpha}\tilde{n}^\alpha)_{\|\sigma} = \bar{f}^{\|\sigma}(V_{;\alpha\beta}(\delta^\beta_\sigma - \tilde{n}^\beta \tilde{n}_\sigma)\tilde{n}^\alpha + V_{;\alpha}\bar{\tilde{n}}^\alpha{}_{;\beta}(\delta^\beta_\sigma - \tilde{n}^\beta \tilde{n}_\sigma), \tag{3.9}$$

where $\bar{\tilde{n}}$ is any local extension of $\tilde{n}$ in N.

On using (3.6) and $(\bar{\tilde{n}}_\alpha \bar{\tilde{n}}^\alpha)_{;\beta}(\delta^\beta_\sigma - \tilde{n}^\beta \tilde{n}_\sigma) = 0$, (3.9) yields

$$\bar{f}^{\|\sigma}(V_{;\alpha}\tilde{n}^\alpha)_{\|\sigma} = \bar{f}^{\|\sigma}(V\,\mathrm{Ric}(g)_{\alpha\beta}(\delta^\beta_\sigma - \tilde{n}^\beta \tilde{n}_\sigma)\tilde{n}^\alpha + V^{\|\mu} A_{\mu\sigma}). \tag{3.10}$$

Hence integrating and using (3.8) with $b^\sigma = V\bar{f}^{\|\sigma}$, we have

$$\int_{T_l} \bar{f}^{\|\sigma}(V_{;\alpha}\tilde{n}^\alpha)_{\|\sigma} = \int_{T_l} ((V\bar{f}^{\|\sigma})_{\|\sigma} H - (V\bar{f}^{\|\sigma})_{\|\alpha} A^\alpha{}_\sigma + A_{\mu\sigma} V^{\|\mu}\bar{f}^{\|\sigma}), \tag{3.11}$$

which on simplifying the right-hand side and integrating the left-hand side by parts yields

$$-\int_{T_l} g(\tilde{n}, \nabla V)\Delta_{T_l}\bar{f} = \int_{T_l} (VH\Delta_{T_l}\bar{f} + H\langle \nabla_{T_l} V, \nabla_{T_l}\bar{f}\rangle - V\langle A, \nabla_{T_l}\nabla_{T_l}\bar{f}\rangle). \tag{3.12}$$

Now $\nabla_{T_l}\nabla_{T_l}\bar{f}$, $\Delta_{T_l}\bar{f}$ and $\langle \nabla_{T_l} V, \nabla_{T_l}\bar{f}\rangle$ are uniformly bounded independent of l. Hence letting $l \to \infty$ and recalling that S is totally geodesic we get

$$\int_S g(n, \nabla V)\Delta_S f = 0. \tag{3.13}$$

Since this holds for arbitrary C^2 function f on S and since $g(n, \nabla V)$ is continuous we have $g(n, \nabla V) = \text{constant}$ on S. Thus (i) is proved.

To prove (ii) first we note that the scalar curvature of g in $N \sim \bar{\mathcal{Q}}$ vanishes. Now using the Gauss–Bonnet theorem and the contracted Gauss equation, we get

$$\int_{T_l} (-\mathrm{Ric}(g)_{\alpha\beta}\tilde{n}^\alpha \tilde{n}^\beta + H^2/2 - |A|^2/2) = 4\pi. \tag{3.14}$$

This implies, by 3.6

$$\int_{T_l} -V^{-1} V_{;\alpha\beta}\tilde{n}^\alpha \tilde{n}^\beta + H^2/2 - |A|^2/2) = 4\pi. \tag{3.15}$$

Now using (2.5) with $\psi = V$ and (3.7) in (3.15), we get

$$\int_{T_l}\left(\frac{|\nabla_{T_l}V|^2}{V^2} + g(\tilde{n},\nabla V)H/V + H^2/2 - |A|^2/2\right) = 4\pi. \tag{3.16}$$

Hence letting l tend to ∞ we get (ii).

Finally to prove (iii) we note that

$$\begin{aligned} g(n,\nabla w)/V &= 2V^{-1}V_{;\alpha\rho}(\delta^\rho_\beta - \tilde{n}^\rho\tilde{n}_\beta)V^{;\beta}\tilde{n}^\alpha + 2V^{-1}(V^{;\beta}\tilde{n}_\beta)V_{;\alpha\rho}\tilde{n}^\alpha\tilde{n}^\rho \\ &= 2\,\mathrm{Ric}(g)_{\alpha\rho}(\delta^\rho_\beta - \tilde{n}^\rho\tilde{n}_\beta)V^{\|\beta}\tilde{n}^\alpha + 2V^{-1}g(\tilde{n},\nabla V)(-\Delta_{T_l}V - g(\tilde{n},\nabla V)H). \end{aligned} \tag{3.17}$$

In the last step above we have used (3.6), (2.5) and (3.7) in addition to replacing $V^{;\beta}$ in the first term by $V^{\|\beta}$.

Hence using (3.8) with $b^\beta = ((cV^2+a)/(1-V^2)^3)V^{\|\beta}$, we get

$$\begin{aligned} &\int_{T_l}\frac{cV^2+a}{V(1-V^2)^3}g(\tilde{n},\nabla w) \\ &= -2\int_{T_l}\left(\frac{cV^2+a}{(1-V^2)^3}V^{\|\beta}\right)_{\|\alpha}A^\alpha{}_\beta + 2\int_{T_l}\left(\frac{cV^2+a}{(1-V^2)^3}V^{\|\beta}\right)_{\|\beta}H \\ &\quad + 2\int_{T_l}\frac{cV^2+a}{V(1-V^2)^3}g(\tilde{n},\nabla V)(-\Delta_{T_l}V - g(\tilde{n},\nabla V)H). \end{aligned} \tag{3.18}$$

Now we have

$$\begin{aligned} 2\int_{T_l}\frac{cV^2+a}{V(1-V^2)^3}g(\tilde{n},\nabla V)(-\Delta_{T_l}V) &= 2\int_{T_l}g(\tilde{n},\nabla V)\left\langle\nabla_{T_l}\left(\frac{cV^2+a}{V(1-V^2)^3}\right),\nabla_{T_l}V\right\rangle \\ &\quad + 2\int_{T_l}\frac{cV^2+a}{V(1-V^2)^3}\langle\nabla_{T_l}V,\nabla_{T_l}g(\tilde{n},\nabla V)\rangle, \end{aligned} \tag{3.19}$$

where the second term in the right-hand side equals $-2\int g(\tilde{n},\nabla V)\Delta_{T_l}f$ for $f = \int((cV^2+a)/V(1-V^2)^3)dV$ and can be evaluated using (3.12). Hence letting l tend to ∞ we get, from (3.18),

$$\lim_{l\to\infty}\int_{T_l}\frac{cV^2+a}{V(1-V^2)^3}g(\tilde{n},\nabla w) = 2m'\int_S\left\langle\nabla_S\left(\frac{cV^2+a}{V(1-V^2)^3}\right),\nabla_S V\right\rangle.$$

This completes the proof of the lemma. □

Theorem 5. *There cannot be any totally geodesic embedded C^2 sphere S in (N,g) such that $S \subset N \sim \mathscr{Q}$.*

Proof of 5. Let us suppose to the contrary that there is a S. Since by hypothesis $\mathscr{Q}$ is connected we have by Remark 3.1 either of the following cases:

Case I: $\mathscr{Q} \subset \bar{N}_2$ or

Case II: $\mathscr{Q} \subset \bar{N}_1$.

We consider Case I first. On N_1 we have $\Delta V = 0$ and on $\partial N_1 = S$ we have

$g(n, \nabla V) =$ constant by (3.2). This implies $g(n, \nabla V) = 0$ on S. Hence

$$\int_{N_1} |\nabla V|^2 = -\int_{N_1} V \Delta V + (1/2) \int_{N_1} \Delta V^2 = \frac{1}{2} \int_S V g(n, \nabla V) = 0,$$

giving $|\nabla V|^2 = 0$ on N_1. Thus V is constant in the interior of N_1 and hence, by continuity, on S. This implies

$$\int_S |\nabla_S V|^2 / V^2 = 0,$$

which contradicts (3.3). Thus $\mathcal{Q} \not\subset \bar{N}_2$.

Now we show that Case II also does not occur and hence get a contradiction. We shall use Robinson's divergence form inequality ([R]) on $N \sim \bar{\mathcal{Q}}$.

On each bounded open subset $\mathcal{N}_4$ of $N \sim \bar{\mathcal{Q}}$ we have the following inequality (provided $V < 1$ on $N \sim \bar{\mathcal{Q}}$ ([R], [MA]))

$$(F V^{-1} w^{;\alpha} + G w V^{;\alpha})_{;\alpha} \geqq 0, \tag{3.20}$$

where

$$F = (cV^2 + a)/(1 - V^2)^3, \tag{3.21}$$

and

$$G = -2c/(1 - V^2)^3 + 6(cV^2 + a)/(1 - V^2)^4, \tag{3.22}$$

c and a being constants such that $F > 0$ on N_2.

First, we show that w is locally C^2 on $N \sim \bar{\mathcal{Q}}$ so that (3.20) makes sense pointwise. Differentiating (3.7) we get $V_{;\alpha}{}^{\alpha}{}_{\beta} = 0$. This implies by commutation law, $V_{;\alpha\beta}{}^{\alpha} = \mathrm{Ric}(g)_{\tau\beta} V^{;\tau}$. Hence,

$$\begin{aligned} \Delta w = 2(V_{;\alpha\beta} V^{;\beta})_{;}{}^{\alpha} &= 2 V_{;\alpha\beta}{}^{\alpha} V^{;\beta} + 2 V_{;\alpha\beta} V^{;\beta\alpha} \\ &= 2\,\mathrm{Ric}(g)_{\tau\beta} V^{;\tau} V^{;\beta} + 2 V_{;\alpha\beta} V^{;\beta\alpha}. \end{aligned} \tag{3.23}$$

Since $\mathrm{Ric}(g)$ is locally $C^{\alpha}, \alpha \in (0,1)$, we get that w is locally $C^{2,\alpha}$.

Now Stoke's formula applied to (3.20) gives

$$\begin{aligned} &\lim_{l \to \infty} \int_{T_l} (F V^{-1} g(\tilde{n}, \nabla w) + G w g(\tilde{n}, \nabla V)) \\ &\quad + \lim_{R \to \infty} \int_{S_R} (F V^{-1} g({}^R n, \nabla w) + G w g({}^R n, \nabla V)) \geqq 0, \end{aligned} \tag{3.24}$$

where T_l and $\tilde{n}$ are as in Lemma 4. S_R is the asymptotic sphere $|x| = R$ and ${}^R n$ is the outward normal to S_R. $\tilde{n}$ is directed outward with respect to the volume bounded by T_l and S_R. Hence by (3.7) and (1.7) we have

$$\lim_{l \to \infty} \int_{T_l} g(\tilde{n}, \nabla V) = -\lim_{R \to \infty} \int_{S_R} g({}^R n, \nabla V) = -4\pi m.$$

Thus by (3.2) we have

$$m' = g(n, \nabla V) | S = -4\pi m / |S|, \tag{3.25}$$

n being the unit normal form on S consistent in direction with $\tilde{n}$. We need to consider only the case $m > 0$ (for $m = 0$, arguments of Case I apply in the domain exterior to S). Hence we assume $V < 1$.

We have

$$\lim_{R\to\infty} \int_{S_R} FV^{-1}g({}^R n, \nabla w) + Gwg({}^R n, \nabla V) = -\pi(c+a)/2m. \tag{3.26}$$

Using (3.21), (3.22), (3.25) and (3.26) in (3.24) we get

$$\lim_{l\to\infty} \int_{T_l} \frac{cV^2+a}{V(1-V^2)^3} g(\tilde{n}, \nabla w) + \int_S \frac{-2cw}{(1-V^2)^3} m' + \int_S \frac{6(cV^2+a)}{(1-V^2)^4} wm' \geqq \pi(c+a)/2m. \tag{3.27}$$

Hence by (3.4) we get

$$m' \int_S \left\langle 2\nabla_S\left(\frac{cV^2+a}{V(1-V^2)^3}\right), \nabla_S V\right\rangle - m' \int_S \frac{2cw}{(1-V^2)^3} + m' \int_S \frac{6(cV^2+a)w}{(1-V^2)^4} \geqq \pi(c+a)/2m. \tag{3.28}$$

Dividing by $|m'|$ and using (3.25) we get,

$$\int_S \left[\left\langle -2\nabla_S\left(\frac{cV^2+a}{V(1-V^2)^3}\right), \nabla_S V\right\rangle + \frac{2cw}{(1-V^2)^3} - \frac{6(cV^2+a)w}{(1-V^2)^4}\right] \geqq (c+a)|S|/8m^2. \tag{3.29}$$

Following Robinson [R] we shall choose two different sets of values for c and a to obtain two inequalities contradicting each other. We put $c = -1$, $a = 1$. We check that $F > 0$ on $N \sim \bar{\mathscr{Q}}$. Then (3.29) becomes

$$\int_S \left[\left\langle -2\nabla_S\left(\frac{1}{V(1-V^2)^2}\right), \nabla_S V\right\rangle - \frac{2w}{(1-V^2)^3} - \frac{6w}{(1-V^2)^3}\right] \geqq 0. \tag{3.30}$$

Simplifying the first term and using $w = |\nabla_S V|^2 + m'^2$ on S, we get

$$\int_S \frac{2(1-9V^2)}{V^2(1-V^2)^3} |\nabla_S V|^2 \geqq 8m'^2 \int_S \frac{1}{(1-V^2)^3} > 8m'^2|S|. \tag{3.31}$$

The last inequality follows because we have $1/(1-V^2)^3 > 1$ on S as $V > 0$. Now using (3.25) and

$$\frac{1-9V^2}{(1-V^2)^3} < 1 \quad \text{on} \quad S, \tag{3.32}$$

we have

$$\int_S \frac{|\nabla_S V|^2}{V^2} > 64\pi^2 m^2/|S|. \tag{3.33}$$

Finally using (3.3) we get

$$|S| > 16\pi m^2. \tag{3.34}$$

We now consider $c = 1$ and $a = 0$ in (3.29). Clearly F is positive. Thus we have

$$\int_S \left[\left\langle -2\nabla_S\left(\frac{V}{(1-V^2)^3}\right), \nabla_S V\right\rangle + \frac{2w}{(1-V^2)^3} - \frac{6V^2 w}{(1-V^2)^4}\right] \geqq |S|/8m^2. \tag{3.35}$$

Simplifying the first term, using $w = |\nabla_S V|^2 + m'^2$ on S and adding the coefficients of $|\nabla_S V|^2$ and m'^2 respectively, we get

$$\int_S \left[\frac{-18V^2}{(1-V^2)^4} |\nabla_S V|^2 + \frac{2m'^2(1-4V^2)}{(1-V^2)^4} \right] \geqq |S|/8m^2. \tag{3.36}$$

giving

$$2m'^2 \int_S \frac{1-4V^2}{(1-V^2)^4} \geqq |S|/8m^2. \tag{3.37}$$

Then using (3.25) and

$$\frac{1-4V^2}{(1-V^2)^4} < 1 \quad \text{on} \quad S, \tag{3.38}$$

we get

$$32\pi^2 m^2/|S| > |S|/8m^2,$$

giving $16\pi m^2 > |S|$ which contradicts (3.34). □

We can now complete the proof of the main Theorem 1.

Proof of 1. By Theorem 2 and Theorem 3 either N is diffeomorphic to $\mathbb{R}^3$ or there exists a C^2 embedded totally geodesic sphere S in $N \sim \mathcal{Q}$. But by Theorem 5 such a sphere cannot exist. Hence the theorem follows. □

4. Application to General Relativity

In this section we apply our main theorem to prove that a geodesically complete, asymptotically Euclidean, static perfect fluid space-time with connected fluid region and satisfying "timelike convergence condition" is diffeomorphic to $\mathbb{R}^3 \times \mathbb{R}$, without using the Poincaré conjecture.

By a static perfect fluid spacetime we mean a geodesically complete space-time $(M, {}^4g)$ such that:

(i) M is a C^∞ manifold diffeomorphic to $N \times \mathbb{R}$, where for each $t \in \mathbb{R}$, $N_t = N \times \{t\}$ is a spacelike three-manifold.

(ii) The Lorentz metric ${}^4g$ can be written as

$${}^4g = -V^2(dt \otimes dt) + g, \tag{4.1}$$

where V is a positive $C^{1,1}$ function and g is a tensor such that g restricted to N is a Riemannian metric on N, and V and g are independent of t. We assume that g is at least $C^{1,1}$.

(iii) $(M, {}^4g)$ satisfies Einstein's equation

$$\mathrm{Ric}({}^4g)_{AB} - \tfrac{1}{2}\mathrm{Scalar}({}^4g)^4g_{AB} = 8\pi((\rho + p)u_A u_B + p^4 g_{AB}), \tag{4.2}$$

where ρ and p are bounded measurable functions and u_A is a unit timelike vector field on M.

By virtue of the Gauss–Codazzi embedding equations for the Lorentz metric ${}^4g$, (4.2) decomposes into

$$\mathrm{Ric}(g)_{\alpha\beta} = V^{-1}V_{;\alpha\beta} + 4\pi(\rho - p)g_{\alpha\beta}, \tag{4.3}$$

and

$$\Delta V = 4\pi V(\rho + 3p) \quad \text{on} \quad N, \tag{4.4}$$

where ; denotes the covariant derivative with respect to g and Δ denotes the Laplacian with respect to g ([L]).

It is clear that ρ and p are independent of t. It follows from (4.2) that if ${}^4g$ satisfies the timelike convergence condition, namely.

$$\mathrm{Ric}({}^4g)(W, W) \geqq 0 \tag{4.5}$$

for all timelike vectors W, then $\rho + 3p \geqq 0$. By continuity (4.5) implies the null convergence condition, namely, $\mathrm{Ric}({}^4g)(K, K) \geqq 0$ for all null vectors K. By virtue of (4.2) the latter condition is satisfied if and only if $\rho + p \geqq 0$.

We also assume that there exists an open connected region $\mathscr{Q} \subset N$ such that $\operatorname{ess\,inf}_K (\rho + p) > 0$ for all compact $K \subset \mathscr{Q}$ and $\rho = p = 0$ in $N \sim \bar{\mathscr{Q}}$.

The functions ρ and p are respectively called the density and the pressure of the fluid. The assumption that the fluid region $\mathscr{Q}$ is connected is needed here in order to apply our main theorem. We also assume that ${}^4g$ satisfies the timelike convergence condition so that by (4.4), ΔV is non-negative. However when $\mathscr{Q}$ is unbounded, the null convergence condition will be sufficient for our purpose. Remark 1.11 is relevant here.

We say that $(M, {}^4g)$ is asymptotically Euclidean if (N, g) satisfies condition (c) of the main Theorem 1 and for some $\mu \in (0, 1)$, V satisfies (by (1.6), this is automatic if $\bar{\mathscr{Q}}$ is compact)

$$1 - V = O(|x|^{-\mu}), \quad \frac{\partial V}{\partial x^\alpha} = O(|x|^{-1-\mu}) \quad \text{as} \quad |x| \to \infty. \tag{4.6}$$

It follows from a result due to Gannon (Proposition 1.2 in [G]) that N is simply connected. (The proof in [G] can be modified to the case of $C^{1,1}$ metric in a way similar to the extension of singularity theorems to the case when the metric is $C^{1,1}$, that is, by taking a smooth sequence of metrics and using a sharpened version of the "Focusing Lemma;" see p. 285 in [HE]. See also Sect. 4.3 in [MA]). As a consequence of this fact and our main theorem, we prove the following theorem. This result has been claimed in [LB] assuming the Poincaré conjecture to be true. In fact the asymptotic conditions and Gannon's Theorem imply that N has the same homotopy as $\mathbb{R}^3$ ([LB]). What we have shown here is that a fake 3-cell cannot occur in N and hence N is diffeomorphic to $\mathbb{R}^3$.

Theorem 6. *A geodesically complete asymptotically Euclidean static perfect fluid space-time having connected fluid region and satisfying the timelike convergence condition is diffeomorphic to* $\mathbb{R}^3 \times \mathbb{R}$.

5. Miscellaneous Remarks and Generalizations

In this section we discuss various generalizations to Theorem 1 and some further applications to general relativity. First we consider an alternative definition of $\mathscr{Q} \subset N$;

$\mathscr{Q}$ is connected, $\operatorname{ess\,inf}_K(\Phi_1 + \Phi_2) > 0$ for all compact sets

$$K \subset \mathscr{Q} \quad \text{and} \quad \Phi_2 = 0 \quad \text{on} \quad N \sim \bar{\mathscr{Q}}.$$

Because of the following lemma the main Theorem 1 continues to hold with $\mathscr{Q}$ defined as above.

Lemma 7. *Let $\mathscr{Q}$ be defined as above. Suppose $\Phi_1 + \Phi_2 \geqq 0$ on N. If there exists an embedded area minimizing (defined in (2.1)) minimal C^2 sphere S on $N \sim \mathscr{Q}$, then $\Phi_1 \equiv 0$ on each component of $N \sim \bar{\mathscr{Q}}$ with closure intersecting S.*

5.1. *Remark.* In the above lemma we continue to assume that N is simply connected so that S separates N. The assumption that V is a positive $C^{1,1}$ function on N is also used.

Proof of 7. Since $\Phi_2 \equiv 0$ on $N \sim \mathscr{Q}$, (1.9) implies that Φ_1 is a Lipschitz function on $N \sim \bar{\mathscr{Q}}$ and

$$\Phi_{1;\beta} = 2V^{-1}\Phi_1 V_{;\beta}. \tag{5.2}$$

Hence $\Phi_1 = CV^2$ on $N \sim \bar{\mathscr{Q}}$, where by virtue of $\Phi_1 + \Phi_2 \geqq 0$, C (which is constant on each component of $N \sim \bar{\mathscr{Q}}$) is non-negative.

Now as in the proof of Theorem 3 we can approximate S by a sequence of C^2 spheres S_t, $t \downarrow 0$. We can take $S_t \subset N \sim \bar{\mathscr{Q}}$ because S separates N and $\mathscr{Q}$ is connected. The fact that S is area-minimizing and Φ_1 is continuous on (each component of) $N \sim \bar{\mathscr{Q}}$ gives (using (2.6)) that $\lim_{t\to 0} \int_{S_t} \Phi_1 V^2 = 0$. Hence $\int_S CV^4 = 0$.

But $V > 0$. Hence on $S, C = 0$ giving $\Phi_1 \equiv 0$ on each component of $N \sim \bar{\mathscr{Q}}$ with closure interseting S. □

5.3. *Remark.* If we use an extra asymptotic condition that the scalar curvature R of g is $0(1)$, then it follows that $\Phi_1 \equiv 0$ on each unbounded component of $N \sim \bar{\mathscr{Q}}$ without the assumption that an area minimizing sphere S exists in $N \sim \mathscr{Q}$. This is because from (1.1) we have $R = \Delta V + 3\Phi_1$ and by hypothesis $\Delta V = 0$. However by Theorems 2 and 3 if such S does not exist then N is topologically Euclidean and hence the conclusion of the main theorem follows without the above mentioned asymptotic decay of R.

5.4. *Remark.* Arguments similar to those used in the proof of Lemma 7 show that we can also define $\mathscr{Q}$ as follows: $\mathscr{Q}$ is connected, $\operatorname{ess\,inf}_K(\Phi_1 + \Phi_2) > 0$ for all compact sets $K \subset \mathscr{Q}$ and $\Phi_1 \equiv 0$ in $N \sim \bar{\mathscr{Q}}$. Then (1.9) implies that $\Phi_{2;\beta} = -2V^{-1}\Phi_2 V_{;\beta}$, giving $\Phi_2 = CV^{-2}$, where C is constant on each component of $N \sim \bar{\mathscr{Q}}$. The same arguments as before then imply that $\Phi_2 \equiv 0$ on each component of $N \sim \bar{\mathscr{Q}}$ with closure intersecting S.

We shall now discuss some generalizations of Eq. (1.1).

Lemma 8. *Theorem 1 continues to hold with the usual conditions on V, g, Φ_1 and Φ_2 if we replace 1.1 by*

$$C_1\Phi_1 g_{\alpha\beta} \geqq \mathrm{Ric}(g)_{\alpha\beta} - V^{-1}V_{;\alpha\beta} \geqq \Phi_1 g_{\alpha\beta} \quad \textit{almost everywhere, where } C_1 \textit{ is a constant.} \tag{5.5}$$

The first inequality in (5.5) is added to ensure that $\Phi_1 \equiv 0$ on $N \sim \bar{\mathcal{Q}}$ implies

$$\mathrm{Ric}(g)_{\alpha\beta} = V^{-1}V_{;\alpha\beta}, \quad \text{on} \quad N \sim \bar{\mathcal{Q}}. \tag{5.6}$$

Proof of 8. Equation (5.6) enables us to apply the reasoning of Sect. 3 (in particular we refer to Theorem 5) to prove that no embedded C^2 totally geodesic sphere exists in $N \sim \mathcal{Q}$. To prove our assertion that Theorem 1 holds with (1.1) replaced by (5.5) we now simply need to use the inequality

$$V^2\,\mathrm{Ric}(g)(n,n) + V\Delta V - VV_{;\alpha\beta}n^\alpha n^\beta \geqq V^2(\Phi_1 + \Phi_2)$$

in (2.6) instead of the equality. The rest of the arguments in the proof of Theorem 3 then apply, and since there is no embedded C^2 totally geodesic sphere in $N \sim \mathcal{Q}$, it follows that N must be diffeomorphic to $\mathbb{R}^3$. □

We shall now apply Theorem 1 to a complete, asymptotically Euclidean (that is, the induced metric satisfies the asymptotic conditions 1.3) simply connected space-like hyper-surface N in a suitable class of space-times $(M, {}^4g)$, not necessarily static. As before the results are different from those in the paper of Frankel and Galloway [FG]; for example, using the asymptotic conditions we can allow certain terms (Φ_1, Φ_2 in (5.11–12)) to vanish identically outside a connected set, whereas the results in [FG] require that ess $\inf(\Phi_1 + \Phi_2) > 0$ for all compact subsets K of the 3-manifold whose topology is to be investigated.

Let $(M, {}^4g)$ and N be as above. In a neighbourhood of N in M we write

$${}^4g = -V^2dt^2 + g_{\alpha\beta}dx^\alpha dx^\beta, \quad \alpha, \beta \in \{1,2,3\},$$

where $\{x^\alpha\}$, $\alpha = 1,2,3$, is a co-ordinate system on N; $g_{\alpha\beta} = g_{\alpha\beta}(t, x^\tau)$ and $V = V(t, x^\tau)$. Suppose g and V restricted to N are respectively $C^{1,1}$ Riemannian metric and $C^{1,1}$ positive function. Also we assume that $\partial^2 g_{\alpha\beta}/\partial t^2$ exist on N. The unit normal time-like vector field u^A on N is given in the above co-ordinate system, by

$$u^0 = V^{-1}; \quad u^\alpha = 0, \quad \alpha = 1,2,3. \tag{5.7}$$

Decomposing $\mathrm{Ric}({}^4g)$ we get

$$\mathrm{Ric}(g)_{\alpha\beta} = V^{-1}V_{;\alpha\beta} + \mathrm{Ric}({}^4g)_{\alpha\beta} + 2\Omega_{\alpha\sigma}\Phi^\sigma_\beta - \Omega\Omega_{\alpha\beta} - V^{-1}\frac{\partial\Omega_{\alpha\beta}}{\partial t}, \tag{5.8}$$

$$V^{-1}\Delta V = \mathrm{Ric}({}^4g)_{00}V^{-2} + V^{-1}g^{\alpha\beta}\frac{\partial\Omega_{\alpha\beta}}{\partial t} - |\Omega|^2, \tag{5.9}$$

where $\Omega_{\alpha\beta} = \frac{1}{2}V^{-1}(\partial g_{\alpha\beta}/\partial t)$ is the second fundamental form of N in M, $\Omega = \Omega_{\alpha\beta}g^{\alpha\beta}$, and $|\Omega|^2 = \Omega_{\alpha\beta}\Omega_{\rho\sigma}g^{\alpha\rho}g^{\beta\sigma}$. As usual Δ and ; denote respectively the Laplacian and covariant differentiation relative to the g metric and the indices are raised by $g^{\alpha\beta}$.

For $p \in N$ there exist $\lambda(p)$ and $\Lambda(p)$ such that

$$\lambda(p)g(X,X)|_p \leqq \left(\mathrm{Ric}(^4g)(X,X) + 2\Omega_{\alpha\sigma}\Omega^{\sigma}{}_{\beta}X^\alpha X^\beta - \Omega\Omega_{\alpha\beta}X^\alpha X^\beta \right.$$

$$\left. - V^{-1}\frac{\partial \Omega_{\alpha\beta}}{\partial t}X^\alpha X^\beta \right)(p) \leqq \Lambda(p)g(X,X)|_p \tag{5.10}$$

for all vectors X tangent to N at p. We shall take

$$\Phi_2 = \mathrm{Ric}(^4g_{00})V^{-2} + V^{-1}g^{\alpha\beta}\frac{\partial \Omega_{\alpha\beta}}{\partial t} - |\Omega|^2 \tag{5.11}$$

and

$$\Phi_1 = \lambda. \tag{5.12}$$

Now we suppose that Φ_1 and Φ_2 satisfies the following conditions:

(i) on N, $\Phi_1 + \Phi_2 \geqq 0$, $\Phi_2 \geqq 0$ and $C\Phi_1 \geqq \Lambda$ for some constant C; and (5.13)

(ii) there exists a connected set $\mathscr{Q} \subset N$ with $\operatorname*{ess\,inf}_K (\Phi_1 + \Phi_2) > 0$ for all compact $K \subset \mathscr{Q}$ and $\Phi_1 = \Phi_2 = 0$ on $N \sim \bar{\mathscr{Q}}$.

Then by virtue of Lemma 8, Theorem 1 applies and gives N to be diffeomorphic to $\mathbb{R}^3$. In particular if N is a Cauchy surface, then by a well known theorem due to Geroch (see Proposition 6.6.8 in [HE]) M will be diffeomorphic to $\mathbb{R}^3 \times \mathbb{R}$.

In case the Cauchy surface N is not *a priori* known to be simply connected, Gannon's Theorem can be used to prove that N is simply connected provided $(M, {}^4g)$ is geodesically complete and satisfies the null convergence condition, and on N, V satisfies the asymptotic condition (4.6), and $\partial g_{\alpha\beta}/\partial t = O(|x|^{-1-\mu})$.

As an example we now consider the case of a static space-time $(M, {}^4g)$ not necessarily perfect fluid. We assume $(M, {}^4g)$ to be asymptotically Euclidean in the sense of (1.3) and (4.6); and to satisfy

$$\mathrm{Ric}(^4g)(K,K) \geqq 0 \tag{5.14}$$

for all null vectors K. Hence any $t = $ constant hyper-surface N is simply connected.

Since the metric is static we have $\Omega_{\alpha\beta} = 0$. Hence (5.8) and (5.9) become

$$\mathrm{Ric}(g)_{\alpha\beta} = V^{-1}V_{;\alpha\beta} + \mathrm{Ric}(^4g)_{\alpha\beta}$$

and

$$V^{-1}\Delta V = \mathrm{Ric}(^4g)_{00}V^{-2}.$$

Let

$$\Phi_2 = \mathrm{Ric}(^4g)_{00}V^{-2}$$

and

$$\Phi_1 = \lambda,$$

where

$$\lambda(p) = \inf((\mathrm{Ric}(^4g)_{\alpha\beta}X^\alpha X^\beta)(p)/g(X,X)|_p),$$

the infimum is over all vectors X tangent to N at p.

Let n be any unit vector tangent to N. Putting $K^A = u^A + n^A$ in (5.14) we have

$$\mathrm{Ric}(g)_{\alpha\beta} n^\alpha n^\beta + V^{-2}\,\mathrm{Ric}({}^4g)_{00} \geqq 0,$$

where we have used the fact $\mathrm{Ric}({}^4g)_{0\beta} = 0$ by Codazzi's equation since N is totally geodesic in M.

Thus $\Phi_1 + \Phi_2 \geqq 0$. If we strengthen the assumption (5.14) to

$$\mathrm{Ric}({}^4g)(W, W) \geqq 0 \tag{5.15}$$

for all time-like vectors W we get in particular (taking $W^A = u^A$)

$$\mathrm{Ric}({}^4g)_{00} V^{-2} \geqq 0.$$

Let $\mathscr{Q}$ be a connected set in N such that on $\mathscr{Q}$, $\operatorname*{ess\,inf}_K (\Phi_1 + \Phi_2) > 0$ for all compact sets $K \subset \mathscr{Q}$; and on $N \sim \bar{\mathscr{Q}}$, $\mathrm{Ric}({}^4g) \equiv 0$. We shall call $\mathscr{Q}$ the "non-vacuum region." Since the conditions (5.13) (i) and (ii) are satisfied, Theorem 1 applies. Hence we have proved the following theorem.

Theorem 9. *Let $(M, {}^4g)$ be an asymptotically Euclidean (in the sense of (1.3) and (4.6)), geodesically complete static space-time satisfying the time-like convergence condition (5.15). If $(M, {}^4g)$ has connected non-vacuum region (defined above) then M is diffeomorphic to $\mathbb{R}^3 \times \mathbb{R}$.*

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