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Debye-Hückel Theory for Charge Symmetric Coulomb Systems

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Abstract. It is proven that the pressure, density and correlation functions of a classical charge symmetric Coulomb system are asymptotic as the plasma parameter ε tends to zero to the approximations predicted by the Debye-Hückel theory. These approximations consist of the ideal gas term plus a term of one lower order in ε. The sine-Gordon transformation and some new correlation inequalities for the associated functional integrals are used.

1. Introduction

We study a classical charge symmetric system in three dimensions in the limit that ε tends to zero. ε is the plasma parameter

$$\varepsilon=\beta/\ell_D, \tag{1.1}$$

where β is the inverse temperature, and ℓ_D is the Debye length

$$\ell_D=(2\beta z)^{-1/2}. \tag{1.2}$$

z is the chemical activity. Debye and Hückel [4] gave a non-rigorous study of this limit. We will prove that certain predictions of their theory are rigorously correct in this limit.

The Debye-Hückel theory gives an approximation for the pressure P as a function of the density σ

$$\frac{1}{kT}P\simeq 2\sigma-\frac{1}{24\pi}\ell_D^{-3}. \tag{1.3}$$

(For example, see p. 229 of [11].) We work in the grand canonical ensemble, so the pressure and density are both functions of z and β. We will show that as ε tends to zero the pressure and density are asymptotically given by

$$\frac{1}{kT}P\sim 2z+\frac{1}{12\pi}\ell_D^{-3}, \qquad \sigma\sim z+\frac{1}{16\pi}\ell_D^{-3}. \tag{1.4}$$

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Combining these approximations yields (1.3).

We will also find the first two terms in the asymptotic behavior of the correlation functions. For example, the correlation function for particles at y_1 and y_2 with charges δ_1 and δ_2 is given by

$$\varrho^{(2)}(y_1,\delta_1;y_2,\delta_2)\sim z^2\left[1-\beta\delta_1\delta_2(-\Delta+\ell_D^{-2})^{-1}(y_1,y_2)+\frac{\beta}{4\pi\ell_D}\right], \tag{1.5}$$

and

$$\varrho^{(2)}(y_1,\delta_1;y_2,\delta_2)-\varrho^{(1)}(y_1,\delta_1)\varrho^{(1)}(y_2,\delta_2)\sim -z^2\beta\delta_1\delta_2(-\Delta+\ell_D^{-2})^{-1}(y_1,y_2).$$

The Debye-Hückel theory predicts that correlation functions of this form should decay exponentially. This is known as Debye screening. Brydges [1] proved that for sufficiently small ε the correlation functions do indeed decay exponentially. This work was generalized by Brydges and Federbush [2] and by Imbrie [8]. Our result implies that as ε tends to zero the correlation functions converge to functions with exponential decay, but this does not imply Debye screening for nonzero ε.

To make the Coulomb system stable we must add a short range potential, e.g., hard cores, to the Coulomb potential. No such short range potential appears in the Debye-Hückel theory, so we will let the short range potential tend to zero as ε tends to zero.

One of the main tools we use is the sine-Gordon transformation. It says that the partition function can be expressed as a functional integral

$$Z=\int d\mu \exp\left\{2z\int_\Lambda :\cos[\sqrt{\beta}\phi(x)]:d^3x\right\}, \tag{1.6}$$

where $d\mu$ is a Gaussian measure whose covariance is essentially $\frac{1}{|x-y|}$. In units with $\ell_D=1$, $\varepsilon\to 0$ implies $\beta\to 0$ and $z\to\infty$ with βz fixed. So

$$:\cos[\sqrt{\beta}\phi(x)]:\simeq 1-\frac{\beta}{2}:\phi^2(x):.$$

The Debye-Hückel approximations all follow from this approximation.

The use of the sine-Gordon transformation introduces functional integrals that must be controlled. We do this using some new correlation inequalities. These inequalities give bounds on the moments of the measures that arise from the sine-Gordon transformation.

A natural approach to the problems studied here would be to use the cluster expansion of Brydges and Federbush [2]. Our approach has advantages and disadvantages with respect to the cluster expansion. Our approach is simpler than the cluster expansion. Moreover, we can allow several types of boundary conditions while the cluster expansion has only been carried out for Dirichlet boundary conditions. The disadvantage of our approach is that it requires charge symmetry. The cluster expansion does not.

This paper is organized as follows. We define the Coulomb system and observables in Sect. 2. In Sect. 3 we state our results. The sine-Gordon transformation and Mayer expansion are used in Sect. 4 to express the observables as functional integrals. Then we give non-rigorous derivations of the results of Sect. 3 using these functional integrals. In Sect. 5 we state and prove the correlation inequalities. Finally, the results stated in Sect. 3 are proved in Sect. 6.

2. Definitions

We consider a system which consists of two species of particles with equal chemical activities z. The species have charges $\pm e$. Let $\beta = \frac{e^2}{kT}$, where k is the Boltzmann constant and T is the temperature.

The particles interact via the two-body potential

$$v(x,\gamma;y,\delta) = \frac{\gamma\delta}{4\pi|x-y|} + v_\varepsilon(x,\gamma;y,\delta), \tag{2.1}$$

where $x, y \in \mathbb{R}^3$ are the positions of the particles and $\gamma, \delta \in \{-1, +1\}$ are their charges. The potential v_ε is a short range potential depending on ε. As ε tends to 0, v_ε tends to zero. [The precise meaning of this statement is given by hypotheses (H1) and (H2) in Sect. 6.]

The potential $\frac{1}{|x-y|}$ is the kernel of $\frac{1}{-\Delta}$, where Δ has free boundary conditions. Physically this means that the box containing the particles is an insulator. Our results are true for other boundary conditions, e.g., Dirichlet and periodic. The kernels of operators involving Δ are simplest with free boundary conditions, so we use free boundary conditions throughout this paper. We leave it to the reader to check that our proofs work for other boundary conditions.

Two examples of v_ε are

I. Hard cores:

$$v_\varepsilon(x,\gamma;y,\delta) = \begin{cases} \infty & \text{if } |x-y| < 2c_0\varepsilon\ell_D, \\ 0 & \text{otherwise}. \end{cases} \tag{2.2}$$

II. Yukawa potential:

$$v_\varepsilon(x,\gamma;y,\delta) = -\gamma\delta\frac{\exp(-|x-y|/c_0\varepsilon\ell_D)}{4\pi|x-y|}, \tag{2.3}$$

c_0 is a constant. ℓ_D is included in the definitions so that c_0 will be dimensionless.

For a volume $\Lambda \subset \mathbb{R}^3$ the grand canonical partition function is

$$Z(\Lambda) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{\gamma_1,\ldots,\gamma_n} \int_\Lambda d^n x \exp[-\beta U_n(x_1,\ldots,x_n;\gamma_1,\ldots,\gamma_n)]. \tag{2.4}$$

Each of $\gamma_1,\ldots,\gamma_n$ is summed over ± 1, and

$$\int_\Lambda d^n x = \int_\Lambda dx_1 \ldots \int_\Lambda dx_n.$$

The potential energy is

$$U_n(x_1,\ldots,x_n;\gamma_1,\ldots,\gamma_n) = \sum_{1\leq i<j\leq n} v(x_i,\gamma_i;x_j,\gamma_j).$$

The pressure is

$$P(\Lambda)=kT\frac{1}{|\Lambda|}\log[Z(\Lambda)], \tag{2.5}$$

where $|\Lambda|$ is the volume of Λ.

The correlation functions are

$$\varrho_\Lambda^{(m)}(y_1,\ldots,y_m;\delta_1,\ldots,\delta_m)=z^m Z(\Lambda)^{-1}\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{\gamma_1,\ldots,\gamma_n}$$
$$\cdot\int_\Lambda d^n x\exp[-\beta U_{n+m}(x_1,\ldots,x_n,y_1,\ldots,y_m;\gamma_1,\ldots,\gamma_n,\delta_1,\ldots,\delta_m)], \tag{2.6}$$

where $y_1,\ldots,y_m$ are distinct points in $\mathbb{R}^3$ and $\delta_1,\ldots,\delta_m\in\{-1,+1\}$.

Because of the charge symmetry the two species have the same density (average number of particles per unit volume). It is given by

$$\sigma(\Lambda)=\frac{1}{|\Lambda|}\int_\Lambda dy\varrho_\Lambda^{(1)}(y;\pm 1). \tag{2.7}$$

We will denote the infinite volume limits of the pressure, correlation functions and density by the same letters without a Λ. For example,

$$P=\lim_{\Lambda\to\mathbb{R}^3}P(\Lambda). \tag{2.8}$$

For simplicity we take the volumes Λ to be boxes with the ratios of the dimensions of the boxes bounded as $\Lambda\to\mathbb{R}^3$.

Lebowitz and Lieb [10] established the existence of the infinite volume limit of the pressure and density. For certain choices of the short range interaction v_ε the existence of the infinite volume limits of all the observables was proven by Fröhlich and Park [6]. With Dirichlet boundary conditions and an essentially arbitrary short range interaction v_ε these limits were shown to exist by Brydges and Federbush [2]. The existence of some infinite volume limit can always be established by a compactness argument.

3. Results

In the theorems of this section the short range potential v_ε can be given by either of our two examples, (2.2) and (2.3). These theorems are true for other choices of v_ε. We state the hypotheses that v_ε must satisfy in Sect. 6. In all the theorems of this section the infinite volume limit is taken before the $\varepsilon\to 0$ limit.

The first theorem says that the pressure is asymptotic to its Debye-Hückel approximation.

Theorem 3.1. $\frac{1}{kT}P\sim 2z+\frac{1}{12\pi}\ell_D^{-3}$ *in the sense that*

$$\lim_{\varepsilon\to 0}\left[\frac{1}{kT}P-2z\right]\ell_D^3=\frac{1}{12\pi}.$$

The correlation functions are also asymptotic to their Debye-Hückel approximations.

Theorem 3.2. *Let* $y_1, \ldots, y_m$ *be distinct points in* $\mathbb{R}^3$ *and* $\delta_1, \ldots, \delta_m \in \{-1, +1\}$. *Then*

$$\varrho^{(m)}(y_1, \ldots, y_m; \delta_1, \ldots, \delta_m) \sim z^m \left[1 - \beta \sum_{1 \leqq i < j \leqq m} \delta_i \delta_j (-\Delta + \ell_D^{-2})^{-1}(y_i, y_j) + \frac{\beta m}{8\pi \ell_D}\right]$$

in the sense that

$$\lim_{\varepsilon \to 0} \frac{\ell_D}{z^m \beta} [\varrho^{(m)}(y_1 \ell_D, \ldots, y_m \ell_D; \delta_1, \ldots, \delta_m) - z^m]$$

$$= - \sum_{1 \leqq i < j \leqq m} \delta_i \delta_j (-\Delta + 1)^{-1}(y_i, y_j) + \frac{m}{8\pi}.$$

Finally, the density is asymptotic to its Debye-Hückel approximation.

Theorem 3.3. $\sigma \sim z + \frac{1}{16\pi} \ell_D^{-3}$ *in the sense that*

$$\lim_{\varepsilon \to 0} (\sigma - z) \ell_D^3 = \frac{1}{16\pi}.$$

4. The Sine-Gordon Transformation and Mayer Expansion

Following Brydges and Federbush [2] we will apply the sine-Gordon transformation to the long range part of the interaction and use a Mayer expansion for the short range part. The details of the Mayer expansion are in Appendix A.

We split the Coulomb interaction into long and short range parts. They are

$$v_L(x, \gamma; y, \delta) = \gamma \delta \frac{1 - \exp(-|x-y|/\mu \ell_D)}{4\pi |x-y|}, \tag{4.1}$$

$$v_Y(x, \gamma; y, \delta) = \gamma \delta \frac{\exp(-|x-y|/\mu \ell_D)}{4\pi |x-y|}. \tag{4.2}$$

So $v = v_L + v_Y + v_\varepsilon$. We denote the total short range interaction by v_S

$$v_S = v_Y + v_\varepsilon. \tag{4.3}$$

The function

$$C(x, y) = \frac{1 - \exp(-|x-y|/\mu \ell_D)}{4\pi |x-y|}$$

is the kernel of the positive operator

$$C = \frac{1}{-\Delta} - \frac{1}{-\Delta + (\mu \ell_D)^{-2}}. \tag{4.4}$$

Hence there exists a Gaussian process with covariance $C(x, y)$, i.e., there exists a probability measure $d\mu$ and a Gaussian random variable $\phi(x)$ for each $x \in \mathbb{R}^3$ such that $\int d\mu \phi(x)\phi(y) = C(x, y)$. See pp. 16–17 of [12].

The partition function can now be written as $Z = \int d\mu Z(\phi)$, where

$$Z(\phi) = \sum_{n=0}^{\infty} \frac{\hat{z}^n}{n!} \sum_{\gamma_1, \dots, \gamma_n} \int_{\Lambda} d^n x \exp\left\{ -\beta \left[\sum_{i<j} v_S(i, j) - \frac{i}{\sqrt{\beta}} \sum_{i=1}^{n} \gamma_i \phi(x_i) \right] \right\}, \tag{4.5}$$

with

$$v_S(i, j) = v_S(x_i, \gamma_i; x_j, \gamma_j), \qquad \hat{z} = z \exp(\beta / 8\pi\mu\ell_D). \tag{4.6}$$

$Z(\phi)$ is a partition function with a convergent Mayer expansion

$$Z(\phi) = \exp\left[\sum_{n=1}^{\infty} K_n(\phi) \right]. \tag{4.7}$$

See (A.1) and (A.4) for the definition of $K_n(\phi)$.

The correlation functions are given by

$$\varrho_{\Lambda}^{(m)}(y_1, \dots, y_m; \delta_1, \dots, \delta_m) = Z(\Lambda)^{-1} \hat{z}^m \int d\mu \prod_{j=1}^{m} \exp[i\sqrt{\beta}\delta_j \phi(y_j)] \bar{Z}(\phi), \tag{4.8}$$

where

$$\begin{aligned} \bar{Z}(\phi) = e^{-\beta E} \sum_{n=0}^{\infty} \frac{\hat{z}^n}{n!} \sum_{\gamma_1, \dots, \gamma_n} \int_{\Lambda} d^n x \exp\Bigg\{ -\beta \Bigg[& \sum_{1 \leqq i < j \leqq n} v_S(i, j) \\ & + \sum_{i=1}^{n} A(x_i, \gamma_i) - \frac{i}{\sqrt{\beta}} \sum_{i=1}^{n} \gamma_i \phi(x_i) \Bigg] \Bigg\}, \end{aligned} \tag{4.9}$$

with

$$\begin{aligned} E &= \sum_{1 \leqq i < j \leqq m} v_S(y_i, \delta_i; y_j, \delta_j), \\ A(x, \gamma) &= \sum_{j=1}^{m} v_S(x, \gamma; y_j, \delta_j). \end{aligned} \tag{4.10}$$

$\bar{Z}(\phi)$ is also a partition function with a convergent Mayer series.

$$\bar{Z}(\phi) = \exp\left[-\beta E + \sum_{n=1}^{\infty} \bar{K}_n(\phi) \right]. \tag{4.11}$$

$\bar{K}_n(\phi)$ is defined by (A.1) and (A.5).

Using the results of the sine-Gordon transformation we can give a non-rigorous derivation of the Debye-Hückel approximations for the pressure, density and correlation functions. We will let $\mu \to 0$ as $\varepsilon \to 0$. Then $v_Y \to 0$, and hence $v_S \to 0$. For $n \geqq 2$, $K_n(\phi)$ contains at least one factor of v_S and so $\to 0$. Thus the important term in the Mayer series is $K_1(\phi)$.

We have

$$K_1(\phi) = 2z \int_{\Lambda} dx : \cos[\sqrt{\beta}\phi(x)] : . \tag{4.12}$$

The normal ordering : : is defined by the requirement that : : be linear and the equation

$$:\exp(\alpha\Psi):=\exp\left(-\frac{\alpha^2}{2}\int d\mu\Psi^2\right)\exp(\alpha\Psi),$$

where $\alpha\in\mathbb{C}$ and Ψ is any Gaussian random variable (see pp. 9–11 of [13]). In particular

$$\begin{aligned}&:\phi^2(x):=\phi^2(x)-C(x,x).\\&:\cos[\sqrt{\beta}\phi(x)]:=\exp\left[\frac{\beta}{2}C(x,x)\right]\cos[\sqrt{\beta}\phi(x)].\end{aligned}\tag{4.13}$$

In units with $\ell_D=1$, $\beta=\varepsilon$ and so $\beta\to 0$. Hence $K_1(\phi)$ should be approximately

$$K_1(\phi)\simeq 2z\int_\Lambda dx\left[1-\frac{\beta}{2}:\phi^2(x):\right].\tag{4.14}$$

Thus

$$\frac{1}{kT}P(\Lambda)\simeq 2z+\frac{1}{|\Lambda|}\log\left\{\int d\mu\exp\left[-\tfrac{1}{2}\ell_D^{-2}\int_\Lambda dx:\phi^2(x):\right]\right\}.\tag{4.15}$$

The Gaussian integral in (4.15) can be calculated. We will show in the proof of Theorem 3.1 in Sect. 6 that

$$\lim_{\mu\to 0}\lim_{\Lambda\to\mathbb{R}^3}\frac{1}{|\Lambda|}\log\left\{\int d\mu\exp\left[-\tfrac{1}{2}\ell_D^{-2}\int_\Lambda dx:\phi^2(x):\right]\right\}=\frac{1}{12\pi}\ell_D^{-3}.$$

So we have the Debye-Hückel approximation $\frac{1}{kT}P\simeq 2z+\frac{1}{12\pi}\ell_D^{-3}$.

For the correlation functions we note that as $v_S\to 0$, $\bar{K}_1(\phi)\simeq K_1(\phi)$. So using (4.14)

$$\begin{aligned}\varrho_\Lambda^{(m)}(y_1,\dots,y_m;\delta_1,\dots,\delta_m)&\simeq Z(\Lambda)^{-1}\hat{z}^m\int d\mu\prod_{j=1}^m\exp[i\sqrt{\beta}\delta_j\phi(y_j)]\\&\quad\cdot\exp\left\{2z\int_\Lambda dx\left[1-\frac{\beta}{2}:\phi^2(x):\right]\right\}\\&=\hat{z}^m\int d\bar{\mu}\prod_{j=1}^m\exp[i\sqrt{\beta}\delta_j\phi(y_j)],\end{aligned}\tag{4.16}$$

where $d\bar{\mu}$ is a Gaussian measure whose covariance in the infinite volume limit is $(C^{-1}+\ell_D^{-2})^{-1}(x,y)$. Some computation shows that as $\mu\to 0$ (4.16) is approximately

$$z^m\left[1-\beta\sum_{1\leqq i<j\leqq m}\delta_i\delta_j(-\Delta+\ell_D^{-2})^{-1}(y_i,y_j)+\frac{\beta m}{8\pi\ell_D}\right].$$

This is the Debye-Hückel approximation for the correlation function $\varrho^{(m)}$. The Debye-Hückel approximation for the density follows from the case of $m=1$.

5. Correlation Inequalities

The sine-Gordon transformation introduces measures of the form $\frac{Z(\phi)}{Z} d\mu$, where $Z(\phi)$ is given by (4.5). We will define a class of measures which includes the above measures. Then we will prove some correlation inequalities that allow us to bound the moments of these measures. Our techniques are reminiscent of those of Fröhlich and Park [6]. These bounds will be used in the next section in our proofs of the theorems of Sect. 3.

Our correlation inequalities hold for any Gaussian process. To state them in their full generality we will make use of the idea of a Gaussian process indexed by a Hilbert space (see pp. 15–20 of [13]). The reader who is not familiar with such Gaussian processes should see Remark 1 below. Let $\mathscr{H}$ be a separable real Hilbert space with inner product $(\,,\,)$. Then the Gaussian process indexed by $\mathscr{H}$ consists of a measure space Ω, a probability measure $d\mu$ and a linear map $\phi : \mathscr{H} \to L^2(d\mu)$, such that for each $\varrho \in \mathscr{H}$, $\phi(\varrho)$ is a Gaussian random variable and $\int d\mu \phi(\varrho)\phi(\varrho') = (\varrho, \varrho')$ for $\varrho, \varrho' \in \mathscr{H}$.

In our applications of our correlation inequalities the Gaussian process will always be the Gaussian process of Sect. 4. This Gaussian process is equivalent to the Gaussian process indexed by the Hilbert space $\mathscr{H}$ consisting of all distributions f on $\mathbb{R}^3$ whose Fourier transform $\hat{f}$ is a function with

$$\|f\|^2 = \int d^3k |\hat{f}(k)|^2 \left[\frac{1}{k^2} - \frac{1}{k^2 + (\mu\ell_D)^{-2}}\right] < \infty .$$

Let δ_x be the delta function centered at x. Then $\phi(\delta_x)$ is equivalent to the random variable that was denoted $\phi(x)$ in Sect. 4.

Definition 5.1. Let $\mathscr{H}$ be a real Hilbert space. Let $(\Omega, d\mu, \phi)$ be the Gaussian process indexed by $\mathscr{H}$. We will say that a measure $\langle\ \rangle$ defined on Ω is a sine-Gordon measure if it can be written in the form

$$\langle F(\phi)\rangle = \int d\mu F(\phi) \int d\nu(\varrho) \exp[i\phi(\varrho)] , \tag{5.1}$$

where $d\nu(\varrho)$ is a finite positive measure on $\mathscr{H}$. Furthermore the $d\nu$ measurable subsets of $\mathscr{H}$ are such that $\phi(\cdot)$ is a jointly measurable function on $\Omega \times \mathscr{H}$, and $d\nu(\varrho)$ is normalized so that

$$\langle 1 \rangle = 1 . \tag{5.2}$$

Remarks. 1. In our applications the measure $\langle\ \rangle$ will be of the form $\frac{Z(\phi)}{Z} d\mu$, where $Z(\phi)$ is defined by (4.5). This is a sine-Gordon measure. The measure $d\nu(\varrho)$ can be thought of as a measure on the configuration space

$$\bigoplus_{n=0}^{\infty} (\mathbb{R}^3 \times \{-1, +1\})^n .$$

Given $\varrho = (x_1, \gamma_1 ; \dots ; x_n, \gamma_n)$,

$$\phi(\varrho) = \sum_{i=1}^{n} \gamma_i \phi(x_i) .$$

And if $\varrho' = (y_1, \delta_1; \ldots; y_m, \delta_m)$, then

$$(\varrho, \varrho') = \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_i \delta_j C(x_i, y_j).$$

The reader who is unfamiliar with Gaussian processes indexed by a Hilbert space can follow our proofs by interpreting $\phi(\varrho)$ and (ϱ, ϱ') in this fashion.

2. In general sine-Gordon measures are complex measures. In the example of the preceding remark the sine-Gordon measure is positive. This follows from Eq. (4.7) for $Z(\phi)$ and the fact that the $K_n(\phi)$ are real. They are real because of the charge symmetry of the system. We will make frequent use of the positivity of this sine-Gordon measure. This is why our techniques only work for charge symmetric systems.

3. The assumption that $\phi(\cdot)$ is jointly measurable on $\Omega \times \mathscr{H}$ insures that the integrals in (5.1) are defined. Given a Hilbert space $\mathscr{H}$ there exists a version of the Gaussian process indexed by $\mathscr{H}$ which is jointly measurable when the measurable subsets of $\mathscr{H}$ are taken to be the Borel sets. (For a similar theorem see pp. 60–62 of Doob [5].) So if $d\nu(\varrho)$ is a Borel measure on $\mathscr{H}$ then the joint measurability assumption of the definition is satisfied simply by choosing the right version of the Gaussian process. This is the case for all the sine-Gordon measures we use in Sect. 6.

Theorem 5.2. *Let $\langle\ \rangle$ be a sine-Gordon measure, $\varrho \in \mathscr{H}$ and $\alpha \in \mathbb{R}$. Then*

$$\langle \cosh[\alpha\phi(\varrho)] \rangle \leqq \int d\mu \cosh[\alpha\phi(\varrho)], \tag{5.3}$$

and [9]

$$(-1)^n \langle :\phi^{2n}(\varrho): \rangle \geqq 0. \tag{5.4}$$

If $\langle\ \rangle$ is also positive, then

$$\langle \phi^{2n}(\varrho) \rangle \leqq d_n \left[\int d\mu \phi^2(\varrho)\right]^n = d_n \|\varrho\|^{2n}, \tag{5.5}$$

and

$$(-1)^n \langle :\phi^{2n}(\varrho): \rangle \leqq d_n \left[\int d\mu \phi^2(\varrho)\right]^n = d_n \|\varrho\|^{2n}, \tag{5.6}$$

where

$$d_n = \frac{(2n)!\, e^n}{2^n n^n}. \tag{5.7}$$

Proof. To prove the first inequality (5.3) we begin with the calculation

$$\int d\mu \cosh[\alpha\phi(\varrho)] \exp[i\phi(\varrho')] = \exp\left[\frac{\alpha^2}{2}(\varrho, \varrho) - \tfrac{1}{2}(\varrho', \varrho')\right] \cos[\alpha(\varrho, \varrho')]. \tag{5.8}$$

This implies

$$\begin{aligned} \int d\mu \cosh[\alpha\phi(\varrho)] \exp[i\phi(\varrho')] &\leqq \exp\left[\frac{\alpha^2}{2}(\varrho, \varrho) - \tfrac{1}{2}(\varrho', \varrho')\right] \\ &= \left\{\int d\mu \cosh[\alpha\phi(\varrho)]\right\} \left\{\int d\mu \exp[i\phi(\varrho')]\right\}. \end{aligned} \tag{5.9}$$

Integrating this inequality with respect to $d\nu(\varrho')$ we obtain (5.3), since $\langle 1 \rangle = 1$.

To prove the second inequality (5.4) it suffices to show

$$(-1)^n \int d\mu : \phi^{2n}(\varrho) : \exp[i\phi(\varrho')] \geqq 0. \tag{5.10}$$

This integral equals

$$(\varrho, \varrho')^{2n} \exp[-\tfrac{1}{2}(\varrho', \varrho')] \geqq 0,$$

which proves (5.4).

Now we assume that $\langle\ \rangle$ is positive. Then

$$\begin{aligned}\frac{\alpha^{2n}}{(2n)!}\langle \phi^{2n}(\varrho)\rangle &\leqq \sum_{m=0}^{\infty} \frac{\alpha^{2m}}{(2m)!}\langle \phi^{2m}(\varrho)\rangle \\ &= \langle \cosh[\alpha\phi(\varrho)]\rangle \\ &\leqq \int d\mu \cosh[\alpha\phi(\varrho)] \\ &= \exp\left[\frac{\alpha^2}{2}(\varrho, \varrho)\right].\end{aligned} \tag{5.11}$$

The third inequality (5.5) follows by taking

$$\alpha = \left[\frac{1}{2n}(\varrho, \varrho)\right]^{-1/2}. \tag{5.12}$$

To prove the last inequality (5.6), note that by (5.4) each term in

$$\langle :\cos[\alpha\phi(\varrho)]:\rangle = \sum_{m=0}^{\infty} \frac{\alpha^{2m}}{(2m)!}(-1)^m \langle :\phi^{2m}(\varrho):\rangle \tag{5.13}$$

is nonnegative. So

$$\begin{aligned}\frac{\alpha^{2n}}{(2n)!}(-1)^n\langle :\phi^{2n}(\varrho):\rangle &\leqq \langle :\cos[\alpha\phi(\varrho)]:\rangle \\ &= \exp\left[\frac{\alpha^2}{2}(\varrho, \varrho)\right]\langle \cos[\alpha\phi(\varrho)]\rangle \\ &\leqq \exp\left[\frac{\alpha^2}{2}(\varrho, \varrho)\right],\end{aligned}$$

where the last inequality uses the positivity of $\langle\ \rangle$. Now choose α as before. □

Remarks. 1. Fröhlich and Park [6] proved inequality (5.4) for $n=1$ for a certain class of sine-Gordon measures.

2. Inequality (5.4) can be used to prove a lower bound on the partition function and hence on the pressure for a special choice of v_ε (see [9]).

6. Proofs

In this section we prove the theorems stated in Sect. 3. We begin by giving the hypotheses that v_ε must satisfy for these theorems. Both of the examples of v_ε in Sect. 2 satisfy these hypotheses.

We assume that $v_\varepsilon = v_\varepsilon^o + v_\varepsilon^r$, where v_ε^r is repulsive, i.e.,

$$v_\varepsilon^r(i,j) \geqq 0\,, \tag{6.1}$$

and v_ε^o satisfies the stability bound

$$\sum_{1 \leqq i < j \leqq n} [v_Y(i,j) + v_\varepsilon^o(i,j)] \geqq -\frac{B(\varepsilon)}{\ell_D} n \tag{6.2}$$

for $\mu > \varepsilon$. $B(\varepsilon)$ is a positive function of ε.

We assume that v_ε^r is independent of the charges of the particles, and v_ε^o is unchanged if the charges of both particles are changed, i.e., $v_\varepsilon^o(x,\gamma;y,\delta) = v_\varepsilon^o(x,-\gamma;y,-\delta)$. Introduce the norms

$$\|v_\varepsilon^o\|_1 = \sup_{y,\delta} \sum_\gamma \int dx |v_\varepsilon^o(x,\gamma;y,\delta)|\,.$$

$$\|v_\varepsilon^r\|_r = \sup_{y,\delta} \sum_\gamma \int dx \{1 - \exp[-\beta v_\varepsilon^r(x,\gamma;y,\delta)]\}\,, \tag{6.3}$$

$$\|v_\varepsilon^o\|_2 = \sup_{y,\delta} \left[\sum_\gamma \int dx |v_\varepsilon^o(x,\gamma;y,\delta)|^2\right]^{1/2}.$$

Note that

$$\|v_Y\|_1 = 2\mu^2 \ell_D^2\,, \tag{6.4}$$

$$\|v_Y\|_2 = \left(\frac{\mu \ell_D}{4\pi}\right)^{1/2}. \tag{6.5}$$

Hypotheses (H1) and (H2) below say how fast v_ε must $\to 0$ as $\varepsilon \to 0$. (H3) says that we have a uniform stability bound as $\varepsilon \to 0$. (H4) and (H5) are weak hypotheses of a technical nature. ℓ_D is included in the hypotheses in various places so that the inequalities will be dimensionless.

Hypotheses on v_ε:

(H1) *There exist* $c_1, \delta_1 > 0$ *such that*

$$\ell_D^{-3} \|v_\varepsilon^r\|_r \leqq c_1 \varepsilon^{2+\delta_1}\,.$$

(H2) *There exist* $c_2, \delta_2 > 0$ *such that*

$$\ell_D^{-2} \|v_\varepsilon^o\|_1 \leqq c_2 \varepsilon^{1+\delta_2}\,.$$

(H3) *There exists* B *such that*

$$\varepsilon B(\varepsilon) \leqq B\,.$$

(H4) *There exists* c_3 *such that*

$$\ell_D^{-1/2} \|v_\varepsilon^o\|_2 \leqq c_3\,.$$

(H5) *For* $x \neq y$

$$\lim_{\varepsilon \to 0} v_\varepsilon(x,\gamma;y,\delta) = 0\,.$$

In Appendix B we verify that the two examples of v_ε given in Sect. 2 satisfy (H1) through (H5). Using the bounds of Appendix A we see that (H1) through (H3) imply that the Mayer series of Sect. 4 converge for sufficiently small ε and μ.

Dimensional considerations suggest that the theory should be invariant under the scaling $\beta \to \beta/\ell$, $z \to z\ell^3$, $v_\varepsilon \to v_\varepsilon^\ell$, with $v_\varepsilon^\ell(x,\gamma;y,\delta) = \ell v_\varepsilon(\ell x,\gamma;\ell y,\delta)$

$$\begin{gathered} \Lambda \to \frac{1}{\ell}\Lambda, \\ y_j \to y_j/\ell, \\ \frac{1}{kT}P \to \frac{1}{kT}P\ell^3, \\ \sigma \to \sigma\ell^3, \\ \varrho^{(m)} \to \varrho^{(m)}\ell^{3m}. \end{gathered} \tag{6.6}$$

Here $\frac{1}{\ell}\Lambda = \left\{\frac{1}{\ell}x : x \in \Lambda\right\}$. This invariance is easily checked by a change of variables in the integrals which define the observables.

In our proofs we will "work in units with $\ell_D = 1$." This simply amounts to carrying out the above scaling with $\ell = \ell_D$ since $\ell_D \to \ell_D/\ell$ under the above scaling. Hypotheses (H1) through (H5) were stated in a dimensionless way. Thus they will continue to hold after the scaling (6.6). In the future we will write v_ε^ℓ simply as v_ε.

Notation. Following the notation of field theory we will let ϕ denote a point in the measure space on which the Gaussian process is defined. $F(\phi)$ will be a function on this measure space, and $\sup_\phi |F(\phi)|$, the supremum of $|F|$ over the measure space.

We will use $O(\varepsilon^p)$ and $o(\varepsilon^p)$ to denote quantities that are $O(\varepsilon^p)$ and $o(\varepsilon^p)$ uniformly in Λ. $o(\Lambda)$ will denote a quantity that $\to 0$ as $\Lambda \to \mathbb{R}^3$. c, c', and δ will denote positive constants. The c, c' or δ in one equation is not necessarily the c, c' or δ in another equation. However, c_0, c_1, c_2, and c_3 do not change from equation to equation.

We will often suppress the argument in integrations with respect to Lebesgue measure on $\mathbb{R}^3$. For example

$$\int dx :\cos[\sqrt{\beta}\,\phi(x)]: = \int :\cos[\sqrt{\beta}\,\phi]:,$$

$$\int dx\psi(x)(-\Delta\psi)(x) = \int \psi(-\Delta)\psi.$$

Proof of Theorem 3.1 (The Pressure). In units with $\ell_D = 1$ the theorem becomes

$$\lim_{\varepsilon \to 0}\left[\frac{1}{kT}P - 2z\right] = \frac{1}{12\pi}. \tag{6.7}$$

Recall that P is the infinite volume pressure. We will work with the finite volume pressure $P(\Lambda)$ throughout the proof. At the end we will take the infinite volume limit. Our estimates will be uniform in Λ and so continue to hold in the infinite volume limit.

We will let $\mu \to 0$ as $\varepsilon \to 0$. Then v_S will $\to 0$. For $n \geqq 2$, $K_n(\phi)$ contains at least one factor of v_S and so will $\to 0$. We would like to let $\mu \to 0$ fast so that $K_n(\phi) \to 0$ fast. However, $C(x,x) = \frac{1}{4\pi\mu}$. So $C(x,x) \to \infty$ as $\mu \to 0$. Thus our bounds on moments of sine-Gordon measures from Theorem 5.2 will be useful only if μ does not $\to 0$ too fast. We let

$$\mu = \varepsilon^{1/4+\delta_4}, \tag{6.8}$$

with $0 < \delta_4 < 1/4$.

We split the proof into two steps. In the first step we estimate the difference between $\frac{1}{kT}P(\Lambda)$ and $\frac{1}{kT}P(\Lambda)$ with $v_S = 0$. Setting $v_S = 0$ is the same as setting $K_n(\phi) = 0$ for $n \geqq 2$. In the second step we estimate the error made by replacing $:\cos[\sqrt{\beta}\phi(x)]:$ by $1 - \frac{\beta}{2} :\phi^2(x):$ in $K_1(\phi)$.

Step 1 (The Short Range Interaction). Define an interpolating function on $[0, 1]$ by

$$I(s) = \int d\mu \exp\left[K_1(\phi) + \sum_{n=2}^{\infty} s^n K_n(\phi)\right]. \tag{6.9}$$

Then $I(1)$ is $Z(\Lambda)$ while $I(0)$ is $Z(\Lambda)$ with v_S set equal to 0. We estimate $\log[I(1)] - \log[I(0)]$ by bounding the logarithmic derivative of $I(s)$.

Define a measure by

$$\langle F(\phi) \rangle_s = I(s)^{-1} \int d\mu F(\phi) \exp\left[K_1(\phi) + \sum_{n=2}^{\infty} s^n K_n(\phi)\right].$$

Then

$$\frac{I'(s)}{I(s)} = \left\langle \sum_{n=2}^{\infty} n s^{n-1} K_n(\phi) \right\rangle_s. \tag{6.10}$$

Since $K_n(\phi)$ is real, $\langle\ \rangle_s$ is a positive measure. We claim it is a sine-Gordon measure. Write

$$\exp\left[K_1(\phi) + \sum_{n=2}^{\infty} s^n K_n(\phi)\right] = \exp[(1-s)K_1(\phi)] \exp\left[\sum_{n=1}^{\infty} s^n K_n(\phi)\right].$$

The second factor on the right hand side is $Z(\phi)$ with $\hat{z}$ replaced by $s\hat{z}$. So by (4.5) it is of the form $\int dv(\varrho) \exp[i\phi(\varrho)]$. The first factor on the right hand side is $Z(\phi)$ with $v_S = 0$ and $\hat{z}$ replaced by $(1-s)\hat{z}$. So it is also of the form $\int dv'(\varrho') \exp[i\phi(\varrho')]$. Hence their product is $\int dv(\varrho) \int dv'(\varrho') \exp[i\phi(\varrho + \varrho')]$, which proves the claim.

For $n \geqq 3$ we simply bound $K_n(\phi)$ as in Appendix A. Using hypotheses (H1), (H2), (H3) and our choice of μ Eqs. (A.7) and (A.8) become

$$|K_n(\phi)| \leqq c\varepsilon^{-1} r^{n-1} |\Lambda|, \tag{6.11}$$

with

$$r = c'[2\varepsilon^{1/2+2\delta_4} + c_1 \varepsilon^{1+\delta_1} + c_2 \varepsilon^{1+\delta_2}].$$

Hence

$$\sum_{n=3}^{\infty} n s^{n-1} |K_n(\phi)| = O(\varepsilon^{\delta}) |\Lambda|. \tag{6.12}$$

When $n=2$ there is only one η and

$$v_S(\eta)=v_Y(1,2)+v_\varepsilon^o(1,2)-\frac{1}{\beta}\frac{u(1,2)}{1+u(1,2)}.$$

The previous argument shows that the second and third terms contribute $O(\varepsilon^\delta)|\Lambda|$ to $|K_2(\phi)|$. The v_Y term requires a new argument.

Define

$$g(t,\phi)=-\frac{\beta\hat{z}^2}{2}\int ds\sum_{\gamma_1,\gamma_2}\int_\Lambda d^2x v_Y(1,2)\exp\{-t\beta s[v_Y(1,2)+v_\varepsilon^o(1,2)]\}U(\sigma)\cos(t\sqrt{\beta}\Phi), \tag{6.13}$$

with $\Phi=\gamma_1\phi(x_1)+\gamma_2\phi(x_2)$. Then the term to be bounded is $g(1,\phi)$. By the hypotheses on v_ε, $U(\sigma)$ is independent of γ_1 and γ_2. But

$$\sum_{\gamma_1,\gamma_2} v_Y(1,2)=0.$$

So $g(0,\phi)=0$.

Hence

$$\begin{aligned} g(1,\phi)&=\int_0^1 dt\frac{dg}{dt}(t,\phi)\\ &=\int_0^1 dt\frac{\beta\hat{z}^2}{2}\int ds\sum_{\gamma_1,\gamma_2}\int_\Lambda d^2x v_Y(1,2)\exp\{-t\beta s[v_Y(1,2)\\ &\quad+v_\varepsilon^o(1,2)]\}U(\sigma)\{\beta s[v_Y(1,2)+v_\varepsilon^o(1,2)]\cos(t\sqrt{\beta}\Phi)\\ &\quad+\sqrt{\beta}\Phi\sin(t\sqrt{\beta}\Phi)\}. \end{aligned} \tag{6.14}$$

Our goal is to bound $\langle K_2(\phi)\rangle_s$. So we need to bound $\langle g(1,\phi)\rangle_s$. Use

$$|\langle\cos(t\sqrt{\beta}\Phi)\rangle_s|\leqq 1. \tag{6.15}$$

Since $|\sin(x)|\leqq|x|$, Theorem 5.2 and the choice of μ (6.8) imply

$$\begin{aligned} |\langle\sqrt{\beta}\Phi\sin(t\sqrt{\beta}\Phi)\rangle_s|&\leqq\beta\langle\Phi^2\rangle_s\\ &\leqq c\varepsilon^{3/4-\delta_4}, \end{aligned} \tag{6.16}$$

where $\delta_4<1/4$.

Next we bound the integrations over x using hypotheses (H3) and (H4), Eqs. (6.4) and (6.5), and the Cauchy Schwartz inequality. The final result is

$$|\langle K_2(\phi)\rangle_s|=O(\varepsilon^\delta)|\Lambda|. \tag{6.17}$$

Equations (6.10), (6.12), and (6.17) imply

$$\frac{1}{|\Lambda|}|\log[I(1)]-\log[I(0)]|=O(\varepsilon^\delta).$$

So

$$\left|\frac{1}{kT}P(\Lambda)-\frac{1}{|\Lambda|}\log\{\int d\mu\exp[K_1(\phi)]\}\right|=O(\varepsilon^\delta). \tag{6.18}$$

Step 2 (The Long Range Interaction). Define a second interpolating function on $[0,1]$ by

$$Z(t)=\exp(2z|\Lambda|)\int d\mu \exp\left\{2zt^{-2}\int_\Lambda [:\cos(t\sqrt{\beta}\phi):-1]\right\}. \tag{6.19}$$

Then

$$Z(1)=\int d\mu \exp[K_1(\phi)], \tag{6.20}$$

and

$$\lim_{t\to 0} Z(t)=\exp(2z|\Lambda|)\int d\mu \exp\left(-1/2\int_\Lambda :\phi^2:\right), \tag{6.21}$$

by a dominated convergence argument

Let

$$\langle F(\phi)\rangle_t = N(t)^{-1}\int d\mu F(\phi)\exp\left[2zt^{-2}\int_\Lambda :\cos(t\sqrt{\beta}\phi):\right], \tag{6.22}$$

where $N(t)$ is defined so that $\langle\ \rangle_t$ is a probability measure. Then

$$\frac{Z'(t)}{Z(t)}=\left\langle \frac{d}{dt}2zt^{-2}\int_\Lambda [:\cos(t\sqrt{\beta}\phi):-1]\right\rangle_t.$$

Expanding $:\cos(t\sqrt{\beta}\phi):$ in a power series this

$$=2z\int_\Lambda dx\sum_{n=2}^{\infty}\frac{(2n-2)t^{2n-3}\beta^n}{(2n)!}(-1)^n\langle :\phi^{2n}(x):\rangle_t. \tag{6.23}$$

Using Theorem 5.2 and the choice of μ (6.8) this is

$$\begin{aligned}&\leqq 2z|\Lambda|\sum_{n=2}^{\infty}(2n-2)\beta^n\frac{e^n}{2^n n^n}\left(\frac{1}{4\pi\mu}\right)^n\\ &=O(\varepsilon^\delta)|\Lambda|.\end{aligned}$$

Hence

$$\frac{1}{|\Lambda|}\left|\log[Z(1)]-\log\left[\lim_{t\to 0}Z(t)\right]\right|=O(\varepsilon^\delta). \tag{6.24}$$

Combining (6.18), (6.20), (6.21), and (6.24) we have

$$\left|\frac{1}{kT}P(\Lambda)-2z-\frac{1}{|\Lambda|}\log\left[\int d\mu \exp\left(-1/2\int_\Lambda :\phi^2:\right)\right]\right|=O(\varepsilon^\delta).$$

Since the $O(\varepsilon^\delta)$ is uniform in Λ we can let $\Lambda\to\mathbb{R}^3$. One can show

$$\begin{aligned}&\lim_{\Lambda\to\mathbb{R}^3}\frac{1}{|\Lambda|}\log\left[\int d\mu \exp\left(-1/2\int_\Lambda :\phi^2:\right)\right]\\ &=\frac{1}{2(2\pi)^3}\int d^3k\left[\frac{1}{k^2}-\frac{1}{k^2+\mu^{-2}}-\log\left(1+\frac{1}{k^2}-\frac{1}{k^2+\mu^{-2}}\right)\right].\end{aligned} \tag{6.25}$$

(See pp. 175–177 of [7] for a similar calculation.) A dominated convergence argument shows that as $\mu \to 0$ the above

$$\to \frac{1}{2(2\pi)^3} \int d^3k \left[\frac{1}{k^2} - \log\left(1 + \frac{1}{k^2}\right)\right] = \frac{1}{12\pi}, \tag{6.26}$$

which completes the proof. $\square$

Proof of Theorem 3.2 (The Correlation Functions). The key idea is to do a complex translation $\phi \to \phi + i\sqrt{\beta}\psi$ in the functional integral expression for $\varrho^{(m)}$, where $i\sqrt{\beta}\psi$ is the stationary point of this integral. We will take $m=2$. The proof of the general case is essentially identical.

Throughout the proof we will work with finite volume correlation functions. We estimate the difference

$$\frac{1}{z^2}[\varrho_\Lambda^{(2)} - z^2] + \beta\delta_1\delta_2(-\Delta + 1)^{-1}(y_1, y_2) - \frac{\beta}{4\pi}.$$

Our estimate is a sum of two terms. One goes to zero faster than β uniformly in Λ. The other term goes to zero as $\Lambda \to \mathbb{R}^3$. So taking the infinite volume limit of our estimate proves the theorem.

We let

$$\mu = \varepsilon^{1/2 + \delta_4}, \tag{6.27}$$

with $0 < \delta_4 < 1/6$. As always we work in units with $\ell_D = 1$. The proof is broken up into seven steps.

Step 1 (Complex Translation). Let C be the covariance operator of $d\mu$ (4.4). Let $K = (C^{-1} + 1)^{-1}$. Let

$$\psi = K(\delta_1\delta_{y_1} + \delta_2\delta_{y_2}), \tag{6.28}$$

where δ_y is the delta function centered at y.

With

$$\begin{aligned} m_\pm^2 &= \frac{1}{2\mu^2}[1 \pm (1 - 4\mu^2)^{1/2}], \\ K &= (1 - 4\mu^2)^{-1/2}\left[\frac{1}{-\Delta + m_-^2} - \frac{1}{-\Delta + m_+^2}\right]. \end{aligned} \tag{6.29}$$

So K has kernel

$$K(x, y) = (1 - 4\mu^2)^{-1/2} \frac{\exp(-m_-|x-y|) - \exp(-m_+|x-y|)}{4\pi|x-y|}. \tag{6.30}$$

and

$$\psi(x) = \delta_1 K(x, y_1) + \delta_2 K(x, y_2). \tag{6.31}$$

We will need two bounds on $\psi(x)$. Since $K(x,y)$ is a positive definite function

$$\begin{aligned}|K(x,y)| &\leqq \sup_w K(w,w)\\ &=(1-4\mu^2)^{-1/2}\frac{m_+-m_-}{4\pi}\\ &\leqq \frac{1}{2\pi\mu}\end{aligned}$$

for sufficiently small μ. As $\mu\to 0$, $m_-\to 1$, so (6.30) implies

$$|K(x,y)| \leqq \frac{\exp(-|x-y|/2)}{2\pi|x-y|}$$

for sufficiently small μ. Thus

$$|\psi(x)| \leqq \frac{1}{\pi\mu}, \tag{6.32}$$

$$|\psi(x)| \leqq \frac{\exp(-|x-y_1|/2)}{2\pi|x-y_1|}+\frac{\exp(-|x-y_2|/2)}{2\pi|x-y_2|} \tag{6.33}$$

for sufficiently small μ.

We perform the complex translation $\phi(x)\to\phi(x)+i\sqrt{\beta}\psi(x)$ in the functional integral for $\varrho_\Lambda^{(2)}$ [see (4.8) and (4.11)]. The result is

$$\begin{aligned}\varrho_\Lambda^{(2)}(y_1,y_2;\delta_1,\delta_2)=Z(\Lambda)^{-1}\hat{z}^2\int d\mu\exp\Big\{&-\beta E+i\sqrt{\beta}[\delta_1\phi(y_1)+\delta_2\phi(y_2)]\\ &-\beta[\delta_1\psi(y_1)+\delta_2\psi(y_2)]+\sum_{n=1}^{\infty}\bar{K}_n(\phi+i\sqrt{\beta}\psi)-i\sqrt{\beta}\int\phi C^{-1}\psi\\ &+\frac{\beta}{2}\int\psi C^{-1}\psi\Big\}.\end{aligned} \tag{6.34}$$

(See p. 171 of [7] for a discussion of translations in Gaussian measures.)

Using the definition of ψ (6.28)

$$\varrho_\Lambda^{(2)}(y_1,y_2;\delta_1,\delta_2)=Z(\Lambda)^{-1}z^2\exp(A+S)\int d\mu\exp[\mathrm{R}(\phi)+\mathrm{iI}(\phi)+\mathrm{K}(\phi)], \tag{6.35}$$

where

$$A=-\frac{\beta}{2}[\delta_1\psi(y_1)+\delta_2\psi(y_2)]+\frac{\beta}{2}[C(y_1,y_1)+C(y_2,y_2)],$$

$$S=-\beta E+2z\int_\Lambda\left[\cosh(\beta\psi)-1-\frac{\beta^2}{2}\psi^2\right]-\frac{\beta}{2}\int_{\Lambda^c}\psi^2,$$

$$\Lambda^c=\mathbb{R}^3\backslash\Lambda,$$

$$R(\phi)=\mathrm{Re}\left[\sum_{n=1}^{\infty}\bar{K}_n(\phi+i\sqrt{\beta}\psi)\right]-\sum_{n=1}^{\infty}K_n(\phi)-2z\int_\Lambda[\cosh(\beta\psi)-1], \tag{6.36}$$

$$I(\phi)=\mathrm{Im}\left[\sum_{n=1}^{\infty}\bar{K}_n(\phi+i\sqrt{\beta}\psi)\right]+\sqrt{\beta}\int\phi\psi,$$

$$K(\phi)=\sum_{n=1}^{\infty}K_n(\phi).$$

Note that $Z(\Lambda)=\int d\mu \exp[K(\phi)]$. The definitions are arranged so that $z^2\exp(A)$ converges to the approximation while S, $R(\phi)$, and $I(\phi)$ converge to zero.

Step 2 (Computing A). Using (6.31),

$$A=-\beta\delta_1\delta_2 K(y_1,y_2)+\frac{\beta}{2}[C(y_1,y_1)-K(y_1,y_1)+C(y_2,y_2)-K(y_2,y_2)].$$

As $\mu\to 0$, $m_+\to\infty$, and $m_-\to 1$. So

$$K(y_1,y_2)\to\frac{\exp(-|y_1-y_2|)}{4\pi|y_1-y_2|}=(-\Delta+1)^{-1}(y_1,y_2).$$

A little work shows $C(y,y)-K(y,y)\to\frac{1}{4\pi}$. Thus

$$A=-\beta\delta_1\delta_2(-\Delta+1)^{-1}(y_1,y_2)+\frac{\beta}{4\pi}+o(\varepsilon). \tag{6.37}$$

Step 3 (Bounding S). We want to show that S is $o(\varepsilon)+o(\Lambda)$. From (4.10),

$$E=v_S(y_1,\delta_1;y_2,\delta_2)=v_Y(y_1,\delta_1;y_2,\delta_2)+v_\varepsilon(y_1,\delta_1;y_2,\delta_2).$$

The first term $\to 0$ as $\mu\to 0$. The second term $\to 0$ as $\varepsilon\to 0$ by hypothesis (H5).

The bounds (6.32) and (6.33) can be used to show

$$\begin{aligned}2z\int_\Lambda\left[\cosh(\beta\psi)-1-\frac{\beta^2}{2}\psi^2\right]&\leqq cz\beta^4\int_\Lambda\psi^4\\&\leqq c'z\beta^{5/2}\int_\Lambda\psi^{5/2}\\&=O(\varepsilon^{3/2}).\end{aligned}$$

Finally,

$$\frac{\beta}{2}\int_{\Lambda^c}\psi^2\to 0\quad\text{as}\quad\Lambda\to\mathbb{R}^3. \tag{6.38}$$

Thus

$$S=o(\varepsilon)+o(\Lambda). \tag{6.39}$$

Step 4 (Bounding $\bar K_n(\phi+i\sqrt{\beta}\psi)-K_n(\phi)$). In this step we show that

$$\sum_{n=2}^{\infty}|\bar K_n(\phi+i\sqrt{\beta}\psi)-K_n(\phi)|=o(\varepsilon). \tag{6.40}$$

We apply Lemma A.1 of Appendix A with

$$v_1(x,\gamma_1)=-\frac{i}{\sqrt{\beta}}\gamma\phi(x), \tag{6.41}$$

$$v_1'(x,\gamma)=\gamma\psi(x)+A(x,\gamma), \tag{6.42}$$

where

$$A(x,\gamma)=v_S(x,\gamma;y_1,\delta_1)+v_S(x,\gamma;y_2,\delta_2). \tag{6.43}$$

Hypotheses (H1), (H2), (H3), the choice of μ and Eq. (A.9) imply

$$r'=o(\varepsilon), \tag{6.44}$$

and

$$\sum_\gamma \int dx\,|\exp[-\beta A(x,\gamma)]-1|=o(\varepsilon^2). \tag{6.45}$$

Bounds (6.32) and (6.33) imply

$$\sum_\gamma \int dx\,|\exp[-\beta\gamma\psi(x)]-1|=O(\varepsilon). \tag{6.46}$$

From (6.32), (6.45), and (6.46),

$$\sum_\gamma \int dx\,|\exp[-\beta v_1'(x,\gamma)]-1|=O(\varepsilon). \tag{6.47}$$

Lemma A.1 now implies

$$\begin{aligned}|\bar{K}_n(\phi+i\sqrt{\beta}\psi)-K_n(\phi)|&\leqq c\hat{z}O(\varepsilon)[o(\varepsilon)]^{n-1}\\&\leqq c'[o(\varepsilon)]^{n-1},\end{aligned}$$

which implies (6.40).

Step 5 (Bounding $R(\phi)$). We split $R(\phi)$ into two parts, $R(\phi)=R_1(\phi)+R_2(\phi)$, with

$$\begin{aligned}R_1(\phi)&=2z\int_\Lambda[:\cos(\sqrt{\beta}\phi):-1][\cosh(\beta\psi)-1],\\R_2(\phi)&=\mathrm{Re}[\bar{K}_1(\phi+i\sqrt{\beta}\psi)]-2z\int_\Lambda:\cos(\sqrt{\beta}\phi):\cosh(\beta\psi)\\&\quad+\mathrm{Re}\left\{\sum_{n=2}^{\infty}[\bar{K}_n(\phi+i\sqrt{\beta}\psi)-K_n(\phi)]\right\}.\end{aligned} \tag{6.48}$$

Using (6.32) and (6.33),

$$2z\int_\Lambda|\cosh(\beta\psi)-1|=O(\varepsilon), \tag{6.49}$$

so

$$\sup_\phi|R_1(\phi)|=O(\varepsilon). \tag{6.50}$$

Using (6.45),

$$\begin{aligned}&\mathrm{Re}[\bar{K}_1(\phi+i\sqrt{\beta}\psi)]-2z\int_\Lambda:\cos(\sqrt{\beta}\phi):\cosh(\beta\psi)\\&=z\sum_\gamma\int_\Lambda dx:\cos(\sqrt{\beta}\phi):\exp[-\beta\gamma\psi(x)]\{\exp[-\beta A(x,\gamma)]-1\}\\&=zo(\varepsilon^2)\\&=o(\varepsilon).\end{aligned}$$

Along with (6.40) this implies

$$\sup_{\phi}|\mathrm{R}_2(\phi)|=\mathrm{o}(\varepsilon). \tag{6.51}$$

Step 6 (Bounding $I(\phi)$). Let $I(\phi)=I_1(\phi)+I_2(\phi)+I_3(\phi)+I_4(\phi)$, with

$$\begin{aligned}
I_1(\phi)&=2z\int_{\Lambda}[\sqrt{\beta}\phi-:\sin(\sqrt{\beta}\phi):]\sinh(\beta\psi),\\
I_2(\phi)&=2z\int_{\Lambda}\sqrt{\beta}\phi[\beta\psi-\sinh(\beta\psi)],\\
I_3(\phi)&=\sqrt{\beta}\int_{\Lambda^c}\phi\psi,\\
I_4(\phi)&=z\sum_{\gamma}\int_{\Lambda}:\sin(\sqrt{\beta}\gamma\phi):\exp(-\beta\gamma\psi)\{\exp[-\beta A(x,\gamma)]-1\}\\
&\quad+\mathrm{Im}\left[\sum_{n=2}^{\infty}\bar{K}_n(\phi+i\sqrt{\beta}\psi)\right].
\end{aligned} \tag{6.52}$$

The argument that proved (6.51) shows

$$\sup_{\phi}|I_4(\phi)|=o(\varepsilon). \tag{6.53}$$

Step 7. Define a probability measure by $\langle F(\phi)\rangle=Z(\Lambda)^{-1}\int d\mu F(\phi)\exp[K(\phi)]$. Then (6.35) becomes

$$\varrho_{\Lambda}^{(2)}(y_1,y_2;\delta_1,\delta_2)=z^2\exp(A+S)\langle\exp[R(\phi)+iI(\phi)]\rangle. \tag{6.54}$$

Equations (6.37) and (6.39) imply that the proof will be completed by showing

$$\langle\exp[R(\phi)+iI(\phi)]\rangle-1=o(\varepsilon)+o(\Lambda). \tag{6.55}$$

Since $\langle\ \rangle$ is even in ϕ and $I(\phi)$ is odd in ϕ,

$$\langle\exp[R(\phi)+iI(\phi)]\rangle=\langle\exp[R(\phi)]\{\cos[I(\phi)]-1\}\rangle+\langle\exp[R(\phi)]-1\rangle.$$

Using (6.50) and (6.51), proving (6.55) reduces to showing

$$\langle\exp[R_1(\phi)]\rangle-1=o(\varepsilon), \tag{6.56}$$

and

$$\langle|\cos[I(\phi)]-1|\rangle=o(\varepsilon)+o(\Lambda). \tag{6.57}$$

Let

$$f(t)=\langle\exp[tR_1(\phi)]\rangle,$$

and

$$\langle F(\phi)\rangle_t=f(t)^{-1}\langle F(\phi)\exp[tR_1(\phi)]\rangle.$$

So

$$\langle\exp[R_1(\phi)]\rangle-1=\int_0^1 f(t)\langle R_1(\phi)\rangle_t dt. \tag{6.58}$$

By (6.50)

$$\sup_{0 \leqq t \leqq 1} f(t) = O(1).$$

So (6.56) will follow from

$$\sup_{0 \leqq t \leqq 1} |\langle R_1(\phi)\rangle_t| = o(\varepsilon). \tag{6.59}$$

$\langle\ \rangle_t$ is a positive sine-Gordon measure. So Theorem 5.2 implies

$$\begin{aligned}|\langle :\cos[\sqrt{\beta}\phi(x)]: -1\rangle_t| &= \sum_{n=1}^{\infty} \frac{\beta^n}{(2n)!}(-1)^n \langle :\phi^{2n}(x):\rangle_t \\ &\leqq \sum_{n=1}^{\infty} \frac{\beta^n e^n}{2^n n^n}\left(\frac{1}{4\pi\mu}\right)^n.\end{aligned} \tag{6.60}$$

By our choice of μ (6.27) this is $O(\varepsilon^{1/2-\delta_4})$ with $\delta_4 < 1/6$. Combining this with (6.49) proves (6.59).

To prove (6.57) we use

$$\begin{aligned}|\cos[I(\phi)]-1| &\leqq |\cos[I(\phi)] - \cos[I_1(\phi)]| + |\cos[I_1(\phi)]-1| \\ &\leqq |I_2(\phi)+I_3(\phi)+I_4(\phi)| + \tfrac{1}{2}I_1(\phi)^2,\end{aligned} \tag{6.61}$$

which follows from $|\cos(\alpha+\theta)-\cos(\alpha)| \leqq |\alpha|$, $|\cos(\theta)-1| \leqq \frac{1}{2}\theta^2$. So (6.57) will follow from

$$\langle |I_i(\phi)|\rangle = o(\varepsilon) + o(\Lambda) \quad \text{for} \quad i = 2, 3, 4, \tag{6.62}$$

$$\langle I_1(\phi)^2\rangle = o(\varepsilon). \tag{6.63}$$

$\langle\ \rangle$ is a positive sine-Gordon measure so Theorem 5.2 and the Cauchy-Schwartz inequality imply

$$\begin{aligned}\langle \sqrt{\beta}\,|\phi(x)|\rangle &\leqq \left(\frac{\varepsilon}{4\pi\mu}\right)^{1/2} \\ &= O(1).\end{aligned} \tag{6.64}$$

Inequalities (6.32) and (6.33) imply

$$2z \int_{\Lambda} |\sinh(\beta\psi) - \beta\psi| = o(\varepsilon). \tag{6.65}$$

These two bounds imply (6.62) for $i=2$.

Using (6.64),

$$\langle |I_3(\phi)|\rangle \leqq O(1) \int_{\Lambda^c} |\psi| = o(\Lambda), \tag{6.66}$$

which proves (6.62) for $i=3$. The case of $i=4$ follows from (6.53).

To prove (6.63)

$$\begin{aligned}&\langle \{:\sin[\sqrt{\beta}\phi(x)]: - \sqrt{\beta}\phi(x)\}^2\rangle \\ &= \langle \{[\exp(\beta/8\pi\mu)-1]\sin[\sqrt{\beta}\phi(x)] + \sin[\sqrt{\beta}\phi(x)] - \sqrt{\beta}\phi(x)\}^2\rangle \\ &\leqq \left\langle \left\{[\exp(\beta/8\pi\mu)-1]\sqrt{\beta}\,|\phi(x)| + \frac{\beta^{3/2}}{6}|\phi(x)|^3\right\}^2\right\rangle,\end{aligned} \tag{6.67}$$

using $|\sin(\theta)| \leqq |\theta|$, $|\sin(\theta)-\theta| \leqq \frac{1}{6}|\theta|^3$. By Theorem 5.2 the above is

$$O\left(\left(\frac{\beta}{\mu}\right)^3\right) = O(\varepsilon^{3/2-3\delta_4}), \tag{6.68}$$

with $\delta_4 < 1/6$. (6.32) and (6.33) imply

$$2z \int_{\Lambda} |\sinh(\beta\psi)| = O(1).$$

Combining this with (6.68) proves (6.63). □

Proof of Theorem 3.3 (The Density). The proof is almost immediate from the proof of the previous theorem for $m=1$. We actually wrote out the proof for $m=2$, but we will refer to it as if it were the proof for $m=1$.

Recall

$$\sigma = \lim_{\Lambda \to \mathbb{R}^3} \frac{1}{|\Lambda|} \int_{\Lambda} dy \varrho_{\Lambda}^{(1)}(y; \pm 1). \tag{6.69}$$

In the previous proof we estimated a quantity like

$$\varrho_{\Lambda}^{(1)}(y; \pm 1) - z - \frac{1}{16\pi}.$$

Our bounds on this expression were independent of Λ and y except for two terms. They were (6.38) $\frac{\beta}{2} \int_{\Lambda^c} \psi^2$, and (6.66) $\langle |I_3(\phi)| \rangle$.

Using (6.66) it suffices to show

$$\lim_{\Lambda \to \mathbb{R}^3} \frac{1}{|\Lambda|} \int_{\Lambda} dy \int_{\Lambda^c} dx |\psi(x)|^p = 0$$

for $p=1,2$. (Remember ψ depends on y.) This follows from (6.33) and our conditions on how $\Lambda \to \mathbb{R}^3$. □

Appendix A. The Mayer Expansion

All the Mayer series in this paper have the same two body interaction $v_S = v_Y + v_\varepsilon^o + v_\varepsilon^r$. They differ in the one body interaction v_1. Following Brydges and Federbush [3] these Mayer series are given by

$$\begin{aligned}\mathscr{K}_n(v_1) = \frac{(-\beta)^{n-1}\hat{z}^n}{n} \sum_{\eta} \int d\sigma f(\eta, \sigma) \sum_{\gamma_1, \ldots, \gamma_n} \int_{\Lambda} d^n x v_S(\eta) \\ \cdot \exp[-\beta v_S(\sigma)] U(\sigma) \exp\left[-\beta \sum_{i=1}^{n} v_1(i)\right],\end{aligned} \tag{A.1}$$

where

$$v_S(\eta)=\prod_{i=1}^{n-1}\left[v_Y(i+1,\eta(i))+v_\varepsilon^o(i+1,\eta(i))\right.$$
$$\left.-\frac{1}{\beta}\frac{u(i+1,\eta(i))}{1+s_{\eta(i)}s_{\eta(i)+1}\cdots s_i u(i+1,\eta(i))}\right],$$
$$u(i,j)=\exp[-\beta v_\varepsilon^r(i,j)]-1,$$
$$v_S(\sigma)=\sum_{1\leqq i<j\leqq n} s_i s_{i+1}\cdots s_{j-1}[v_Y(i,j)+v_\varepsilon^o(i,j)], \tag{A.2}$$
$$U(\sigma)=\prod_{1\leqq i<j\leqq n}[1+s_i s_{i+1}\cdots s_{j-1}u(i,j)],$$
$$\int d\sigma=\prod_{i=1}^{n=1}\int_0^1 ds_i,$$
$$f(\eta,\sigma)=\prod_{i=2}^{n-1} s_{\eta(i)}s_{\eta(i)+1}\cdots s_{i-1}.$$

Empty products are taken to be 1. The sum over η is a sum over all functions $\eta:\{1,2,\ldots,n-1\}\to\{1,2,\ldots,n-1\}$, such that

$$\eta(i)\leqq i. \tag{A.3}$$

The three Mayer series we use are as follows. $K_n(\phi)$ is $\mathscr{K}_n(v_1)$ with

$$v_1(i)=v_1(x_i,\gamma_i)=\frac{-i}{\sqrt{\beta}}\gamma_i\phi(x_i). \tag{A.4}$$

$\bar{K}_n(\phi)$ is $\mathscr{K}_n(v_1)$ with

$$v_1(i)=\frac{-i}{\sqrt{\beta}}\gamma_i\phi(x_i)+\sum_{j=1}^{m} v_S(x_i,\gamma_i;y_j,\delta_j). \tag{A.5}$$

$\bar{K}_n(\phi+i\sqrt{\beta}\psi)$ is $\mathscr{K}_n(v_1)$ with

$$v_1(i)=\frac{-i}{\sqrt{\beta}}\gamma_i\phi(x_i)+\gamma_i\psi(x_i)+\sum_{j=1}^{m} v_S(x_i,\gamma_i;y_j,\delta_j). \tag{A.6}$$

As shown in [3]

$$|\mathscr{K}_n(v_1)|\leqq\hat{z}\exp[\beta\|v_1\|_-+\varepsilon B(\varepsilon)]2|\Lambda|r^{n-1}, \tag{A.7}$$

where

$$r=\hat{z}\beta\exp[1+\beta\|v_1\|_-+\varepsilon B(\varepsilon)]\left[\|v_Y\|_1+\|v_\varepsilon^o\|_1+\frac{1}{\beta}\|v_\varepsilon^r\|_r\right]; \tag{A.8}$$

$\|\ \|_1$ and $\|\ \|_r$ are defined by (6.3). $\|v_1\|_-$ is the sup norm of the negative part of the real part of v_1. For each of our three Mayer series

$$\beta\|v_1\|_-=O(1). \tag{A.9}$$

To see this note that the stability bound (6.2) with $n=2$ implies

$$\beta \sum_{j=1}^{m} v_S(x,\gamma;y_j,\delta_j) \geqq -2m\varepsilon B(\varepsilon) \geqq -2mB$$

by hypothesis (H3). And ψ is always such that $\|\beta\psi\|_\infty = O(1)$. The condition for convergence of the Mayer series is $r<1$.

In the proof of Theorem 3.2 we need a bound on the difference of two Mayer series.

Lemma A.1.

$$|\mathscr{K}_n(v_1+\bar{v}_1)-\mathscr{K}_n(v_1)| \leqq 2\hat{z}\exp[\beta\|v_1\|_- + \varepsilon B(\varepsilon)] \sum_\gamma \int dx \cdot |\exp[-\beta\bar{v}_1(x,\gamma)]-1|(r')^{n-1}, \tag{A.10}$$

with

$$r' = 2[\exp(\beta\|\bar{v}_1\|_-)+1]r. \tag{A.11}$$

Proof. The difference between the two $\mathscr{K}_n$'s is equal to the expression for $\mathscr{K}_n(v_1)$ with the factor

$$\left\{\exp\left[-\beta\sum_{i=1}^{n}\bar{v}_1(i)\right]-1\right\}$$

included in the integral. We rewrite this factor as

$$\prod_{i=1}^{n}\{\exp[-\beta\bar{v}_1(i)]-1+1\}-1 = \sum_{S\neq\phi}\prod_{i\in S}\{\exp[-\beta\bar{v}_1(i)]-1\}, \tag{A.12}$$

where S is summed over all nonempty subsets of $\{1,2,\dots,n\}$.

Each term in this sum contains at least one factor $\exp[-\beta\bar{v}_1(i_o)]-1$. We bound any other factors of $\exp[-\beta\bar{v}_1(i)]-1$ by $\exp(\beta\|\bar{v}_1\|_-)+1$. Then we bound the integrations over x in the usual way except that we bound the integration over x_{i_o} last. (Think of i_o as the base of the tree graph η.) This last integration gives a factor of

$$\sum_\gamma \int dx\,|\exp[-\beta\bar{v}_1(x,\gamma)]-1|.$$

Since (A.12) has 2^n-1 terms, the lemma follows. □

Appendix B

In this appendix we verify that the two examples of v_ε given in Sect. 2 satisfy hypotheses (H1) through (H5) of Sect. 6.

I. Hard Cores (2.2). Let

$$v_\varepsilon^r = v_\varepsilon, \qquad v_\varepsilon^o = 0; \tag{B.1}$$

(H1), (H2), (H4), and (H5) are immediate. Unfortunately (H3) is not true. We must replace our system with an equivalent one in which (H3) holds.

We do this using a generalization to the Yukawa potential of Newton's theorem for the Coulomb potential. Let $|x_1 - x_2| \geqq 2c_0\varepsilon\ell_D$. Then the potential energy due to the Yukawa potential

$$\frac{\exp(-|x-y|/\mu\ell_D)}{4\pi|x-y|} \tag{B.2}$$

of charges γ_1 and γ_2 at x_1 and x_2 is equal to the potential energy due to this Yukawa potential of two spheres of radius $c_0\varepsilon\ell_D$ with centers at x_1 and x_2 and total charges $\hat{\gamma}_1$ and $\hat{\gamma}_2$ distributed uniformly on their surfaces, where

$$\hat{\gamma}_i = \frac{c_0\varepsilon}{\mu\sinh\left(\dfrac{c_0\varepsilon}{\mu}\right)}\gamma_i.$$

Thus we can redefine $v_Y(x_1, \gamma_1; x_2, \gamma_2)$ to be the potential energy due to the potential (B.2) of two spheres of radius $c_0\varepsilon\ell_D$ at x_1 and x_2 with charges $\hat{\gamma}_1$ and $\hat{\gamma}_2$ uniformly distributed on their surfaces. For the long range part of the interaction v_L we still treat the charges as point charges. So the sine-Gordon transformation is not affected by our redefinition of v_Y.

v_Y is positive definite so the stability bound (6.2) holds with $\dfrac{B(\varepsilon)}{\ell_D}$ equal to 1/2 of the self energy of these spheres. (H3) follows by computing this self energy.

Since we have changed the definition of v_Y we must recompute $\|v_Y\|_1$ and $\|v_Y\|_2$. We leave it to the reader to check that these norms behave essentially as before.

II. Yukawa Potential (2.3). Let $v_\varepsilon^o = v_\varepsilon$, $v_\varepsilon^r = 0$; (H1) and (H5) are immediate. (H2) and (H4) take a little calculation.

We can assume that $\mu > c_0\varepsilon$, since in the proofs $\varepsilon \to 0$ faster than μ. Hence $v_Y + v_\varepsilon^o$ is positive definite. (H3) follows by computing $v_Y(i, i) + v_\varepsilon^o(i, i)$.

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References

1. Brydges, D.: A rigorous approach to Debye screening in dilute Coulomb systems. Commun. Math. Phys. **58**, 313–350 (1978)
2. Brydges, D., Federbush, P.: Debye screening. Commun. Math. Phys. **73**, 197–246 (1980)
3. Brydges, D., Federbush, P.: A new form of the Mayer expansion in classical statistical mechanics. J. Math. Phys. **19**, 2064–2067 (1978)
4. Debye, P., Hückel, E.: Zur Theorie der Elektrolyte. Phys. Z. **24**, 185–206 (1923)
5. Doob, J.L.: Stochastic processes. New York: Wiley 1953
6. Fröhlich, J., Park, Y.M.: Correlation inequalities and the thermodynamic limit for classical and quantum continuous systems. Commun. Math. Phys. **59**, 235–266 (1978)
7. Glimm, J., Jaffe, A.: Quantum physics: a functional integral point of view. Berlin, Heidelberg, New York: Springer 1981
8. Imbrie, J.: Debye screening for jellium and other Coulomb systems. Commun. Math. Phys. **87**, 515–565 (1983)

9. Kennedy, T.: A lower bound on the partition function for a classical charge symmetric system. J. Stat. Phys. **28**, 633–638 (1982)
10. Lebowitz, J.L., Lieb, E.H.: The constitution of matter: existence of thermodynamics for systems composed of electrons and nuclei. Adv. Math. **9**, 316–398 (1972)
11. McConkey, G., Prock, A.: Topics in chemical physics (based on lectures by P. Debye). New York: Elsevier 1962
12. Simon, B.: Functional integration and quantum physics. London, New York: Academic Press 1979
13. Simon, B.: The $P(\phi)_2$ Euclidean (quantum) field theory. Princeton: Princeton University Press 1974

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