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Resonances for the AC-Stark Effect*

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Abstract. The resonance problem for the AC-Stark effect is discussed. We prove that all bound states of the system $-(1/2)\Delta + V(x)$ will turn into resonances after an AC-electric field is switched on and the order of the imaginary part of a resonance is determined by the number of the photons it takes to ionize the bound state which is turning into the resonance; if two bound states have energy difference of the photon, there exists a state which oscillates between the two states for a long time.

1. Introduction

Suppose that a quantum particle of mass m and charge e in a potential field $V(x)$ is subject to an alternating electric field $\mu E \cos \omega t$. Then the Schrödinger equation for the motion of the particle is written as

$$i\hbar\, \partial u/\partial t = [-(\hbar^2/2m)\Delta + V(x) - \mu e E x \cos \omega t]\, u. \tag{1.1}$$

Here $\mu > 0$ is the strength of the field, $E \in \mathbb{R}^3, |E| = 1$ is the direction, ω is the frequency, $\hbar = h/2\pi$ and h is Planck's constant;

$$-\Delta = -(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2).$$

The purpose of this paper is to study the resonances for Eq. (1.1). We shall argue that the resonances should be defined as the poles of the resolvent of the "coupled photon-particle Hamiltonian" $-i\hbar\,\partial/\partial t - (\hbar^2/2m)\Delta + V(x - \mu e m^{-1} \omega^{-2} E \cos \omega t)$ in the second Riemann sheet and show, in particular, the following two results under suitable conditions on $V(x)$:

(A) For sufficiently small μ and almost all ω, all the bound states $\{(\phi_j(x), -k_j^2)\}$ of $H = -(\hbar^2/2m)\Delta + V(x)$ will turn into the resonances $\{(\phi_j(t,x,\mu), \lambda_j)\}$ and the imaginary part of λ_j is determined as $\operatorname{Im} \lambda_j = C_j(\omega)\mu^{2n} + O(\mu^{2n+1})$, $C_j(\omega) < 0$, where n is the smallest integer such that $-k_j^2 + n\hbar\omega > 0$: $(\phi_j(t,x,\mu), \phi_j(x)) = e^{-i\lambda_j t/\hbar} + O(\mu)$ uniformly in $t \geqq 0$ (Theorem 3.5 and 3.6).

(B) If two bound states $(\phi_j(x), -k_j^2)$ and $(\phi_i(x), -k_i^2)$ of H have the energy

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difference $-k_j^2 + k_i^2 = \pm \hbar\omega$, then the time dependent wave function for (1.1) with initial state $\phi_j(x)$ will oscillate between $\phi_j(x)$ and $\phi_i(x)$ for a long time (Theorem 3.6). We shall employ a technique which is a synthesis of the complex scaling method of Aguilar–Balslev–Combes ([1], [2]) and the Floquet theory ([3], [9], [21]). Hereafter we shall take the unit and coordinates such that $\hbar = m = -e = 1$ and $E = (1,0,0)$ and assume $0 \leqq \mu \leqq \Omega < \infty$. We consider Eq. (1.1) in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^3)$.

Throughout the paper we assume that the potential $V(x)$ satisfies the following condition (A_n) $(n = 0, 1, \ldots, \infty)$ with sufficiently large n. Some additional assumptions will be made later. We write as $C_0(\mathbb{R}^3)$ the Banach space of all continuous functions $f(x)$ on $\mathbb{R}^3$ such that $f(x) \to 0$ as $|x| \to \infty$ equipped with the norm $\|f\|_\infty = \sup\{|f(x)| : x \in \mathbb{R}^3\}$. If $f(x) \in C_0(\mathbb{R}^3)$, the function $f(e^\theta x + \rho E)$ is a $C_0(\mathbb{R}^3)$-valued continuous function of $(\theta, \rho) \in \mathbb{R}^2$.

Assumption (A_n). (1) $V(x) \in C_0(\mathbb{R}^3)$ and is a real-valued function.

(2) There exists a constant $0 < a < \pi/4$ such that for any fixed $-\omega^{-2}\Omega \leqq \rho \leqq \omega^{-2}\Omega$, the $C_0(\mathbb{R}^3)$-valued function $V(e^\theta x + \rho E)$ of $\theta \in \mathbb{R}^1$ can be extended as an analytic function to the strip $\mathbb{C}_a = \{z \in \mathbb{C} : -a < \operatorname{Im} z < a\}$ of the complex plane.

(3) For any fixed $\theta \in \mathbb{C}_a$, the $C_0(\mathbb{R}^3)$-valued function $V(e^\theta x + \rho E)$ of $(-\omega^{-2}\Omega \leqq \rho \leqq \omega^{-2}\Omega)$ is a C^n-function and $(\partial^k/\partial\rho^k)V(e^\theta x + \rho E)$ $(0 \leqq k \leqq n$ or $0 \leqq k < \infty$ in the case $n = \infty)$ is again a $C_0(\mathbb{R}^3)$-valued analytic function of $\theta \in \mathbb{C}_a$. Typical examples of the potentials which satisfy Assumption (A_∞) are smeared Coulomb or Yukawa potentials:

$$V(x) = \frac{Z}{|x|} * ((2\pi\varepsilon)^{-3/2} e^{-|x|^2/2\varepsilon}), \quad V(x) = \frac{Ze^{-\gamma|x|}}{|x|} * ((2\pi\varepsilon)^{-3/2} e^{-|x|^2/2\varepsilon}).$$

Although with some extra work we can accommodate some singularity for the potential $V(x)$ in the direction perpendicular to E, the analyticity in the E direction is essential.

Before explaining the problem more precisely, we first make a transformation for Eq. (1.1). Notice that if $V(x) \equiv 0$, then (1.1) can be solved explicitly as

$$u(t, x) = T(t)(\exp(-itH_0) T(0)^{-1} u(0, \cdot))(x), \tag{1.2}$$

where $H_0 = -\frac{1}{2}\Delta$ is the free Hamiltonian and

$$\begin{aligned} T(t) f(x) = \exp(&-i\mu E x \sin \omega t/\omega \\ &+ i\mu^2 \sin 2\omega t/8\omega^3 - i\mu^2 t/4\omega^2) \times f(x - \mu E \cos \omega t/\omega^2). \end{aligned} \tag{1.3}$$

Equations (1.2) and (1.3) show that the effect of the AC-electric field on the free particle amounts to the addition of the harmonic oscillation to the free trajectory. Suggested by this, we define the new wave function $u_D(t, x)$ by

$$u(t, \cdot) = T(t) u_D(t, \cdot).$$

and write Eq. (1.1) in terms of $u_D(t, x)$:

$$i\partial u_D/\partial t = H(t, \mu) u_D \equiv [-(\tfrac{1}{2})\Delta + V(x + \mu\omega^{-2} E \cos \omega t)] u_D \tag{1.4}$$

(see Kitada–Yajima [14] §7). This transformation eliminates the high singularity of $\mu Ex\cos\omega t$ by introducing the oscillating potential which is more tractable. We may apply a standard theorem (Kato [13]) to see that Eq. (1.4) generates a unitary propagator $U(t,s,\mu)$ and the solution to the original equation (1.1) can be given as

$$u(t) = T(t)U(t,s,\mu)T(s)^{-1}u(s).$$

Hereafter we shall discard the trivial oscillation $T(t)$ and consider Eq. (1.4) only. We write $u_D = u$, $H(t,\mu=0) = H$ and $U(t,s,\mu=0) = \exp(-i(t-s)H)$. We remark that in the limit $\omega \to 0$, Eq. (1.2) reduces to the free propagator for the DC-Stark problem given by Avron–Herbest [24] and (1.4) to the time dependent Schrödinger equation for DC-Stark Hamiltonian in the moving frame:

$$i\partial u/\partial t = [-(\tfrac{1}{2})\Delta + V(x - \tfrac{1}{2}\mu E t^2)]u.$$

after an additional translation by $\frac{1}{2}\mu\omega^{-2}E$ is made at initial time.

We now explain the problem. To avoid unnecessary complexity we assume here the potential is short range: $|V(x)| \leqq c(1+|x|)^{-1-\varepsilon}$, $\varepsilon > 0$. When $\mu = 0$, the solutions to (1.4) are well understood: The spectrum $\sigma(H)$ of H consists of the absolutely continuous part $\sigma_{ac}(H) = [0,\infty)$ and the point spectrum $\sigma_p(H) = \{-k_j^2\}_j$. The absolutely continuous subspace $\mathscr{H}_{ac}(H)$ for H consists of the scattering states: for any $f \in \mathscr{H}_{ac}(H)$ there corresponds $f_\pm \in \mathscr{H}$ such that as $\pm t \to \infty$, $\|\exp(-itH)f - \exp(-itH_0)f_\pm\| \to 0$; for each $-k_j^2 \in \sigma_p(H)$, the eigenfunction ϕ_j, $H\phi_j = -k_j^2\phi_j$, is well-localized (see Kuroda [14] for example). If $\mu \neq 0$, this picture is still preserved. Considering the Floquet operator $U(s+2\pi/\omega,s,\mu)$, which is unitary, in place of H, we have that $\sigma(U(s+2\pi/\omega,s,\mu)) = \sigma_{ac}(U(s+2\pi/\omega,s,\mu)) \cup \sigma_p(U(s+2\pi/\omega,s,\mu))$; if $f \in \mathscr{H}_{ac}(U(s+2\pi/\omega,s,\mu))$ there corresponds $f_\pm \in \mathscr{H}$ such that $\|U(t,s,\mu)f - \exp(-i(t-s)H_0)f_\pm\| \to 0$ as $t \to \pm\infty$; if $U(s+2\pi/\omega,s,\mu)f = e^{-i(2\pi/\omega)\lambda}f(\lambda \in \mathbb{R})$, $U(t,s,\mu)f = e^{-i(t-s)\lambda}f(t)$ with $f(t)$ periodic in t, and $U(t,s,\mu)f$ stays essentially in a bounded region of the configuration space for all time (Yajima [21], Howland [9] and Kitada–Yajima [14]). We believe, however, that these bound states are in fact absent and that after the field is switched on the eigenvalue $-k_j^2$ of the unperturbed operator will disappear forming a resonance pole λ_j in the unphysical sheet of the complex plane. Correspondingly, although $U(t,0,\mu)\phi_j$ will be eventually free as $t \to \infty$, it behaves like a bound state for a long time. It is also believed from the analogy from classical mechanics, that there is a resonance phenomenon between two states whose difference of energies is exactly equal to $n\hbar\omega$, exibiting the "photon property" of the electro-magnetic field (see Landau–Lifshitz [16], also Sargent–Scully [18]). Our subject here is to provide a sound mathematical justification to these phenomena.

It should be clear from the above argument that the problem is virtually equivalent to the spectral problem for the Floquet operator $U(s+2\pi/\omega,s,\mu)$ associated with Eq. (1.4). According to the Floquet theory ([3]), the spectral property of $U(s+2\pi/\omega,s,\mu)$ can be studied by means of the operator $-i\partial/\partial t + H(t,\mu)$ with periodic boundary condition. Thus we introduce the new Hilbert space $\mathscr{K} = L^2(\mathbb{T}_\omega) \otimes \mathscr{H}$, $\mathbb{T}_\omega = \mathbb{R}/(2\pi/\omega)\mathbb{Z}$ is the circle, and the selfadjoint operator $K(\mu) = -i\partial/\partial t + H(t,\mu)$ there. (The Hilbert space of space-time variables and the operator $-i\partial/\partial t + H(t)$ were first used by Howland in the quantum scattering theory [8].) The unitary group generated by $K(\mu)$ is given as

$$\exp(-i\sigma K(\mu))f(t) = U(t, t-\sigma, \mu)f(t-\sigma), f \in \mathscr{K}, -\infty < \sigma < \infty,$$

and the spectral equivalence of $K(\mu)$ and $U(s+2\pi/\omega, s, \mu)$ is described as follows: If $\mathscr{U}_s$ is the unitary operator on $\mathscr{K}$ defined as

$$(\mathscr{U}_s f)(t) = U(t, s, \mu)f(t) \text{ for } s < t \leqq s + 2\pi/\omega. \tag{1.5}$$

and extended by periodicity elsewhere, then for all s,

$$\exp(-i(2\pi/\omega)K(\mu)) = \mathscr{U}_s(\mathbb{1} \otimes U(s+2\pi/\omega, s, \mu))\mathscr{U}_s^*. \tag{1.6}$$

Furthermore if $K(\mu)\phi(t) = \lambda\phi(t)$, then $\phi(t)$ is $\mathscr{H}$-valued continuous function and $U(t, 0, \mu)\phi(0) = e^{-i\lambda t}\phi(t)$, and conversely if $U(2\pi/\omega, 0, \mu)\phi_0 = e^{-i\lambda(2\pi/\omega)}\phi_0$, then $\phi(t) = e^{i\lambda t}U(t, 0)\phi_0$ is the eigenfunction of $K(\mu)$ with eigenvalue λ (see Lemma 2.9). Here $K(\mu)$ is unitarily equivalent to the original $-i\partial/\partial t - (\frac{1}{2})\Delta + V(x) + \mu Ex\cos\omega t$ (modulo the energy shift by $-\mu^2/4\omega^2$) via the transformation $\tilde{T}(t)$ which is obtained from $T(t)$ by eliminating $-it\mu^2/4\omega^2$ in (1.3), and it may be interpreted as the 'photon-particle' Hamiltonian: $\mathscr{K} = \bigoplus_{n=-\infty}^{\infty} (\langle e^{i\omega nt}\rangle \otimes \mathscr{H})$ and $\langle e^{i\omega nt}\rangle \otimes \mathscr{H}$ is regarded as the (n-photon) + (particle) state space and $-i\partial/\partial t$ the photon energy operator. We shall take this point of view in what follows.

If $\mu = 0$, $K(\mu = 0) = -i\partial/\partial t + H$ has the spectrum $\sigma(K(0)) = \bigcup_{n=-\infty}^{\infty} \{n + \sigma(H)\}$ and all the eigenvalues of $K(0)$ are embedded in the continuum. If $\mu \neq 0$, we expect these eigenvalues will dissolve into the continuum. This dissolution of the bound states and the appearance of the resonances are best described by means of the complex scaling technique ([1], [2]). For $\theta \in \mathbb{C}_a$, we define the operators on $\mathscr{H}$

$$H_0(\theta) = e^{-2\theta}H_0, \tag{1.7}$$

$$H(t, \theta, \mu) = H_0(\theta) + V(e^\theta x + \mu\omega^{-2}E\cos\omega t), H(\theta) = H(t, \theta, \mu = 0), \tag{1.8}$$

and the operators on $\mathscr{K}$

$$K_0(\theta) = -i\partial/\partial t + H_0(\theta), K(\theta, \mu) = -i\partial/\partial t + H(t, \theta, \mu). \tag{1.9}$$

When $\mu = 0$, $\sigma(K(\theta, \mu = 0)) = \bigcup_{n=-\infty}^{\infty} [\{n\omega + \sigma_p(H(\theta))\} \cup \{n\omega + e^{-2\theta}\mathbb{R}^+\}]$, and by the dilation analytic theory for $H(\theta)$, $\sigma_p(H(\theta)) \cap \mathbb{R} = \sigma_p(H)$ is θ-independent (see Prop. 2.1 and Lemma 2.7). Since the perturbation $V(e^\theta x + \mu\omega^{-2}E\cos\omega t)$ is $K(\theta, 0)$-compact (Lemma 2.8), the essential spectrum $\sigma_{\text{ess}}(K(\theta, \mu)) = \bigcup_{n=-\infty}^{\infty} \{n\omega + e^{-2\theta}\mathbb{R}^+\}$ is independent of μ, whereas the eigenvalues $-k_j^2$ of $H(\theta)$ will be shifted down into the lower complex plane $\lambda_j(\mu)$ (Im $\theta > 0$). These complex eigenvalues $\lambda_j(\mu)$ are θ-independent and may be computed by the perturbation series. We shall find that Im $\lambda_j(\mu) = C_j(\omega)\mu^{2n} + O(\mu^{2n+1})$, $C_j(\omega) < 0$, where n is the number of photons it takes to ionize the bound state $\phi_j(x)$, i.e. the smallest integer such that $-k_j^2 + n\omega > 0$. We call $\lambda_j(\omega)$'s the resonance energies and the corresponding eigenfunction $\phi_j(t, x; \theta)$ the resonance state. Suppose now that $-k_j^2$ and $-k_i^2$ are simple eigenvalues of H with eigenfunctions $\phi_j(x)$ and $\phi_i(x)$. If $-k_j^2 + k_i^2 = n\omega, n \in \mathbb{Z}$, then $-k_j^2$ is no longer simple as an eigenvalue of $K(\theta,0)$. It is doubly degenerated with eigenfunctions $\phi_j(x)$ and $e^{in\omega t}\phi_i(x)$. According to the perturbation theory, the eigenvalue $-k_j^2$ will then split into two levels and we shall find that $U(t,s,\mu)\phi_j$ oscillates for a long time between $\phi_j(x)$ and $\phi_i(x)$. This exhibits the resonances between two states $\phi_j(x)$ and $\phi_i(x)$.

If the electric field is direct, i.e. $\omega=0$, the resonance problem for (1.1) has been studied by several authors. Herbst [5], Graffi–Grecchi [4], [5] and Herbst–Simon [7] discussed it by the complex scaling technique and Yajima [22] and Jensen [11] by the weighted space method. On the other hand if the field is alternating, the only existing theory is the time-dependent perturbation theory by Dirac (see Langhoff–Epstein–Karplus [17] for the review and the literature) and the mathematical justification has been in order. We should mention here that Howland announced a similar idea for defining the resonances for the AC-Stark problem (see [10]).

We use the following notation and conventions: $\mathbb{R}$ is the real line, $\mathbb{R}^+=[0,\infty)$. $\mathbb{Z}$ is the set of all integers. $\mathbb{C}$ is the complex plane. $\mathbb{C}_a=\{z\in\mathbb{C}:-a<\operatorname{Im}z<a\}$, $\mathbb{C}_a^\pm=\{z\in\mathbb{C}:0<\pm\operatorname{Im}z<a\}$, $\bar{\mathbb{C}}_a^\pm=\{z\in\mathbb{C}:0\leqq\pm\operatorname{Im}z<a\}$. For $1\leqq p\leqq\infty$, $L^p(\mathbb{R}^3)$ is the Banach space of p-summable functions on $\mathbb{R}^3$, $\mathscr{H}=L^2(\mathbb{R}^3)$. For $0\leqq s<\infty$, $H^s(\mathbb{R}^3)$ is the Sobolev space of order s. For a Banach space X, $\mathscr{B}(X)$ denotes the Banach algebra of all bounded operators on X. $\|\ \|$ stands for the norm of vectors as well as the norm of operators in $\mathscr{H}$ or $\mathscr{K}$. For a closed operator T on X, $\sigma(T)$, $\sigma_{\text{ess}}(T)$ and $\sigma_p(T)$ stand for the spectrum, the essential spectrum and the point spectrum. $\sigma_d(T)=\sigma(T)\backslash\sigma_{\text{ess}}(T)$ and $\rho(T)=\mathbb{C}\backslash\sigma(T)$ is the resolvent set. $D(T)$ is the domain of T. $\mathscr{F}_t$ stands for the Fourier transform with respect to the variable t. It is a unitary transform from $L^2(\mathbb{T}_\omega)$ to $l^2(\mathbb{Z})$. We shall use the terminology of Kato [12], Chapter IX for the semi-groups of operators. The statements or formulas which contain $\pm$ should be understood as two statements or formulas, one for upper signs and the other for lower signs.

2. Preliminaries

We collect here several preliminary results which will be needed in the following sections. We always assume that the potential $V(x)$ satisfies at least Assumption (A_1), although some of the results hold under a weaker Assumption (A_0). We first recall the following well-known theorem of Aguilar–Combes [1].

Proposition 2.1. ([1]). *The family of operators $H(\theta)=-(\frac{1}{2})e^{-2\theta}\Delta+V(e^\theta x)$, $\theta\in\mathbb{C}_a$ with domain $D(H(\theta))=H^2(\mathbb{R}^3)$ is a selfadjoint holomorphic family of type (A) in Kato's sense [12] and satisfies the following properties:*

(1) $\sigma_{\text{ess}}(H(\theta))=e^{-2\theta}\mathbb{R}^+$.

(2) *$\sigma_d(H(\theta))$ is invariant in θ*: $\sigma_d(H(\theta))=\bigcup_{0<\pm\operatorname{Im}\theta'<\pm\operatorname{Im}\theta}\sigma_d(H(\theta'))$ for $\theta\in\mathbb{C}_a^\pm$.

(3) $\sigma_d(H(\theta))\cap\mathbb{R}=\sigma_p(H)$ *and for* $\theta\in\mathbb{C}_a^\pm$, $\sigma_d(H(\theta))\backslash\mathbb{R}\subset\{z\in\mathbb{C}:\mp2\operatorname{Im}\theta<\pm\arg z<0\}$. *$\overline{\sigma_d(H(\theta))}\backslash\sigma_d(H(\theta))$ consists at most of $\{0\}$ in the extended complex plane $\mathbb{C}\cup\{\infty\}$.*

(4) *The eigenfunction $\phi(x)$ of H with eigenvalue $-k^2<0$ is dilation analytic,* i.e. *the $\mathscr{H}$-valued function $\phi_\theta=e^{3\theta/2}\phi(e^\theta x)$ of $\theta\in\mathbb{R}$ can be extended to $\mathbb{C}_a$ as an $\mathscr{H}$-valued analytic function. ϕ_θ is the eigenfunction of $H(\theta)$ with the same eigenvalue.*

(5) *$\lambda \in \sigma_d(H(\theta)) \cap \mathbb{R}$ is a semi-simple eigenvalue of $H(\theta)$.*

For $p>0$, we write as $L^p(\mathbb{R}^3)+L_\varepsilon^\infty(\mathbb{R}^3)$ the set of all functions W such that for any small $\varepsilon>0$ there are two functions W_1 and W_2 which satisfy $W(x)=W_1(x)+W_2(x)$, $W_1 \in L^p$, $W_2 \in L^\infty$ and $\|W_2\|_\infty<\varepsilon$.

Lemma 2.2. *Let $W \in L^p(\mathbb{R}^3)+L_\varepsilon^\infty(\mathbb{R}^3)$ with $p>3$. Then for any $\theta \in \mathbb{C}_a^\pm$, $W(H_0(\theta)-\lambda \mp i)^{-1}$ $(\lambda \in \mathbb{R})$ is a compact operator in $\mathscr{H}$ and*

$$\lim_{\lambda \to \pm\infty} \| W(H_0(\theta)-\lambda \mp i)^{-1} \| = 0. \tag{2.1}$$

Proof. Since $\|(H_0(\theta)-\lambda \mp i)^{-1}\| \leqq 1$ for any $\lambda \in \mathbb{R}$, and $\theta \in \overline{\mathbb{C}_a^\pm}$, a simple approximation argument shows that it suffices to show the lemma for $W \in L^p(\mathbb{R}^3)$ with support which has a finite measure. Then the operator $W(H_0(\theta)-\lambda \mp i)^{-1} = e^{2\theta} W(H_0 - e^{2\theta}(\lambda+i))^{-1}$ has an L^2-integral kernal $e^{2\theta} W(x) \exp((2e^{2\theta}(\lambda \pm i))^{1/2} \times |x-y|)/2\pi|x-y|$ with $\mathrm{Re}(2e^{2\theta}(\lambda \mp i))^{1/2}<0$ and is of Hilbert–Schmidt class. By the resolvent equation, we have

$$\begin{aligned} &\| W(H_0(\theta)-\lambda \mp i)^{-1}\|^2 = \| W(H_0(\theta)-\lambda \mp i)^{-1}(H_0(\bar\theta)-\lambda \pm i)^{-1} W^* \| \\ &= e^{4\mathrm{Re}\,\theta} |2\mathrm{Im}\, e^{2\theta}(\lambda \pm i)|^{-1} \| W\{(H_0-e^{2\theta}(\lambda \pm i))^{-1} - (H_0 - e^{2\bar\theta}(\lambda \mp i))^{-1}\} W^* \| \end{aligned} \tag{2.2}$$

Since $\|A(H_0-z)^{-1}B\| \to 0$ as $|z| \to \infty$ with $\mathrm{Im}\, z \neq 0$ for any $A(x)$ and $B(x) \in L^r(\mathbb{R}^3) \cap L^s(\mathbb{R}^3)$ with $1 \leqq r < 3 < s$ (cf. Ginibre–Moulin [28], Prop. 3.1, for example), the right hand side of (2.2) converges to zero as $\lambda \to \pm\infty$. This proves (2.1). □

Since $V(t,x,\theta,\mu) \equiv V(e^\theta x + \mu\omega^{-2} E \cos\omega t)$ is a $C_0(\mathbb{R}^3)$-valued analytic function of $\theta \in \mathbb{C}_a$, $H(t,\theta,\mu) = H_0(\theta) + V(t,x,\theta,\mu)$ also satisfies Prop. 2.1 for any fixed t and μ. The following two lemmas guarantee the existence of the propagator $U(t,s,\theta,\mu)$ for the evolution equation

$$i\partial u/\partial t = H(t,\theta,\mu)u. \tag{2.3}$$

We write

$$S(\theta) f(x) = e^{3\theta/2} f(e^\theta x), \quad \theta \in \mathbb{R}^1 \tag{2.4}$$

and

$$M = M_\omega = \sup\{\| V(e^\theta x + \rho E)\|_\infty, \theta \in \mathbb{C}_a, |\rho| \leqq \omega^{-2}\}. \tag{2.5}$$

We use Kato's terminology for the semi-groups (Kato [12], Chapter IX).

Lemma 2.3. (1) *For any fixed $t \in \mathbb{R}$, $\theta \in \mathbb{C}_a^\pm$ and $0 \leqq \mu \leqq \Omega$, $\pm iH(t,\theta,\mu)+M$ with domain $H^2(\mathbb{R}^3)$ is maximal accretive and $\pm iH(t,\theta,\mu)$ generates a C_0-semi-group $\exp(\mp i\sigma H(t,\theta,\mu))$, $\sigma \geqq 0$ on $\mathscr{H}$:*

$$\begin{cases} \|(\pm iH(t,\theta,\mu)-z)^{-1}\| \leqq |\mathrm{Re}(z+M)|^{-1}, \mathrm{Re}\, z < -M; \\ \|\exp(\mp i\sigma H(t,\theta,\mu)\| \leqq \exp(M\sigma). \end{cases} \tag{2.6}$$

If $\theta \in \mathbb{C}_a^\pm$, $\exp(\mp i\sigma H(t,\theta,\mu))$ is a holomorphic semi-group of type $\mathscr{H}(2\theta-\delta,\gamma_\delta)$ with any $\delta>0$ and some $\gamma_\delta>0$.

(2) *For any fixed z with $\mathrm{Re}\, z < -M$, $(\pm iH(t,\theta,\mu)-z)^{-1}$ is differentiable in*

$(t,\mu)\in\mathbb{R}^1\times[0,\Omega]$, *continuous in* $(t,\theta,\mu)\in\mathbb{R}^1\times\bar{\mathbb{C}}_a^{\pm}\times[0,\Omega]$ *and is analytic in* $\theta\in\mathbb{C}_a^{\pm}$ *as a* $\mathscr{B}(\mathscr{H})$*-valued function.*

(3) *The* $\mathscr{B}(\mathscr{H})$*-valued function* $\exp(\mp i\sigma H(t,\theta,\mu))$ *is strongly continuous in* $(\sigma,t,\theta,\mu)\in\mathbb{R}_+\times\mathbb{R}\times\bar{\mathbb{C}}_a^{\pm}\times[0,\Omega]$ *and is analytic in* $\theta\in\mathbb{C}_a^{\pm}$. *For* $\theta'\in\mathbb{R}$ *and* $\theta\in\bar{\mathbb{C}}_a^{\pm}$,

$$S(\theta')\exp(\mp i\sigma H(t,\theta,\mu))S(\theta')^{-1}=\exp(\mp i\sigma H(t,\theta+\theta',\mu)). \tag{2.7}$$

Proof. We prove the lemma for upper signs. The other case can be proved similarly. Clearly $iH_0(\theta)$ is maximal accretive for $\theta\in\bar{\mathbb{C}}_a^+$ and if $\theta\notin\mathbb{R}$ it is a generator of a holomorphic semi-group of type $\mathscr{H}(2\theta,0)$. Phillips' theorem for the perturbation of semi-groups ([12], pp. 495–497) then implies the statement (1) for $iH(t,\theta,\mu)$. Prop. 2.1 with $V(t,x,\theta,\mu)$ replacing $V(e^\theta x)$ implies the analyticity of $(iH(t,\theta,\mu)-z)^{-1}$ in $\theta\in\mathbb{C}_a^+$. The continuity in (t,θ,μ) and the differentiability in (t,μ) are obvious from Assumption (A_1). It follows from the property (2) and the strong convergence of the semi-group ([12], Chap. IX. Theorem 2.18), $\exp(-i\sigma H(t,\theta,\mu))$ is strongly continuous in all variables. Equation (2.7) follows from the equation $S(\theta')H(t,\theta,\mu)S(\theta')^{-1}=H(t,\theta+\theta',\mu)$. Finally we prove the analyticity of $\exp(-i\sigma H(t,\theta,\mu))$ in $\theta\in\mathbb{C}_a^+$ for $0<\delta<\operatorname{Im}\theta<a$ with any fixed $\delta>0$. We take a contour $\Gamma=\{\exp(i(\pi-\delta)/2)\lambda-L,\ \lambda\geqq 0\}\cup\{\exp(-i(\pi-\delta)/2)\lambda-L,\ \lambda\geqq 0\}$ with sufficiently large $L\geqq 0$ and write

$$\exp(-i\sigma H(t,\theta,\mu))=\frac{1}{2\pi i}\int_\Gamma e^{-\sigma z}(iH(t,\theta,\mu)-z)^{-1}dz. \tag{2.8}$$

By Lemma 2.2, $(H(t,\theta,\mu)-z)^{-1}=(H_0(\theta)-z)^{-1}(1+V(e^\theta x+\mu\omega^{-2}E\cos\omega t)(H_0(\theta)-z)^{-1})^{-1}$ is uniformly bounded on an ε-neighbourhood Γ_ε of Γ. Hence $d^n/d\theta^n(iH(t,\theta,\mu)-z)^{-1}$ is also uniformly bounded on Γ and the analyticity follows from (2.8). □

Lemma 2.4. *Let* $\theta\in\bar{\mathbb{C}}_a^{\pm}$ *and* $0\leqq\mu\leqq\Omega$. *Then Eq. (2.3) generates a unique propagator* $\{U(t,s,\theta,\mu):\pm t\geqq\pm s\}$ *such that:*

(1) $U(s,s,\theta,\mu)=\mathbb{1}$, $U(t,r,\theta,\mu)\,U(r,s,\theta,\mu)=U(t,s,\theta,\mu)$ *for* $\pm t\geqq\pm r\geqq\pm s$;

(2) $U(t,s,\theta,\mu)H^2(\mathbb{R}^3)\subset H^2(\mathbb{R}^3)$; *if* $f\in H^2(\mathbb{R}^3)$, $U(t,s,\theta,\mu)f$ *is differentiable in* (t,s) *and*

$$i(\partial/\partial t)U(t,s,\theta,\mu)f=H(t,\theta,\mu)U(t,s,\theta,\mu)f; \tag{2.9}$$

$$-i(\partial/\partial s)U(t,s,\theta,\mu)f=U(t,s,\theta,\mu)H(s,\theta,\mu)f; \tag{2.10}$$

$$\|U(t,s,\theta,\mu)\|\leqq\exp(M|t-s|). \tag{2.11}$$

(3) $U(t+2\pi/\omega,\ s+2\pi/\omega,\theta,\mu)=U(t,s,\theta,\mu)$.

(4) *For* $\theta'\in\mathbb{R}^1$,

$$U(t,s,\theta+\theta',\mu)=S(\theta')\,U(t,s,\theta,\mu)\,S(\theta')^{-1}. \tag{2.12}$$

(5) $U(t,s,\theta,\mu)$ *is strongly continuous in* (t,s,θ,μ) *for* $\pm t\geqq\pm s$, $\theta\in\bar{\mathbb{C}}_a^{\pm}$ *and* $0\leqq\mu\leqq\Omega$. *For any fixed* $\pm t\geqq\pm s$, $0\leqq\mu\leqq\Omega$, *it is analytic in* $\theta\in\mathbb{C}_a^{\pm}$.

(6) *For* $\theta\in\mathbb{R}$, $\{U(t,s,\theta,\mu):(t,s)\in\mathbb{R}^2\}$ *is a unitary propagator.*

Proof. The existence and the uniqueness of the propagator $U(t,s,\theta,\mu)$ which satisfies (1) and (2) are direct consequences of Lemma 2.3 and Kato's theorem ([13]). (3) is a consequence of the uniqueness of the propagator and the periodicity of the Hamiltonian $H(t,\theta,\mu)$. We prove (4) and (5) for $\theta\in\bar{\mathbb{C}}_a^+$ and $0\leqq s\leqq t<T<\infty$. The other case can be proved similarly. For integers $n\geqq 1$ we set $H_n(t,\theta,\mu)=H(kT/n,\theta,\mu)$ for $kT/n\leqq t<(k+1)T/n$, $k=0,1,\dots,n-1$; $U_n(t,s,\theta,\mu)=\exp(-i(t-s)H(kT/n,\theta,\mu))$ if $kT/n\leqq s<t\leqq(k+1)T/n$ and $U_n(t,s,\theta,\mu)=\exp(-i(t-lT/n)H(lT/n,\theta,\mu))\ \exp(-i(T/n)H((l-1)T/n,\theta,\mu))\dots\exp(-i(T/n)H((k+1)T/n,\theta,\mu))\exp(-i((k+1)T/n-s)H(kT/n,\theta,\mu))$ if $kT/n\leqq s<(k+1)T/n\leqq lT/n<t<(l+1)T/n$. Then it is easy to see by Lemma 2.3 that for $\theta'\in\mathbb{R}$

$$U_n(t,s,\theta+\theta',\mu)=S(\theta')U_n(t,s,\theta,\mu)S(\theta')^*; \tag{2.13}$$

$U_n(t,s,\theta,\mu)$ is strongly continuous in (t,s,θ,μ), analytic in $\theta\in\mathbb{C}_a^+$ and $\|U_n(t,s,\theta,\mu)\|\leqq\exp((t-s)M)$. Moreover for any $f\in H^2(\mathbb{R}^3)$, there are constants $\tilde{M}$ and $C>0$ such that

$$\begin{aligned}&\|(U_n(t,s,\theta,\mu)-U_m(t,s,\theta,\mu))f\|\\&\leqq C\exp(\tilde{M}(t-s))\int_s^t\|(H_n(r,\theta,\mu)-H_m(r,\theta,\mu))\|_{H^2\to L^2}\|f\|_{H^2}dr.\end{aligned}\tag{2.14}$$

Since the right hand side of (2.14) converges to zero as $n,m\to\infty$ uniformly in (t,s,θ,μ) on every compact subset, we have statement (5). By taking the limit $n\to\infty$ in (2.13), we have Eq. (2.12). (See Tanabe [20], Chapter 4.4 for the details.) Statement (6) is obvious. □

Using the propagator $\{U(t,s,\theta,\mu)\}$ constructed in Lemma 2.4 for $\theta\in\bar{\mathbb{C}}_a^\pm$ and $0\leqq\mu\leqq\Omega$, we define a one parameter family of operators $\{\mathscr{U}(\sigma,\theta,\mu):\pm\sigma\geqq 0\}$ on $\mathscr{K}=L^2(\mathbb{T}_\omega)\otimes\mathscr{H}$ by

$$\mathscr{U}(\sigma,\theta,\mu)f(t)=U(t,t-\sigma,\theta,\mu)f(t-\sigma),\quad f\in\mathscr{K}. \tag{2.15}$$

Lemma 2.5. *For $\theta\in\bar{\mathbb{C}}_a^\pm$, let $\{\mathscr{U}(\pm\sigma,\theta,\mu):\sigma\geqq 0\}$ be defined by (2.15) and $K(\theta,\mu)$ be the maximal operator defined by (1.9). Then*

(1) *$\{\mathscr{U}(\pm\sigma,\theta,\mu):\sigma\geqq 0\}$ is a C_0-semi-group on $\mathscr{K}$ and*

$$\|\mathscr{U}(\pm\sigma,\theta,\mu)\|\leqq\exp(M|\sigma|). \tag{2.16}$$

(2) *$\pm iK(\theta,\mu)+M$ is maximal accretive and*

$$\mathscr{U}(\pm\sigma,\theta,\mu)=\exp(\mp i\sigma K(\theta,\mu)). \tag{2.17}$$

(3) *For $\theta\in\mathbb{R}$, $\{\mathscr{U}(\sigma,\theta,\mu):-\infty<\sigma<\infty\}$ is a unitary group and $K(\theta,\mu)$ is self-adjoint.*

Proof. The first statement is obvious from Lemma 2.4. We prove the second for upper signs. Since $iK_0(\theta)$ is clearly maximal accretive and $V(t,x,\theta,\mu)$ is bounded, the maximal operator $iK(\theta,\mu)=iK_0(\theta)+iV(t,x,\theta,\mu)$ is the closed extension of $iK(\theta,\mu)$ defined on $\mathscr{D}=C^1(\mathbb{T}_\omega,\mathscr{H})\cap C(\mathbb{T}_\omega,H^2(\mathbb{R}^3))$ and $iK(\theta,\mu)+M$ is maximal accretive. Since $\mathscr{D}$ is invariant under $\{\mathscr{U}(\sigma,\theta,\mu)\}$ by Lemma 2.4, $\mathscr{D}$ is a core for its generator; for

$f \in \mathscr{D}$, $i(d/d\sigma)\mathscr{U}(\sigma,\theta,\mu)f|_{\sigma=0} = K(\theta,\mu)f$ by (2.10). Thus we have (2.17). Statement (3) follows from Lemma 2.4, (6) and Stone's theorem. □

For the unperturbed operator $K_0(\theta) = (-i\partial/\partial t) \otimes \mathbb{1} + \mathbb{1} \otimes H_0(\theta)$ we clearly have, for $\operatorname{Im}\theta \neq 0$, $D(K_0(\theta)) = L^2(\mathbb{T}_\omega) \otimes H^2(\mathbb{R}^3) \cap H^1(\mathbb{T}_\omega) \otimes L^2(\mathbb{R}^3)$.

Lemma 2.6. (1) *Let $0 \leqq \mu \leqq \Omega$ be fixed. For $\theta \in \mathbb{C}_a^\pm$, $K(\theta,\mu)$ is a holomorphic family of operators of type (A) with the common domain*

$$D(K(\theta,\mu)) = L^2(\mathbb{T}_\omega) \otimes H^2(\mathbb{R}^3) \cap H^1(\mathbb{T}_\omega) \otimes L^2(\mathbb{R}^3).$$

(2) *For $\theta \in \mathbb{C}_a^\pm$ and $\theta' \in \mathbb{R}$, $K(\theta+\theta',\mu) = (\mathbb{1} \otimes S(\theta'))K(\theta,\mu)(\mathbb{1} \otimes S(\theta')^{-1})$.*

The spectra for unperturbed operators $K_0(\theta)$ and $K(\theta,\mu=0)$ are easy to locate.

Lemma 2.7. *Let $\theta \in \mathbb{C}_a$. Then*

(1) $\sigma(K_0(\theta)) = \sigma_{\text{ess}}(K_0(\theta)) = \bigcup_{n=-\infty}^{\infty} \{n\omega + e^{-2\theta}\mathbb{R}^+\}$;

(2) $\sigma(K(\theta,0)) = \sigma_{\text{ess}}(K(\theta,0)) \cup \sigma_d(K(\theta,0))$ *and*

$$\sigma_{\text{ess}}(K(\theta,0)) = \sigma(K_0(\theta)) = \bigcup_{n=-\infty}^{\infty} \{n\omega + e^{-2\theta}\mathbb{R}^+\};$$

$$\sigma_d(K(\theta,0)) = \bigcup_{n=-\infty}^{\infty} \{n\omega + \sigma_d(H(\theta))\}.$$

Proof. By Fourier transform $K_0(\theta)$ is unitarily equivalent to the multiplication operator by $n\omega + e^{-2\theta}\xi^2$ on the space $l^2(\mathbb{Z}) \otimes L^2(\mathbb{R}^3)$. This implies statement (1). Similarly regarding $l^2(\mathbb{Z}) \otimes L^2(\mathbb{R}^3) = \bigoplus_{n=-\infty}^{\infty} L^2(\mathbb{R}^3)$, we see that $K(\theta,0)$ is unitarily equivalent to $\bigoplus_{n=-\infty}^{\infty} (n\omega + H(\theta))$. Since $(H(\theta)-z)^{-1} = (H_0(\theta)-z)^{-1}(1 + V(e^\theta x) \times (H_0(\theta)-z)^{-1})^{-1}$ for $z \notin \sigma(H(\theta))$, Lemma 2.1 and 2.2 imply the second statement. □

To locate the essential spectrum of $K(\theta,\mu)$, we need the following lemma. We write the multiplication operator on $\mathscr{K}$ by $V(e^\theta x + \mu\omega^{-2}E\cos\omega t)$ as $\mathscr{V}(\theta,\mu)$.

Lemma 2.8. *Let $0 \leqq \mu \leqq \Omega$ and $\theta \in \bar{\mathbb{C}}_a^\pm$. Then the following statements hold:*

(1) *For any $z \notin \sigma(K_0(\theta))$, $\mathscr{V}(\theta,\mu)(K_0(\theta)-z)^{-1}$ is a compact operator on $\mathscr{K}$.*

(2) *For $\pm\operatorname{Im} z > 0$, $\mathscr{V}(\theta,\mu)(K_0(\theta)-z)^{-1}$ is analytic in $\theta \in \mathbb{C}_a^\pm$, norm continuous in $\theta \in \bar{\mathbb{C}}_a^\pm$ and $\|\mathscr{V}(\theta,\mu)(K_0(\theta)-z)^{-1}\| \to 0$ as $\operatorname{Im} z \to \pm\infty$.*

(3) *If $f \in \mathscr{K}$ satisfies $f + \mathscr{V}(\theta,\mu)(K_0(\theta)-z)^{-1}f = 0$ with $z \notin \sigma(K_0(\theta))$, then z is an eigenvalue of $K(\theta,\mu)$ with the eigenfunction $(K_0(\theta)-z)^{-1}f$.*

Proof. We prove the lemma for upper signs. The other case is similar.

(1) Since $\mathscr{V}(\theta,\mu)(K_0(\theta)-z)^{-1}$ is analytic in $z \in \rho(K_0(\theta))$ it suffices to show $\mathscr{V}(\theta,\mu)(K_0(\theta) \pm i)^{-1}$ is compact. We prove—case only. Let us write $W(x,\theta) = \sup\{|V(e^\theta x + \mu\omega^{-2}E\cos\omega t)| : t \in \mathbb{R}\}$ and denote the multiplication operator by $W(x,\theta)$ in $\mathscr{K}$ as $\mathscr{W}(\theta)$. Clearly $W(\cdot,\theta) \in L^p(\mathbb{R}^3) + L_\varepsilon^\infty(\mathbb{R}^3)$ for any $0 < p < \infty$ and

$\mathscr{W}(\theta)(K_0(\theta)-i)^{-1}=\mathscr{F}_t^{-1}\left(\bigoplus_{n=-\infty}^{\infty} W(x,\theta)(H_0(\theta)+n\omega-i)^{-1}\right)\mathscr{F}_t$, where we regarded $l^2(\mathbb{Z})\otimes\mathscr{H}=\bigoplus_{n=-\infty}^{\infty}\mathscr{H}$. Hence by Lemma 2.2, $\mathscr{W}(\theta)(K_0(\theta)-i)^{-1}$ is compact on $\mathscr{K}$ which in turn implies the compactness of $\mathscr{V}(\theta,\mu)(K_0(\theta)-i)^{-1}=\mathscr{V}(\theta,\mu)\mathscr{W}(\theta)^{-1}\cdot\mathscr{W}(\theta)(K_0(\theta)-i)^{-1}$.

(2) Except for the norm continuity of $\mathscr{V}(\theta,\mu)(K_0(\theta)-z)^{-1}$ at the boundary $\mathbb{R}$ of $\bar{\mathbb{C}}_a^{\pm}$, the statement is clear by Assumption (A_0). For proving the norm continuity, it suffices to show it for $(K_0(\theta)-z)^{-1}\mathscr{V}(\theta,\mu), \theta\in\bar{\mathbb{C}}_a^-, \operatorname{Im} z<0$. Let $\theta_0\in\mathbb{R}$ and suppose $\theta\in\bar{\mathbb{C}}_a^-$ approaches θ_0. Then for $\operatorname{Im} z<0$, we can check by an elementary computation that

$$\|(K_0(\theta)-z)^{-1}(K_0(\theta_0)-z)\|=\sup_{n,p}|n\omega+e^{-2\theta_0}p^2/2-z|/|n\omega+e^{-2\theta}p^2/2-z|$$

is uniformly bounded in θ and for $f\in C^\infty(\mathbb{T}_\omega, C_0^\infty(\mathbb{R}^3))$, $(K_0(\theta)-z)^{-1}(K_0(\theta_0)-z)f\to f$ strongly as $\theta\to\theta_0$. Thus $(K_0(\theta)-z)^{-1}(K_0(\theta_0)-z)$ approaches the identity operator strongly. Since $(K_0(\theta_0)-z)^{-1}\mathscr{V}(\theta,\mu)$ is a compact operator and is norm continuous by statement (1) and Assumption (A_0), we see that

$$(K_0(\theta)-z)^{-1}\mathscr{V}(\theta,\mu)=(K_0(\theta)-z)^{-1}(K_0(\theta_0)-z)\cdot(K_0(\theta_0)-z)^{-1}\mathscr{V}(\theta,\mu)$$

approaches $(K_0(\theta_0)-z)^{-1}\mathscr{V}(\theta_0,\mu)$ in norm as $\theta\to\theta_0$.

(3) Statement (3) is obvious. □

As the last topic of this section, we discuss the relation of the eigenvalues of $K(\theta,\mu)$ and $U(s+2\pi/\omega,s,\theta,\mu)$. We assume $\theta\in\bar{\mathbb{C}}_a^+$ here for simplicity.

Lemma 2.9. *Let $\theta\in\bar{\mathbb{C}}_a^+$ and $0\leqq\mu\leqq\Omega$. Suppose that $K(\theta,\mu)f=\lambda f$. Then $f=f(t)$ is an $\mathscr{H}$-valued continuous function and $f(t)=e^{i\lambda(t-s)}U(t,s,\theta,\mu)f(s)$. In particular, $U(s+2\pi/\omega,s,\theta,\mu)f(s)=e^{-i\lambda(2\pi/\omega)}f(s)$. Conversely if $U(s+2\pi/\omega,s,\theta,\mu)\phi=e^{-i(2\pi/\omega)\lambda}\phi$, then $f(t)\equiv e^{i\lambda(t-s)}U(t,s,\theta,\mu)\phi\in D(K(\theta,\mu))$ and $K(\theta,\mu)f=\lambda f$.*

Proof. If $K(\theta,\mu)f=\lambda f$, we have $\exp(-i\sigma K(\theta,\mu))f(t)=U(t,t-\sigma,\theta,\mu)f(t-\sigma)=e^{-i\lambda\sigma}f(t)$ for all $\sigma>0$, or $U(t+\sigma,t,\theta,\mu)f(t)=e^{-i\lambda\sigma}f(t+\sigma)$ for all $\sigma>0$ and a.e.t. Thus by Fubini's theorem and the strong continuity of the propagator, $f(t)$ is continuous and the first statement holds. For proving the converse it suffices to note that for all $\sigma\geqq 0$ and $t\in\mathbb{T}_\omega$,

$$\exp(-i\sigma K(\theta,\mu))f(t)=e^{i\lambda(t-\sigma-s)}U(t,t-\sigma,\theta,\mu)U(t-\sigma,s,\theta,\mu)\phi=e^{-i\lambda\sigma}f(t). \qquad \square$$

3. Theorems

Using the lemmas obtained in the previous section, we first show the following spectral properties of $K(\theta,\mu)$.

Theorem 3.1. *Let Assumption (A_1) be satisfied. Then for $0\leqq\mu\leqq\Omega$, $\{K(\theta,\mu), \theta\in\mathbb{C}_a^{\pm}\}$ is a holomorphic family of operators on $\mathscr{K}$ of type (A) with the common domain $L^2(\mathbb{T}_\omega)\otimes H^2(\mathbb{R}^3)\cap H^1(\mathbb{T}_\omega)\otimes L^2(\mathbb{R}^3)$ and for $\pm\operatorname{Im} z>0$, $(K(\theta,\mu)-z)^{-1}$ is a $\mathscr{B}(\mathscr{K})$-*

valued strongly continuous function of $\theta\in\bar{\mathbb{C}}_a^\pm$. *Furthermore the following statements hold for any* $0\leqq\mu\leqq\Omega$ *and* $\theta\in\mathbb{C}_a^\pm$.

(1) $\sigma_{\mathrm{ess}}(K(\theta,\mu))=\bigcup_{n=-\infty}^{\infty}\{n\omega+e^{-2\theta}\mathbb{R}^+\}$.

(2) $\sigma_d(K(\theta,\mu))$ *is a discrete subset of* $\bar{\mathbb{C}}^\mp\setminus\bigcup_{n=-\infty}^{\infty}\{n\omega+e^{-2\theta}\mathbb{R}^+\}$ *with possible accumulation points in* $\{n\omega:n\in\mathbb{Z}\}$. *Any* $\lambda\in\sigma_d(K(\theta,\mu))$ *is an eigenvalue of* $K(\theta,\mu)$ *with finite algebraic multiplicity.*

(3) $\sigma_d(K(\theta,\mu))$ *is invariant in* $\theta\in\mathbb{C}_a^\pm$ *as long as it is free from* $\sigma_{\mathrm{ess}}(K(\theta,\mu))$

(4) $\sigma_d(K(\theta,\mu))\cap\mathbb{R}=\sigma_p(K(\theta=0,\mu))$ *and* $\lambda\in\sigma_d(K(\theta,\mu))\cap\mathbb{R}$ *is a semi-simple eigenvalue of* $K(\theta,\mu)$ *with finite multiplicity.*

Proof. We prove the theorem for upper signs. The holomorphy of $K(\theta,\mu)$ is proved in Lemma 2.6. By Lemma 2.8, (2) and the resolvent equation, we have that if $\operatorname{Im}z>0$ is sufficiently large then $z\in\rho(K(\theta,\mu))$ and

$$\begin{aligned}(K(\theta,\mu)-z)^{-1}&=(K_0(\theta)-z)^{-1}-(K(\theta,\mu)-z)^{-1}\mathscr{V}(\theta,\mu)(K_0(\theta)-z)^{-1}\\&=(K_0(\theta)-z)^{-1}(1+\mathscr{V}(\theta,\mu)(K_0(\theta)-z)^{-1})^{-1}.\end{aligned}\tag{3.1}$$

By the first equation of (3.1) and Lemma 2.8, (1), $(K(\theta,\mu)-z)^{-1}-(K_0(\theta)-z)^{-1}$ is compact. It follows then by Weyl's theorem and Lemma 2.7, (1), $\sigma_{\mathrm{ess}}(K(\theta,\mu))=\bigcup_{n=-\infty}^{\infty}\{n\omega+e^{-2\theta}\mathbb{R}^+\}$. On the other hand by the analytic Fredholm theory (Steinberg [19]) and Lemma 2.8, (1)–(3), the last expression (3.1) extends as a meromorphic function of z to $\rho(K_0(\theta))$ with a discrete set of poles at the eigenvalues of $K(\theta,\mu)$ and that the residues at the poles are all of finite type. This proves that $\sigma_d(K(\theta,\mu))$ is a discrete set of the complement of $\bigcup_{n=-\infty}^{\infty}\{n\omega+e^{-2\theta}\mathbb{R}^+\}$ and $\lambda\in\sigma_d(K(\theta,\mu))$ is an eigenvalue of $K(\theta,\mu)$ with finite algebraic multiplicity. Since for $\theta'\in\mathbb{R}$, $K(\theta+\theta',\mu)$ is unitarily equivalent to $K(\theta,\mu)$ by Lemma 2.6, (2), and the eigenvalues of $K(\theta+\theta',\mu)$ are analytic functions of $\theta'^{1/k}$ with some integer k by the analytic perturbation theory ([12]), $\sigma_d(K(\theta,\mu))$ is θ-invariant and its possible limit points are a subset of $\{n\omega:n\in\mathbb{Z}\}$. Moreover, since $K(\theta,\mu)$ is selfadjoint for $\theta\in\mathbb{R}$ and $\mathscr{V}(\theta,\mu)(K_0(\theta)-z)^{-1}$ is norm continuous in $\theta\in\bar{\mathbb{C}}_a^+$ for $\operatorname{Im}z>0$ by Lemma 2.8, (2), for any $\theta\in\bar{C}_a^+$, $(K(\theta,\mu)-z)^{-1}$ cannot have a pole for $\operatorname{Im}z>0$. Thus $\sigma_d(K(\theta,\mu))$ is confined in the closed lower half plane and this completes the proof of statements (2) and (3). Since the proof of statement (4) is virtually the same as that of Lemma II, 2 of Aguilar–Combes [1], we omit it here. □

Now we make the information about $\sigma_d(K(\theta,\mu))$ more precise using the perturbation series. Under Assumption (A_1), we have

$$\begin{aligned}V(t,x,\theta,\mu)&=V(e^\theta x)+\mu\omega^{-2}\cos\omega t(\partial V/\partial x_1)(e^\theta x)+W_2(t,x,\theta,\mu)\\&\equiv T_0(\theta)+\mu T_1(\theta)+W_2(\theta,\mu),\end{aligned}\tag{3.2}$$

where $\mu^{-1}\|W_2(\theta,\mu)\|_{\mathscr{B}(\mathscr{K})}\to 0$ as $\mu\to 0$. We write $W_1(\theta,\mu)=\mu T_1(\theta)+W_2(\theta,\mu)$.

Theorem 3.2. *Let Assumption* (A_1) *be satisfied,* $\theta \in \mathbb{C}_a^{\pm}$ *and let* $\lambda \in \mathbb{C}^{\pm}$ *be an eigenvalue of* $H(\theta)$ *with (algebraic) multiplicity* $m_0(\lambda)$.

(1) *Suppose that* $\lambda + n_j\omega, j = 1, \ldots, l$, *are the eigenvalues of* $H(\theta)$ *which differ from* λ *by integral multiples of* ω *and* $m_j(\lambda)$ *are their multiplicities. Then for sufficiently small* $0 \leqq \mu$, *there exist exactly* $\sum_{j=0}^{l} m_j(\lambda) = N(\lambda)$ *eigenvalues* $\lambda_1(\mu), \ldots, \lambda_N(\mu)$ *of* $K(\theta, \mu)$ *(counting the multiplicities) such that* $\lambda_j(\mu) \to \lambda$ *as* $\mu \to 0$.

(2) *Suppose that* $\lambda < 0$, $m_0(\lambda) = 1$ *and* $l = 0$ *in* (1) *and that* $H(\theta)\phi_\theta = \lambda\phi_\theta, \phi_\theta$ *being the eigenfunction which satisfies Prop. 2.1,* (4). *Then* $\lambda_1(\mu) = \lambda + o(\mu)$ *and the corresponding eigenfunction* $\phi_\theta(t, x, \mu)$ *of* $K(\theta, \mu)$ *can be chosen as*

$$\begin{aligned}\phi_\theta(t,x,\mu) = \phi_\theta(x) - 2^{-1}\omega^{-2}\mu\{&e^{-i\omega t} \otimes (H(\theta) - \lambda - \omega)^{-1}(\partial V/\partial x_1)(e^\theta x)\phi_\theta(x) \\ &+ e^{i\omega t} \otimes (H(\theta) - \lambda + \omega)^{-1}(\partial V/\partial x_1)(e^\theta x)\phi_\theta(x)\} + o(\mu),\end{aligned} \tag{3.3}$$

where $o(\mu)$ *stands for an* $\mathscr{H}$*-valued continuous function* $f_\theta(t, x, \mu)$ *such that* $\|f(t, \cdot, \mu)\|_{\mathscr{H}} = o(\mu)$ *as* $\mu \to 0$ *uniformly in* $t \in \mathbb{T}_\omega$.

(3) *Suppose that* $\lambda < 0, m_0(\lambda) = 1, l = 1$ *with* $n_1 = \pm 1$ *and* $m_1(\lambda) = 1$ *in* (1) *and that* $H(\theta)\phi_\theta^{(1)} = \lambda\ \phi_\theta^{(1)}, H(\theta)\phi_\theta^{(2)} = (\lambda \pm \omega)\phi_\theta^{(2)}$, *where* $\phi_\theta^{(j)}$ *satisfies Prop. 2.1,* (4). *Then the eigenvalues* $\lambda_j(\mu)$ *are given as*

$$\lambda_j(\mu) = \lambda - (-1)^j 2^{-1}\omega^{-2}\mu(\phi^{(1)},(\partial V/\partial x_1)(x)\phi^{(2)})_{\mathscr{H}} + o(\mu),(j = 1, 2). \tag{3.4}$$

If $(\phi^{(1)}, (\partial V/\partial x_1)\phi^{(2)}) \neq 0$, *the corresponding eigenfunctions* $\phi_\theta^{(j)}(t, x, \mu)$ *are chosen as*

$$\phi_\theta^{(j)}(t, x, \mu) = 2^{-1}(\phi_\theta^{(1)}(x) + (-1)^{j-1}e^{-i\omega t}\phi_\theta^{(2)}(x)) + O(\mu), \tag{3.5}$$

where $O(\mu)$ *stands for an* $\mathscr{H}$*-valued continuous function* $f_\theta^{(j)}(t, x, \mu)$ *such that* $\|f_\theta^{(j)}(t, \cdot, \mu)\|_{\mathscr{H}} = O(\mu)$ *as* $\mu \to 0$ *uniformly in* $t \in \mathbb{T}_\omega$.

Proof. Let us write as $e_k(t) = e^{ik\omega t}$ and the projection to the space spanned by $e_k(t)$ in $L^2(\mathbb{T}_\omega)$ as Q_k, $k = 0, \pm 1, \ldots$. $R(z, \theta, \mu) = (K(\theta, \mu) - z)^{-1}$.

(1) Since $K(\theta, 0) = \mathscr{F}_t^{-1}\left(\bigoplus_{-\infty}^{\infty}\{n\omega + H(\theta)\}\right)\mathscr{F}_t$, it follows from Lemma 2.1 and 2.2 that λ is an eigenvalue of $K(\theta, \mu)$ with multiplicity N. Hence by (3.2) and the bounded perturbation theory ([12], Chapt. III), we have statement (1).

(2) We set $P_0(\theta)u = (u, \phi_{\bar\theta})\phi_\theta$, $u \in \mathscr{H}$ and $\Phi_\theta(0) = \Phi_\theta(t, x, 0) \equiv (\omega/2\pi)^{1/2}e_0 \otimes \phi_\theta(x)$. We write the reduced resolvent of $K(\theta, 0)$ at $z = \lambda$ as $R_\theta(\lambda)$:

$$R_\theta(\lambda) = \sum_{k \neq 0}(\mathbb{1} \otimes (H(\theta) + k\omega - \lambda)^{-1})(Q_k \otimes \mathbb{1}) + Q_0 \otimes (H(\theta) - \lambda)^{-1}(\mathbb{1} - P_0(\theta)) \tag{3.6}$$

Using the second resolvent equation for $R(z, \theta, \mu)$ twice and then the first resolvent equation for $R(z, \theta, 0)$, we have

$$\begin{aligned}R(z,\theta,\mu) = R(z,\theta,0) &- \mu R(z,\theta,0)T_1(\theta)R(z,\theta,0) \\ &- R(i,\theta,0)[(\mathbb{1} + (z - i)R(z,\theta,0))(W_2(\theta,\mu) \\ &- W_1(\theta,\mu)R(z,\theta,\mu)W_1(\theta,\mu))R(z,\theta,0)].\end{aligned} \tag{3.7}$$

Following the standard argument ([12], Chapt. II), we see that there is a small circle $\Gamma_\delta = \{z : |z - \lambda| = \delta\}$ such that for sufficiently small $|\mu| \leqq \varepsilon$, $K(\theta, \mu)$ has only one eigenvalue $\lambda_1(\mu)$ in $|z-\lambda|<2\delta$; $|\lambda_1(\mu)-\lambda|<\delta/2$ and $\|R(z,\theta,\mu)\|$ for $z\in\Gamma_\delta$ is uniformly bounded. Then

$$R^{(2)}(z,\theta,\mu)=(1+(z-i)R(z,\theta,0))(W_2(\theta,\mu) - W_1(\theta,\mu)R(z,\theta,\mu)W_1(\theta,\mu))\times R(z,\theta,0)$$

satisfies

$$\sup\{\|R^{(2)}(z,\theta,\mu)\|, z\in\Gamma_\delta\}=o(\mu) \text{ as } \mu\to 0 \tag{3.8}$$

and the eigenfunction $\Phi_\theta(\mu)=\Phi_\theta(t,x,\mu)$ of $K(\theta,\mu)$ with eigenvalue $\lambda_1(\mu)$ may be chosen as

$$\begin{aligned}\Phi_\theta(\mu)&=-\frac{1}{2\pi i}\int_{\Gamma_\delta} R(z,\theta,\mu)\Phi_\theta(0)\,dz\\ &=\Phi_\theta(0)-\mu R_\theta(\lambda)T_1(\theta)\Phi_\theta(0)+R(i,\theta,0)(2\pi i)^{-1}\int_{\Gamma_\delta} R^{(2)}(z,\theta,\mu)\Phi_\theta(0)\,dz.\end{aligned} \tag{3.9}$$

Since $R(i,\theta,0)$ is a bounded operator from $\mathscr{K}$ into $H^1(\mathbb{T}_\omega)\otimes L^2(\mathbb{R}^3)$ by Lemma 2.6, (3.8), (3.9) and Sobolev's embedding theorem show that

$$\sup_t \|\Phi_\theta(t,\cdot,\mu)-\Phi_\theta(t,\cdot,0)-\mu(R_\theta(\lambda)T_1(\theta)\Phi_\theta(0))(t,\cdot)\|_{\mathscr{H}}=o(\mu) \tag{3.10}$$

as $\mu\to 0$. Equations (3.10) and (3.6) imply (3.3). For the eigenvalue $\lambda_1(\mu)$ we apply Kato [12], Theorem VIII. 2.6 to obtain

$$\lambda_1(\mu)=\lambda-\mu((Q_0\otimes P_0(\theta))T_1(\theta)\Phi_\theta(0),\ \Phi_{\bar\theta}(0))+o(\mu). \tag{3.11}$$

Since the second summand in (3.11) obviously vanishes $\lambda_1(\mu)=\lambda+o(\mu)$.

(3) We prove the case $n_1=1$ only. Since $m(\lambda)=1$, λ is an isolated semi-simple eigenvalue of $K(\theta,0)$ of multiplicity two with the eigenfunctions $e_0(t)\otimes\phi_\theta^{(1)}(x)$ and $e_{-1}(t)\otimes\phi_\theta^{(2)}(x)$. By statement (1), there are two eigenvalues $\lambda_1(\mu)$ and $\lambda_2(\mu)$ of $K(\theta,\mu)$, which approach λ as $\mu\to 0$. We write the total eigenprojection to these eigenvalues as $P(\theta,\mu)$. Having in mind the standard reduction procedure for degenerate perturbation theory ([12], Chapt. II, §3), we set

$$K^{(1)}(\theta,\mu)=\mu^{-1}(K(\theta,\mu)-\lambda)P(\theta,\mu). \tag{3.12}$$

By the resolvent equation, we may write as

$$\begin{aligned}K^{(1)}(\theta,\mu)=P(\theta,0)T_1(\theta)P(\theta,0)+(2\pi i\mu)^{-1}\int_\Gamma (z-\lambda)R(z,\theta,0)\\ \{W_2(\theta,\mu)-W_1(\theta,\mu)R(z,\theta,0)W_1(\theta,\mu)\}R(z,\theta,\mu)dz,\end{aligned} \tag{3.13}$$

where Γ is a small circle around $z=\lambda$. As in the proof of (2), the second term in the right of (3.13), which we write as $\tilde K^{(2)}(\theta,\mu)$, satisfies

$$\|\tilde K^{(2)}(\theta,\mu)\|=o(1) \text{ as } \mu\to 0. \tag{3.14}$$

The unperturbed operator $K^{(1)}(\theta,0)=P(\theta,0)T_1(\theta)P(\theta,0)$ of the reduced operator $K^{(1)}(\theta,\mu)$ has in $P(\theta,0)\mathscr{K}$ the matrix representation with respect to the basis

$\{e_0(t)\otimes\phi_\theta^{(1)}(x),\ e_{-1}(t)\otimes\phi_\theta^{(2)}(x)\}$:

$$\begin{bmatrix} 0 & 2^{-1}\omega^{-2}(\phi^{(1)},(\partial V/\partial x_1)(x)\phi^{(2)}) \\ 2^{-1}\omega^{-2}(\phi^{(1)},(\partial V/\partial x_1)(x)\phi^{(2)}) & 0 \end{bmatrix}. \tag{3.15}$$

Thus $P(\theta,0)\ T_1(\theta)P(\theta,0)$ has the eigenvalues $\lambda_\pm = \pm(1/2\omega^2)(\phi^{(1)},(\partial V/\partial x_1)\phi^{(2)})$ with the corresponding eigenfunctions $\Phi_\pm(\theta)=(\frac{1}{2})(e_0(t)\otimes\phi_\theta^{(1)}(x)\pm e_{-1}(t)\otimes\phi_\theta^{(2)}(x))$. Therefore the eigenvalues $\lambda_1(\mu)$ and $\lambda_2(\mu)$ of $K(\theta,\mu)$ are given by (3.4) and the corresponding eigenfunctions $\phi_\theta^{(j)}(t,x,\mu)$ are given as

$$\Phi_\theta^{(j)}(t,x,\mu)=\Phi_\pm(\theta)+(2\pi i)^{-1}\int_{\Gamma_j}(K^{(1)}(\theta,0)-z)^{-1}\tilde{K}^{(2)}(\theta,\mu)(K^{(1)}(\theta,\mu)-z)^{-1}\Phi_\pm(\theta)dz, \tag{3.16}$$

where Γ_j is a small circle around $\lambda_\pm$. Since $(K^{(1)}(\theta,0)-z)^{-1}$ is clearly a $\mathscr{B}(\mathscr{K},C(\mathbb{T}_\omega,\mathscr{H}))$-valued analytic function of $z\in\Gamma_j$, (3.14) and (3.15) imply (3.5). □

If we assume a higher smoothness Assumption (A_∞), we may compute the eigenvalues of $K(\theta,\mu)$ up to any order of $\mu>0$ and the problem of proving $\operatorname{Im}\lambda_j(\mu)\neq 0$ can be reduced to explicit evaluation of certain integrals. We write for integers $n>0$,

$$\begin{aligned} V(t,x,\theta,\mu) &= \sum_{k=0}^{n}\mu^k(\omega^{-2}\cos\omega t)^k(\partial^k V/\partial x_1^k)(e^\theta x)/k! + W_{n+1}(t,x,\theta,\mu) \\ &= \sum_{k=0}^{n}(\mu/2\omega^2)^k(e_{-1}(t)+e_{+1}(t))^k\otimes Q_k(x,\theta)+W_{n+1}(t,x,\theta,\mu) \\ &\equiv \sum_{k=0}^{n}\mu^k T_k(\theta)+W_{n+1}(\theta,\mu), \end{aligned} \tag{3.17}$$

where $\| W_{n+1}(\theta,\mu)\| = O(\mu^{n+1})$ as $\mu\to 0$. We assume for simplicity

Assumption (S). All negative eigenvalues of $H = -(\frac{1}{2})\Delta + V(x)$ are simple.

We remark that for generic potentials Assumption (S) is satisfied and that under Assumption (S) except for a countable set of ω, $N(\lambda)=1$ for all $\lambda\in\sigma_p(H)$ since $\sigma_p(H)$ is a discrete set. We fix $\lambda\in\sigma_p(H)$ and write as

$$T_k = T_k(\theta),\ S_k = R_\theta(\lambda)^k \text{ for } k=1,2,\dots;\ S_0 = -P(\theta,0) = -|\Phi_\theta(0)\rangle\langle\Phi_{\bar\theta}(0)|,$$
$$S=S_1;\ R(n\omega,\lambda,\theta)=(H(\theta)-\lambda-\eta\omega)^{-1}, n=1,2,\dots,$$

where we used the notation of the proof of Theorem 3.2, (3). By Kato [12], Chapt. II, the perturbation series for the eigenvalue $\lambda_1(\mu)$ is given by

$$\lambda_1(\mu)=\lambda+C_1\mu+C_2\mu^2+\cdots+C_l\mu^l+O(\mu^{l+1}),\ l=1,2,\dots, \tag{3.18}$$

with

$$C_l=\sum_{p=1}^{l}\frac{(-1)^p}{p}\sum_{\substack{v_1+\cdots+v_p=l\\ k_1+\cdots+k_p=p-1}}\operatorname{tr} T_{v_1}S_{k_1}\dots T_{v_p}S_{k_p}. \tag{3.19}$$

Since one of S_{k_j} in (3.19) must equal to $-P(\theta,0)$ and

$$\begin{aligned} &-\operatorname{tr} T_{v_1}S_{k_1}\dots T_{v_j}P(\theta,0)T_{v_{j+1}}\dots T_{v_p}S_{k_p} \\ &\quad = -(T_{v_{j+1}}S_{k_{j+1}}\dots T_{v_p}S_{k_p}T_{v_1}\dots T_{v_j}\Phi_\theta(0),\Phi_{\bar\theta}(0))_{\mathscr{K}}, \end{aligned} \tag{3.20}$$

we can rewrite the sum (3.19) as

$$\begin{aligned} C_l = &\sum_{p=1}^{l}(-1)^p \sum_{v_1+\dots+v_p=l} (T_{v_1}ST_{v_2}\dots ST_{v_p}\Phi_\theta(0),\Phi_{\bar\theta}(0))_{\mathscr{K}} \\ &+(\text{terms with more than one } S_{k_j}=-P(\theta,0)). \end{aligned} \tag{3.21}$$

By (3.6) we have for $v_2+\dots+v_p=n_p$, $k_1,\dots k_{p-1}\neq 0$

$$\begin{aligned} &S_{k_1}T_{v_2}\dots S_{k_{p-1}}T_{v_p}\Phi_\theta(0) \\ &=(\omega/2\pi)^{1/2}(1/2\omega^2)^{n_p}\sum_{\substack{0\leqq l_j\leqq v_j \\ j=2,\dots,p}}\binom{v_2}{l_2}\dots\binom{v_p}{l_p}\, e_{2(l_2+\dots+l_p)-n_p}(t)\otimes \\ &\quad R(2(l_2+\dots+l_p)\omega-n_p\omega,\lambda,\theta)^{k_1}Q_{v_2}(\theta)R(2(l_3+\dots+l_p)\omega \\ &\quad -(n_p-v_2)\omega,\lambda,\theta)^{k_2}\dots R(2l_p\omega-v_p\omega,\lambda,\theta)^{p-1}Q_{v_p}(\theta)\phi_\theta. \end{aligned} \tag{3.22}$$

Lemma 3.3. *If $v_1+\dots+v_p$ is odd, then*

$$(T_{v_1}S_{k_1}T_{v_2}\dots S_{k_{p-1}}T_{v_p}\Phi_\theta(0),\Phi_{\bar\theta}(0))=0. \tag{3.23}$$

Proof. If none of $k_j=0$, (3.23) immediately follows from (3.22) since $2(l_1+\dots+l_p)-(v_1+\dots+v_p)$ never vanishes. The case when some of $k_j=0$ is reduced to the case where none of $k_j=0$. □

Lemma 3.4. *Let n be the smallest integer such that $\lambda+n\omega>0$ and $v_1+\dots+v_p\leqq 2n$. Suppose that there exist no $1\leqq i\leqq p$ such that $v_1+\dots+v_i=n=v_{i+1}+\dots+v_p$. Then*

$$\operatorname{Im}(T_{v_1}S_{k_1}T_{v_2}\dots S_{k_{p-1}}T_{v_p}\Phi_\theta(0),\Phi_{\bar\theta}(0))=0. \tag{3.24}$$

Proof. Under the condition of the lemma, for some $j, v_1+\dots+v_j<n$ and $v_{j+2}+\dots+v_p<n$. Then

$$\begin{aligned} &(T_{v_1}S_{k_1}T_{v_2}\dots S_{k_{p-1}}T_{v_p}\Phi_\theta(0),\Phi_{\bar\theta}(0)) \\ &\quad =(S_{k_{j+1}}\dots S_{k_{p-1}}T_{v_p}\Phi_\theta(0),T^*_{v_j}S^*_{k_j}\dots S^*_{k_1}T^*_{v_1}\Phi_{\bar\theta}(0)). \end{aligned} \tag{3.25}$$

By (3.22) and Lemma 2.1 this is an analytic function of $\theta\in\mathbb{C}_a$ which is independent of $\operatorname{Re}\theta$ and hence is independent of $\theta\in\mathbb{C}_a$. Setting $\theta=0$ and appealing to the fact that $R(k\omega,\lambda,0)$ is a real operator for $k<n$, we have the lemma. □

Lemma 3.3 implies that $C_j=0$ for odd j's and Lemma 3.4 implies that $\operatorname{Im}C_{2j}=0$ for $j<n, n=n(\lambda,\omega)$ being the smallest integer such that $\lambda+n\omega>0$. Equations (3.22) and (3.24) also imply that the second summand of (3.21) is real and moreover

$$\operatorname{Im}C_{2n}=-\operatorname{Im}(R(n\omega,\lambda,\theta)\phi(n,\omega,\lambda,\theta),\phi(n,\omega,\lambda,\bar\theta))_{\mathscr{H}}, \tag{3.26}$$

$$\phi(n,\omega,\lambda,\theta)=(2\omega^{-2})^n\sum_{p=1}^{n}(-1)^p\sum_{v_1+\dots+v_p=n}Q_{v_1}(\theta)R(n\omega-v_1\omega,\lambda,\theta)Q_{v_2}(\theta)$$
$$\dots Q_{v_{p-1}}(\theta)R(v_p\omega,\lambda,\theta)Q_{v_p}(\theta)\phi_\theta. \tag{3.27}$$

By Lemma 2.1 the inner product in (3.26) is independent of $\theta\in\mathbb{C}_a$ and mimicking B. Simon's computation [27], we have

$$\begin{aligned}\operatorname{Im}C_{2n}&=-\operatorname{Im}\lim_{\varepsilon\downarrow 0}(R(n\omega,\lambda\pm i\varepsilon,\theta)\phi(n,\omega,\lambda,\theta),\phi(n,\omega,\lambda,\bar{\theta}))\\&=\mp\pi(dE(\lambda+n\omega)/d\lambda\phi(n,\omega,\lambda,0),\phi(n,\omega,\lambda,0)),\end{aligned}\tag{3.28}$$

where $\{E(\lambda)\}$ is the spectral resolution of H. Note that, again by Lemma 2.1, regarded as a function of ω, the last expression of

$$\begin{aligned}&2i\pi((dE(\lambda+n\omega)/d\lambda)\phi(n,\omega,\lambda,0),\phi(n,\omega,\lambda,0))\\&=((H-\lambda-n\omega-i0)^{-1}-(H-\lambda-n\omega+i0)^{-1})\phi(n,\omega,\lambda,0),\\&\quad\phi(n,\omega,\lambda,0))\\&=((H(\theta)-\lambda-n\omega)^{-1}\phi(n,\omega,\lambda,\theta),\phi(n,\omega,\lambda,\bar{\theta}))\\&\quad-((H(\bar{\theta})-\lambda-n\omega)^{-1}\phi(n,\omega,\lambda,\bar{\theta}),\phi(n,\omega,\lambda,\theta)),\lambda+n\omega\notin\sigma_p(H),\end{aligned}\tag{3.29}$$

can be analytically extended from each of the real intervals $(-\infty,-\lambda/n)$, $(-\lambda/n,-\lambda/(n-1)),\dots,(-\lambda/2,-\lambda),(-\lambda,\infty)$ to a complex domain: from $(-\infty,-\lambda/n)$ to $\{z:|\arg(\lambda+nz)|>|2\operatorname{Im}\theta|\}$, from $(-\lambda/k,-\lambda/k-1)$ to $\{z:|\arg(\lambda+kz)|<|2\operatorname{Im}\theta|$ and $|\arg(\lambda+(k-1)z)|>|2\operatorname{Im}\theta|\}(k=2,\dots,n)$, $(-\lambda,\infty)$ to $\{z:|\arg(\lambda+jz)|<|2\operatorname{Im}\theta|\}$. Each of these branch can vanish only on a countable set of ω's unless it vanishes identically where it is defined. Remark that the branch from $(-\infty,-\lambda/n)$ vanishes identically except on its poles. Thus we have proved the following theorem.

Theorem 3.5. *Let $V(x)$ satisfy Assumption (A_∞) and (S), $\theta\in\mathbb{C}_a^\pm$, $\lambda<0$ be an eigenvalue of H and for $n=1,2,\dots\phi(n,\omega,\lambda,\theta)$ be the function defined by (3.27). Then the eigenvalue $\lambda_1(\mu)$ of $K(\theta,\mu)$ has a perturbation expansion*

$$\lambda_1(\mu,\omega)=\lambda+C_2(\lambda,\omega)\mu^2+C_4(\lambda,\omega)\mu^4+\cdots+C_{2n}(\lambda,\omega)\mu^{2n}+O(\mu^{2n+1})\tag{3.30}$$

up to any order. Suppose further that each branch of

$$I_n(\lambda,\omega)=\mp\pi((dE(\lambda+n\omega)/d\lambda)\phi(n,\omega,\lambda,0),\phi(n,\omega,\lambda,0))$$

does not vanish identically for $\omega>-\lambda/n,n=1,2,\dots$. Then for all but a countable set of $\omega>0$

$$\begin{cases}\operatorname{Im}C_{2j}\ (\lambda,\omega)=0\ \text{for}\ j=1,\dots,n_0-1,\\ \operatorname{Im}C_{2n_0}\ (\lambda,\omega)=I_{n_0}(\lambda,\omega)\lessgtr 0.\end{cases}\tag{3.31}$$

Here $n_0=n_0(\lambda,\omega)$ is the smallest integer such that $\lambda+n\omega>0$.

Finally we would like to study the implications of Theorem 3.2 and Theorem 3.5 for the original propagator $U(t,s,\mu)$. For this purpose we assume the following

Assumption (SM)$^{\pm}$ There exists a point $\theta_0 \in \mathbb{C}_a^{\pm}$ such that

$$|V(t,x,\theta_0,\mu)| \leqq C(1+|x|)^{-2-\varepsilon}, \varepsilon > 0$$

for all $(t,x) \in \mathbb{R}^4$, $0 \leqq \mu \leqq \Omega$.
We write as $A(\theta,\mu)$ and $B(\theta,\mu)$ the multiplication operators by $|V(t,x,\theta,\mu)|^{1/2}$ and $V(t,x,\theta,\mu)|V(t,x,\theta,\mu)|^{-1/2}$ respectively. By the proof of Lemma 3.3 of Yajima [21] it follows under Assumption (SM) that the operator-valued function $Q(z,\theta_0,\mu) = A(\theta_0,\mu)(K_0(\theta_0)-z)^{-1}B(\theta_0,\mu)$ has the following properties:

1) $Q(z,\theta_0,\mu)$ is a compact operator-valued analytic function of $z \in \rho(K_0(\theta_0))$.

2) $\|Q(z,\theta_0,\mu)\| \to 0$ as $\operatorname{Im} z \to \pm\infty$ (according to $\theta_0 \in \mathbb{C}_a^{\pm}$).

3) $\mathscr{B}(\mathscr{K})$-valued function $Q(\lambda \pm i\varepsilon, \theta_0, \mu)$ has a continuous boundary value $Q(\lambda \pm i0, \theta_0, \mu)$ as $\varepsilon \downarrow 0$.

4) $\exp(-in\omega t)Q(z,\theta_0,\mu)\exp(in\omega t) = Q(z+n\omega,\theta_0,\mu)$.

We further assume

Assumption (R). For $\theta_0 \in \mathbb{C}_a^{\pm}$ of Assumption (SM)$^{\pm}$, $(1+Q(0 \pm i0,\theta_0,\mu))^{-1}$ exists in $\mathscr{B}(\mathscr{K})$ for all small $0 \leqq \mu$.

Theorem 3.6. *Let Assumptions* (A_1), (SM)$^{\pm}$ *and* (R) *be satisfied and* $\lambda < 0$ *be an eigenvalue of* H.

1) *Suppose* λ *is as in Theorem 3.2,* (2) *and* $H\phi = \lambda\phi$. *Then as* $\mu \to 0$, $(U(t,s,\mu)\phi,\phi) = e^{-i\lambda_1(\mu)(t-s)} + O(\mu)$ *uniformly in* $\pm t > \pm s$.

2) *Suppose* λ *is as in Theorem 3.2,* (3) *and* $H\phi^{(1)} = \lambda\phi^{(1)}, H\phi^{(2)} = (\lambda \pm \omega)\phi^{(2)}$. *Then as* $\mu \to 0$

$$(U(t,s,\mu)\phi^{(1)}, \phi^{(1)}) = \tfrac{1}{2}\{e^{-i\lambda^{(1)}(\mu)(t-s)} + e^{-i\lambda^{(2)}(\mu)(t-s)}\} + O(\mu),$$
$$(U(t,s,\mu)\phi^{(1)}, \phi^{(2)}) = \tfrac{1}{2}\{e^{-i\lambda^{(1)}(\mu)(t-s)} - e^{-i\lambda^{(2)}(\mu)(t-s)}\}e^{-i\omega t} + O(\mu),$$

uniformly in $\pm t > \pm s$.

For proving the theorem we admit the following lemma for a moment.

Lemma 3.7. *Let Assumptions* $(SM)^{\pm}$ *and* (R) *be satisfied. Then there exists a constant* $C > 0$ *such that*

$$\|U(t,s,\theta_0,\mu)\| \leqq C \text{ for } \pm s < \pm t. \tag{3.32}$$

Proof of Theorem 3.6. We prove the theorem only for upper signs with $s = 0$. We write $\theta_0 = \theta$.

1) By Lemma 2.9 and Theorem 3.2, (2) we have

$$U(t,0,\theta,\mu)(\phi_\theta(x) + \psi_\theta(x,\mu)) = e^{-i\lambda_1(\mu)t}(\phi_\theta(x) + \psi_\theta(t,x,\mu)),$$

where $\|\psi_\theta(\cdot,\mu)\|_{\mathscr{H}} = O(\mu)$ and $\|\psi_\theta(t,\cdot,\mu)\|_{\mathscr{H}} = O(\mu)$ uniformly in t. Therefore

$$(U(t,0,\mu)\phi,\phi) = (U(t,0,\theta,\mu)\phi_\theta, \phi_{\bar\theta}) = (e^{-i\lambda_1(\mu)t}\phi_\theta, \phi_{\bar\theta}) + O(\mu) = e^{-i\lambda_1(\mu)t} + O(\mu),$$

uniformly in t. Here we used Lemma 2.4 to obtain the first equality; and Lemma 3.7 to obtain the uniformity in t of $O(\mu)$ in the second equation.

2) By Lemma 2.9, Theorem 3.2, (3), and Lemma 3.7

$$U(t,0,\theta,\mu)(\phi_\theta^{(1)}(x)+\phi_\theta^{(2)}(x)) = e^{-i\lambda^{(1)}(\mu)t}(\phi_\theta^{(1)}(x)+e_{-1}\otimes\phi_\theta^{(2)}(x)) + O(\mu),$$
$$U(t,0,\theta,\mu)(\phi_\theta^{(1)}(x)-\phi_\theta^{(2)}(x)) = e^{-i\lambda^{(2)}(\mu)t}(\phi_\theta^{(1)}(x)-e_{-1}\otimes\phi_\theta^{(2)}(x)) + O(\mu), \quad (3.33)$$

where $O(\mu)$ stands for functions $f_j(t,x,\mu)$ such that $\sup_{t\geqq 0}\|f_j(t,\cdot,\mu)\| = O(\mu)$ as $\mu\to 0$. Thus solving (3.33) for $U(t,0,\theta,\mu)\phi_\theta^{(1)}(x)$ and taking the inner products with $\phi_\theta^{(1)}(x)$ and $\phi_\theta^{(2)}(x)$, we have

$$(U(t,0,\theta,\mu)\phi_\theta^{(1)}(x),\phi_\theta^{(1)}(x)) = \tfrac{1}{2}(e^{-i\lambda^{(1)}(\mu)t}+e^{-i\lambda^{(2)}(\mu)t}) + O(\mu).$$

and

$$(U(t,0,\theta,\mu)\phi_\theta^{(1)}(x),\phi_\theta^{(2)}(x)) = \tfrac{1}{2}(e^{-\lambda^{(1)}(\mu)t}-e^{-i\lambda^{(2)}(\mu)t})\,e^{-i\omega t} + O(\mu).$$

Since the inner products in the left hand side are independent of θ by Lemma 2.4, we have statement (2) of the theorem.

Proof of Lemma 2.7. We prove the lemma for upper signs with $s=0$ only. We write $\theta=\theta_0$ and prove that

$$\|\mathscr{U}(\sigma,\theta,\mu)\| \leqq \mathscr{C} \quad \text{for} \quad \sigma\geqq 0, \quad 0\leqq\mu\leqq\Omega. \quad (3.34)$$

Once (3.34) is proved, we obtain (3.32) as follows: For any $f\in\mathscr{H}$, we set $\tilde{f}(t) = U(t,0,\theta,\mu)f$ for $0\leqq t\leqq 2\pi/\omega$ and extend it elsewhere periodically. Then by (3.34) and Lemma 2.5

$$\begin{aligned}
&(2\pi/\omega)\|U(\sigma+2\pi/\omega,0,\theta,\mu)f\|_{\mathscr{H}}^2 \\
&= \int_0^{2\pi/\omega} \|U(\sigma+2\pi/\omega,\sigma+t,\theta,\mu)U(\sigma+t,t,\theta,\mu)\tilde{f}(t)\|_{\mathscr{H}}^2\,dt \\
&\leqq \exp(4\pi M/\omega)\int_0^{2\pi/\omega}\|U(t,t-\sigma,\theta,\mu)\tilde{f}(t-\sigma)\|_{\mathscr{H}}^2\,dt \\
&\leqq C^2\exp(4\pi M/\omega)\|\tilde{f}\|_{\mathscr{K}}^2 \\
&\leqq C^2\exp(8\pi M/\omega)(2\pi/\omega)\|f\|_{\mathscr{H}}^2. \qquad (3.35)
\end{aligned}$$

Equation (3.35) and Lemma 2.5 imply (3.32). For proving (3.34) we use the smooth operator technique of Kato [26]. By the properties (1)–(4) above and the argument in the proof of Theorem 3.1, $(1+Q(z,\theta,\mu))^{-1}$ is a meromorphic function of $z\in\rho(K_0(\theta))$ with possible poles at the eigenvalues of $K(\theta,\mu)$ which are confined in the closed lower half plane; for $z\in\rho(K(\theta,\mu))$,

$$\begin{aligned}
(K(\theta,\mu)-z)^{-1} &= (K_0(\theta)-z)^{-1} \\
&\quad - (K_0(\theta)-z)^{-1}B(\theta,\mu)(1+Q(z,\theta,\mu))^{-1}A(\theta,\mu)(K_0(\theta)-z)^{-1}. \qquad (3.36)
\end{aligned}$$

Let us write the real poles of $(1+Q(z,\theta,\mu))^{-1}$ as $-\sum_{j=1}^{\infty}(z-\lambda_j)^{-1}\tilde{Q}(\theta,\mu,\lambda_j)$, and the corresponding one for $(K(\theta,\mu)-z)^{-1}$ as $-\sum_{j=1}^{\infty}(z-\lambda_j)^{-1}P(\theta,\mu,\lambda_j)$. Then by property (4), $P_n(\theta,\mu)=\sum_{n\omega\leqq\lambda_j<(n+1)\omega} P(\theta,\mu,\lambda_j)$ satisfies

$$P_n(\theta,\mu)=e^{in\omega t}P_0(\theta,\mu)e^{-in\omega t}, \tag{3.37}$$

and by Theorem 3.2

$$\|P_0(\theta,\mu)\|<\infty \tag{3.38}$$

since H has only finite number of eigenvalues. Writing $(1+Q(z,\theta,\mu))_c^{-1}\equiv(1+Q(z,\theta,\mu))^{-1}+\sum_{j=1}^{\infty}(z-\lambda_j)^{-1}\tilde{Q}(\theta,\mu,\lambda_j)$, we have by Assumption (R) that $(1+Q(z,\theta,\mu))_c^{-1}$ has continuous boundary values from the upper half plane and that

$$\|(1+Q(z,\theta,\mu))_c^{-1}\|\leqq C_1<\infty, z\in\bar{\mathbb{C}}^+, 0\leqq\mu\leqq\Omega. \tag{3.39}$$

We also note that as shown in the appendix,

$$\sup_{\eta\geq 0}\int_{-\infty}^{\infty}\|A(\theta,\mu)(K_0(\theta)-\lambda-i\eta)^{-1}f\|_{\mathscr{K}}^2\,d\lambda\leqq C\|f\|_{\mathscr{K}}^2, \tag{3.40}$$

$$\sup_{\eta\geqq 0}\int_{-\infty}^{\infty}\|B(\theta,\mu)^*(K_0(\theta)-\lambda-i\eta)^{-1*}f\|_{\mathscr{K}}^2\,d\lambda\leqq C\|f\|_{\mathscr{K}}^2. \tag{3.41}$$

Now by the semi-group theory we have for $\eta>M$,

$$(\exp(-i\sigma K(\theta,\mu))f,g)_{\mathscr{K}}=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}e^{-i\sigma(\lambda+i\eta)}((K(\theta,\mu)-\lambda-i\eta)^{-1}f,g)d\lambda. \tag{3.42}$$

Plugging (3.36) into the right of (3.42) and using (3.39)–(3.41), we can see that

$$\begin{aligned}(\exp(-i\sigma K(\theta,\mu))f,g)_{\mathscr{K}}&=(\exp(-i\sigma K_0(\theta))f,g)_{\mathscr{K}}\\&+\sum_{n=-\infty}^{\infty}\sum_{n\omega\leqq\lambda_j\leqq(n+1)\omega}e^{-i\sigma\lambda_j}(P(\theta,\mu,\lambda_j)f,g)\\&+\frac{1}{2\pi i}\lim_{\eta\downarrow 0}\int_{-\infty}^{\infty}e^{-i\sigma(\lambda+i\eta)}((1+Q(\theta,\mu,\lambda+i\eta))_c^{-1}A(\theta,\mu)(K_0(\theta)-\lambda-i\eta)^{-1}f,\\&\qquad B(\theta,\mu)^*(K_0(K_0(\theta)-\lambda-i\eta)^{-1*}g)d\lambda.\end{aligned} \tag{3.43}$$

By (3.37)–(3.41) the right hand side is majorized in modules by a constant times $\|f\|\ \|g\|$. This completes the proof of the lemma. □

Appendix

Here we prove (3.40) and (3.41). We use the following elementary lemma.

Lemma A.1. *For any $0\leqq\phi\leqq\pi/2$,*

$$\int_0^{\infty}\left|\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{-ie^{i\phi}xy}f(y)dy\right|^2dx\leqq C\int_0^{\infty}|f(y)|^2dy, \tag{A.1}$$

where the constant $C>0$ is independent of ϕ and $f\in L^2(0,\infty)$.

Proof. By the density argument, we need to prove (A.1) only for $f\in\mathbb{C}_0^{\infty}(0,\infty)$.

We have for $0<\phi\leqq\pi/4$,

$$\int_0^\infty\left|\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-ie^{i\phi}xy}f(y)dy\right|^2 dx$$

$$=e^{i\phi}\int_0^\infty dx\left[\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-ie^{2i\phi}xy}f(y)dy\cdot\overline{\int_0^\infty e^{-ixy}f(y)dy}\right]$$

$$\leqq\left(\int_0^\infty dx\left|\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-ie^{2i\phi}xy}f(y)dy\right|^2\right)^{1/2}\left(\int_0^\infty|f(y)|^2dy\right)^{1/2},$$

by a change of variable $x\to e^{i\phi}x$ and Plancharel's formula. It suffices to show (A.1) for $\pi/4\leqq\phi\leqq\pi/2$. For these ϕ, we have

$$\int_0^\infty\left|\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-ie^{i\phi}xy}f(y)dy\right|^2 dx=\int\frac{f(y)\overline{f(z)}}{2\pi i(e^{i\phi}y-e^{-i\phi}z)}dydz,$$

and

$$\int_0^\infty\frac{y^{-1/2}dy}{|e^{i\phi}y-e^{-i\phi}|}=\int_0^\infty\frac{y^{-1/2}dy}{(y^2-2(\cos 2\phi)y+1)^{1/2}}\leqq\int^\infty\frac{y^{-1/2}dy}{\sqrt{y^2+1}}<\infty.$$

Thus by Hardy–Littlewood–Polya [25] we have (A.1). □

Lemma A.2. *Let H be a selfadjoint operator and A be on H-smooth operator (Kato [26]), i.e.* $\int^\infty\|A\exp(-itH)u\|^2dt\leqq\|A\|_H^2\|u\|^2$. *Then for any $0\leqq\pm\phi\leqq\pi/2$,*

$$\pm\int_0^{\pm\infty}\|A\exp(-ite^{i\phi}H)u\|_{\mathscr{H}}^2dt\leqq C\|u\|_{\mathscr{H}},$$

where the constant $C>0$ is independent of $0\leqq\pm\phi\leqq\pi/2$ and $u\in\mathscr{H}$.

Proof. Let $E(\lambda)$ be the spectral resolution for H. By Kato's criterion for smoothness, $AE(\lambda)u$ is absolutely continuous and $dAE(\lambda)u/d\lambda\equiv\tilde{u}(\lambda)$ is an $\mathscr{H}$-values square integrable function with $\int\|\tilde{u}(\lambda)\|^2d\lambda\leqq C\|A\|_{\mathscr{H}}^2\|u\|^2$. Since

$$A\exp(-ite^{i\phi}H)u=\int_0^\infty e^{-ite^{i\phi}\lambda}\tilde{u}(\lambda)d\lambda,$$

the lemma follows from a vector-valued version of Lemma A.1. □

Now we can prove (3.40) and (3.41). We prove (3.40) only. Let us write as M the multiplication operator by $(1+|x|^2)^{-(1+\varepsilon)/2}$. It is well-known that M is H_0-smooth operator ([28]). By the Fourier inversion formula, Assumption (SM) and Lemma A.2,

$$\int_{-\infty}^\infty\|A(\theta,\mu)(K_0(\theta)-\lambda-i\eta)^{-1}f\|_{\mathscr{H}}^2d\lambda\leqq(2\pi)^{-1}\int_0^\infty\|A(\theta,\mu)e^{-i\sigma K_0(\theta)}f\|^2d\sigma$$

$$\leqq C\int_0^\infty\|Me^{-i\sigma K_0(\theta)}f\|_{\mathscr{H}}^2d\sigma$$

$$\leqq C\int_0^\infty d\sigma \int_0^{2\pi/\omega} \| Me^{-i\sigma e^{-2\theta} H_0} f(t-\sigma)\|_{\mathscr{H}}^2 \, dt$$

$$= C\int_0^\infty d\sigma \int_0^{2\pi/\omega} \| Me^{-i\sigma e^{-2\theta} H_0} f(t)\|_{\mathscr{H}}^2 \, dt$$

$$\leqq C\|f\|_{\mathscr{K}}.$$

Here we used the periodicity of f to obtain the equality in the fourth step.

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