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Remark on the Absence of Absolutely Continuous Spectrum for *d*-Dimensional Schrödinger Operators with Random Potential for Large Disorder or Low Energy

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Abstract. We show that there is no absolutely continuous part in the spectrum of the Anderson tight-binding model for large disorder or low energy. The proof is based on the exponential decay of the Green's function proved by Fröhlich and Spencer. The extension of this result to the continuous case is also discussed.

1. Introduction

In the last few years disordered systems have been one of the most actively investigated subjects in solid state physics. Following Anderson's approach [1] it is possible to describe the behavior of an electron in a crystal with randomly distributed impurities by means of a Hamiltonian on $l^2(\mathbb{Z}^d)$ of the form:

$$H(v) = -\Delta + v, \tag{1}$$

where Δ is the finite difference Laplacian on $\mathbb{Z}^d$:

$$(\Delta\psi)(n) = \sum_{\substack{m\in\mathbb{Z}^d \\ |m-n|=1}} (\psi(m)-\psi(n)), \qquad n\in\mathbb{Z}^d,$$

and $v = \{v(n)\}_{n\in\mathbb{Z}^d}$ are independent identically distributed (i.i.d.) random variables.

After Anderson's paper the Schrödinger equation with random potential and its discrete analog (1) have been extensively investigated: especially in connection with metal insulator transition.

In the one dimensional case it has been shown [2, 3] that H has a dense pure point spectrum with exponentially localized eigenfunctions.

However, in more than one dimension the strongest result in this direction is the absence of diffusion for sufficiently large disorder or low energies, proved by

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Fröhlich and Spencer [4]. In order to obtain this result they prove that the Green's function of Anderson's Hamiltonian (1) decays exponentially for long distances with probability one.

In this paper we derive from their result the absence of an absolutely continuous component of the spectrum of H in the same range of the parameters.

2. Notations and the Result of Fröhlich and Spencer

Let $v=\{v(n)\}_{n\in\mathbb{Z}^d}$ be i.i.d. random variables with common distribution:

$$d\lambda(v)=\frac{d\lambda(v)}{dv}dv\,,$$

with

$$\sup_v \frac{d\lambda}{dv}\equiv\left|\frac{d\lambda}{dv}\right|_\infty=\delta^{-1}<+\infty\,.$$

The random potential v belongs to the probability space $\Omega=\prod_{n\in\mathbb{Z}^d}(\mathbb{R},d\lambda(v(n)))$ with product measure:

$$dP(v)=\prod_{n\in\mathbb{Z}^d}d\lambda(v(n))\,.$$

For the reader's convenience, we recall the result of Fröhlich and Spencer on the exponential decay of the Green's function in a form suitable for our purposes.

Theorem 1 (*FS*'s result). i) *For any $p>0$ and any $m>0$ there exists a constant $C(p,m)$ such that if $|E|+\delta>C(p,m)$. Then the following event holds with probability at least $1-l^{-p}$:*

There exists a set A_l containing the origin such that:

$$\tfrac{1}{2}l\leqq\min_{b\in\partial A_l}(0,b)\leqq\max_{b\in\partial A_l}(0,b)\leqq l\,,$$

and for all $\varepsilon\neq0$ and all x, y satisfying $|x-y|\geqq l^{3/4}$

$$|(H_{A_l}(v)-E-i\varepsilon)^{-1}(x,y)|\leqq\exp(-m|x-y|)$$

ii) $E\notin\sigma(H_{A_l}(v))$, P-a.s.

Here $H_{A_l}(v)$ denotes the restriction of the Hamiltonian $H(v)$ to the set A_l with Dirichlet boundary condition at the boundary of A_l,

$$\partial A_l=\{(n,n');\,|n-n'|=1\,,\ n\in A_l,\,n'\notin A_l\}$$

and $\sigma(H_{A_l}(v))$ denotes its spectrum.

3. Absence of Absolutely Continuous Spectrum

We state now our main result:

Theorem 2. *There exist positive constants δ_0, E_0 such that if $\delta>\delta_0$ then*

$$\sigma_{\mathrm{ac}}(H(v))=\phi\,,\qquad P\text{-a.s.}\,,$$

and if $d\lambda(v)$ is gaussian:

$$\{E; |E| \geqq E_0\} \cap \sigma_{ac}(H(v)) = \phi, \qquad P\text{-a.s.},$$

where $\sigma_{ac}(H(v))$ denotes the absolutely continuous part of the spectrum of $H(v)$.

Proof. In order to prove this theorem we begin by showing that the result of Fröhlich and Spencer implies that with probability one there are no polynomially bounded solutions of the equation $(H(v)-E)\psi E=0$ (Lemma 1). However, we know (Lemma 2) that for any realization of the potential v and for almost all energies E [with respect to the spectral measure ϱ_v of $H(v)$], there exist polynomially bounded generalized eigenfunctions of $H(v)$ with eigenvalue E. The proof of the theorem then follows the argument used by Pastur [2] for the one dimensional case: if there is an absolutely continuous component of the spectrum of $H(v)$ for almost all v, then from Lemmas 1 and 2 we obtain a contradiction.

Let us fix the energy E in such a way that $|E|+\delta \geqq C(p,m)$ with $C(p,m)$ the constant appearing in Theorem 1 and let $l_n = 2^n$ be a sequence of length scales.

We denote by $\Omega_{E,\delta}$ the set of realizations of the potential v for which there exists an integer $n_0(E,\delta,v)$ such that for any $n > n_0$ there exists a set A_{l_n} satisfying conditions (i), (ii) of Theorem 1. Using Theorem 1 and the Borel-Cantelli lemma we obtain:

$$P(\Omega_{E,\delta}) = 1. \tag{2}$$

The following result is now an easy consequence of (2) and Theorem 1:

Lemma 1. *For any $v \in \Omega_{E,\delta}$ the equation:*

$$(H(v)-E)\psi = 0, \tag{3}$$

has no polynomially bounded solutions.

Proof of the Lemma. Fix $v \in \Omega_{E,\delta}$ and assume (3) has a polynomially bounded solution ψ_E. Then for any $n > n_0(E,\delta,v)$ ψ_E satisfies:

$$\psi_E(x) = \sum_{(z,z') \in \partial A_{l_n}} (H_{A_{l_n}} - E)^{-1}(x,z)\,\psi_E(z') \tag{4}$$

for any $x \in A_{l_n}$.

This equation can easily be obtained by considering ψ_E as the unique solution of the following problem:

$$(H(v)-E)u = 0 \quad \text{in} \quad A_{l_n}, \qquad u \restriction_{\partial A_{l_n}} = \psi_E.$$

Using now the polynomial boundedness of ψ_E and the exponential decay of $(H_{A_{l_n}} - E)^{-1}(x,z)$ for $|x-z| \geqq l^{3/4}$ we get that

$$|\psi_E(x)| \leqq C \exp(-m l_n^{3/4})\, l_n^{\alpha}, \tag{5}$$

for some $C > 0$, $\alpha > 0$ and any x such that $\operatorname{dist}(x, \partial A_{l_n}) \geqq l_n^{3/4}$. The arbitrariness of n together with (5) implies now that $\psi_E \equiv 0$.

We now recall the following result concerning the generalized solutions of $H(v)$ (see Berezanskii [5] and Simon [6]):

Lemma 2. *For any v there exists a spectral measure $d\varrho_v$ of $H(v)$ and for almost every E with respect to ϱ_v there exist solutions of the equation $(H(v)-E)\,\psi_E=0$ which satisfy*

$$|\psi_E(x)| \leqq C(1+|x|^2)^{d/2+\eta} \quad \textit{for any} \quad \eta>0\,,$$

and some constant $C>0$.

For definiteness let us consider the case $\delta > \delta_0 = C(p,m)$, the case $d\lambda(v)$ gaussian and $|E| \geqq E_0 = C(p,m)$ being similar. It is well known [7] that $\sigma_{ac}(H(v))$ is P-almost surely independent of v, so let us assume that

A) $\quad \sigma_{ac}(H(v)) = \Delta \neq \emptyset$ for any $v \in \bar{\Omega} \subset \Omega$ with $P(\bar{\Omega})=1$.

Consider now the space $M = \Omega x \mathbb{R}$ and define on M the measure $\tilde{P} = P \otimes \mu$, where μ denotes the Lebesgue measure. Let now $M_0 \subset M$ be defined by $M_0 \equiv \{(v,E);\ v \in \Omega_{E,\delta}\}$. We observe that M_0 is a $\tilde{P}$ measurable set; this follows from the definition of $\Omega_{E,\delta}$ and from the fact that $(H(v)-E-i\varepsilon)^{-1}(x,y)$ is a continuous function in ε for $\varepsilon \neq 0$ and a jointly measurable function in E and v ([7]). In the continuous case which will be discussed in the next section one also needs the continuity of the Green's function in x,y for $x \neq y$ (see [6]).

$$\tilde{P}(M_0) = \int_\Delta d\mu(E) \int_\Omega dP \chi_{H_0 \to M_0}(E,v) = \int_\Delta d\mu(E) \int_\Omega dP \chi_{H_0 \to M_0}(E,v) = \int_\Delta d\mu(E) P(\bar{\Omega} \cap \Omega_{E,\delta})\,. \tag{6}$$

By assumption A) and by (2), $P(\bar{\Omega} \cap \Omega_{E,\delta}) = 1\ \forall E$. Furthermore $\mu(\Delta) > 0$, since we are assuming that for all $v \in \bar{\Omega}$ the spectral measure ϱ_v has an absolutely continuous component. Thus the right-hand side of (6) is strictly positive.

On the other hand, by Fubini's theorem:

$$\tilde{P}(M_0) = \int_{\bar{\Omega}} dP \int_\Delta d\mu(E) \chi_{M_0}(E,v)\,. \tag{7}$$

By assumption A) and Lemma 2 we know that for μ-almost all $E \in \Delta$ and all $v \in \bar{\Omega}$ there exists a polynomially bounded solution of the equation: $(H(v)-E)\psi_E = 0$. However, using Lemma 1 and the definition of M_0 this is impossible, that is $\chi_{M_0}(E,v)=0$ for $v \in \bar{\Omega}$ and μ-almost all $E \in \Delta$.

Thus the right-hand side of (7) is zero and we get a contradiction.

4. Extension to the Continuous Case $\Delta + V$ on $L^2(\mathbb{R}^d)$

Here we discuss briefly the extension of our main result to the continuous version of the Anderson model.

Let $\{C_i\}_{i \in \mathbb{Z}^d}$ be a covering of $\mathbb{R}^d$ with unit cubes around the sites of $\mathbb{Z}^d$, and let $\{q_i(\omega)\}_{i \in \mathbb{Z}^d}$ be i.i.d. random variables with values in $[0,1]$ such that

$$P(q_0(\omega) \in [a,b]) = \int_a^b f(q)\,dq\,,$$

$$|f|_\infty < +\infty\,, \quad \text{and} \quad 0 < P(q_0(\omega) \leqq 1/2) \leqq \alpha < 1\,.$$

Let also $\varphi \in C^{\infty}(C_0)$ be such that:

i) $\varphi(x) > 0 \; \forall x \neq 0, \quad x \in C_0$,

ii) $\varphi(0) = 0$ and the origin is a quadratic minimum of φ, i.e. there exists $1/2 > \eta > 0$ and $C(\eta) > 0$ with $\varphi(x) \geqq C(\eta) x^2 \; \forall |x| < \eta$.

We now define the selfadjoint random Schrödinger operator on $L^2(\mathbb{R}^d)$:

$$H(\omega) = -\Delta + \sum_{i \in \mathbb{Z}^d} q_i(\omega) \varphi(x-i), \tag{8}$$

where we set $\varphi(x-i) = 0$ if $x - i \notin C_0$.

Since the random variables $\{q_i(\omega)\}_{i \in \mathbb{Z}^d}$ can be arbitrarily small with positive probability it follows from Weyl's result that $\sigma(H(\omega)) = [0, +\infty)$, P-a.s. In the continuous case, by using Green's formulas, (4) reads:

$$\psi_E(x) = \int_{\partial A_{l_n}} dz (\partial_{n_z} (H_{A_{l_n}} - E)^{-1}(x, z)) \psi_E(z),$$

where

$$\partial_{n_z} (H_{A_{l_n}} - E)^{-1}(x, z), \quad x \in A_{l_n}, \quad z \in \partial A_{l_n}$$

denotes the outward normal derivative at z of $(H_{A_{l_n}}(\omega) - E)^{-1}(x, y)$. When the function φ is replaced by the characteristic function of the cube C_0 the continuous analogue of the result of Fröhlich and Spencer given in the previous section was proved in [8] (see Theorem 3.6) for the Green's function and for its gradient for all energies $0 \leqq E \leqq E^*(|f|_\infty, \alpha)$ with the threshold $E^*(|f|_\infty, \alpha)$ given by:

$$E^*(|f|_\infty, \alpha) = \min \left\{ E_0(\alpha), \ln \left(\frac{|f|_\infty}{E_1(\alpha)} \right)^{-d/2} \right\},$$

$E_0(\alpha)$ and $E_1(\alpha)$, being suitable constants *independent* of $|f|_\infty$.

Since Lemma 2 on polynomially bounded eigenfunctions holds for Schrödinger operators on $L^2(\mathbb{R}^d)$ (see e.g. Simon [6]), by the same proof given for Anderson's model we would get the absence of absolutely continuous spectrum in $[0, E^*(|f|_\infty, \alpha)]$ for

$$\bar{H}(\omega) \equiv -\Delta + \sum_{i \in \mathbb{Z}^d} q_i(\omega) \chi_{C_0}(x-i).$$

It is interesting to observe that if in $E^*(|f|_\infty, \alpha)$ we increase $|f|_\infty$ while keeping α fixed $\left(\text{e.g. } f_a = \frac{\alpha}{a} \chi_{[0,a]} + 2(1-\alpha) \chi_{[1/2, 1]} \text{ and } a \ll 1 \right)$ the classically allowed region:

$$\left\{ x \in \mathbb{R}^d; \sum_i q_i(\omega) \chi_{C_0}(x-i) \leqq E^*(|f|_\infty, \alpha) \right\},$$

contains with probability one an infinite cluster of nearest neighbour cubes C_i if α was chosen greater than the percolation probability for the site percolation model on $\mathbb{Z}^d$. In order to extend the result to the more general case (8) the only missing step in [8] is the proof of the basic probabilistic estimate in this case. This is done in the next lemma (see e.g.: Lemma 2.4 in [4] and Lemma 3.2 in [8]):

Lemma 3. *Let $\tilde{\Lambda} \subset \mathbb{Z}^d$ be bounded, let $\Lambda = \bigcup_{i \in \tilde{\Lambda}} C_i$, and let $H_\Lambda(\omega)$ be the restriction of the Hamiltonian* (8) *to $L^2(\Lambda)$ with Dirichlet boundary conditions. Then for any $k < E$:*

$$P(\operatorname{dist}(\sigma(H_\Lambda(\omega)), E) < k) \leqq \operatorname{const} k |\tilde{\Lambda}|^3 ,$$

where

$$|\tilde{\Lambda}| = \# \{i \in \tilde{\Lambda}\} .$$

Sketch of the Proof. Let $\mu_n(H_\Lambda(\omega))$ denote the n^{th} eigenvalue (counting multiplicity) of $H_\Lambda(\omega)$, and let $N(E, H_\Lambda(\omega)) = \# \{n; \mu_n(H_\Lambda(\omega)) < E\}$. Then following Wegner [9] we write:

$$\begin{aligned} P(\operatorname{dist}(\sigma(H_\Lambda(\omega)), E) < k) &= P(N(E+k, H_\Lambda(\omega)) - N(E-k, H_\Lambda(\omega)) \geqq 1) \\ &\leqq \int dP(\omega) \int_{|E'-E| \leqq k} \frac{d}{dE'} N(E', H_\Lambda(\omega)) . \end{aligned} \tag{9}$$

To bound the right-hand side of (9) we use the estimate on the $\mu_n(H_\Lambda(\omega))$ valid for any sufficiently small ε if $\mu_n(H_\Lambda(\omega)) \leqq a$,

$$\begin{aligned} \mu_n(H_\Lambda(\omega)) - \varepsilon &\geqq \mu_n \left(-\Delta_\Lambda + \sum_i \left(q_i - \frac{\varepsilon}{\lambda(a)} \right) \varphi(x-i) \right), \\ \mu_n(H_\Lambda(\omega)) + \varepsilon &\leqq \mu_n \left(-\Delta_\Lambda + \sum_i \left(q_i + \frac{\varepsilon}{\lambda(a)} \right) \varphi(x-i) \right), \end{aligned} \tag{10}$$

with $\lambda(a)$ given by:

$$\lambda(a) = \frac{K_1(\eta)}{|\tilde{\Lambda}|(4a + K_2(\eta))} ,$$

where $K_1(\eta)$ and $K_2(\eta)$ are positive constants depending only on η (see the definition of φ). In turn (10) follows in a rather straightforward way from the min-max principle and the estimate:

$$\int_\Lambda dx\, g^2(x) \sum_{i \in \tilde{\Lambda}} \varphi(x-i) \geqq \lambda(a) \int_\Lambda dx\, g^2(x) \quad \text{if} \quad g \in H_0^1(\Lambda) \quad \text{and} \quad \int_\Lambda dx |(\nabla g)(x)|^2 \leqq a . \tag{11}$$

In order to derive (11) we used the quadratic nature of the minimum of φ together with the Heisenberg inequality. Using (10) we bound the right-hand side of (9) by:

$$- \sup_{|a-E|<k} \lambda(a)^{-1} \int dP(\omega) \int_{E-k}^{E+k} \sum_{i \in \tilde{\Lambda}} \frac{\partial}{\partial q_i} N(E', H_\Lambda(\omega)) dE' .$$

The rest of the proof follows now word by word that of Lemma 3.2 in [8].

As a conclusion we can now state the analogue of Theorem 2:

Theorem 3. *Let $H(\omega)$ be given by* (8). *Then*

$$\sigma_{\text{ac}}(H(\omega)) \cap [0, E^*(|f|_\infty, \alpha)] = \phi , \qquad P\text{-a.s.}$$

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