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# Estimates on the Vorticity of Solutions to the Navier-Stokes Equations*

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**Abstract.** We estimate the vorticity of the flow of an incompressible, viscous, three dimensional fluid near the boundary of its container. We obtain a bound that is valid outside a small subset of space-time with special properties.

## 1. Introduction

It is the basis of the Prandtl boundary layer theory that vorticity is introduced into solutions of the Navier-Stokes equations through a boundary layer. Therefore, it is important to obtain estimates of the size of the vorticity close to the boundary. The theorem below yields the following type of estimate: We fix a small positive number $\tau$ and examine points $(x, t)$ in space-time, where $x$ lies at a distance $\tau$ from the boundary. We also assume that the time elapsed since the beginning of the flow is at least $\tau^2$. Then the size of the vorticity at $(x, t)$ is at most $O(\tau^{-2})$ unless $(x, t)$ happens to lie in a certain set. This set is the union of cylinders of size $\tau$. The number of different cylinders is at most $O(\tau^{-5/3})$. Since the cylinders are subsets of space-time, their union is a small set. However, the important point is not the measure of this set. The interesting thing is the clustering of this set into lumps of size $\tau$. Outside of these lumps we have uniform estimates on the vorticity.

The proof of the main theorem in [1] involved the construction of a similar set. There the set was the union of finitely many cylinders $A_i$ where $\sum_i$ (diameter of $A_i)^{5/3}$ was bounded by a constant that depended only on the initial kinetic energy. In addition, the maximum of the diameters of the $A_i$ could be made arbitrarily small. The theorem below is an improvement on this.

One can go further and state that the vorticity is continuous at the points where we can estimate its size. This is a consequence of the local boundedness

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of the velocity vector field at these points (this is brought out in the proof) and of the argument that led to partial continuity of the vorticity in [1].

In this paper the vorticity is the curl in the distribution sense of the velocity vector field. It is not difficult to show that it is actually a classical curl at the points at which it is estimated. In other words, the spatial partial derivatives of the velocity vector field exist in the classical sense at each such point.

**Theorem.** *Let $U$ be an open bounded subset of $R^3$ such that the boundary of $U$ has Lebesgue measure zero. Suppose that $w^0: R^3 \to R^3$ is an $L^2$ function such that $w^0(x)=0$ for $x \notin U$ and $\operatorname{div}(w^0)=0$. Then there exists a weak solution $u: U \times (0, \infty) \to R^3$ to the Navier-Stokes equations of incompressible fluid flow in $U$ with adherence at the boundary of $U$, initial condition $w^0$, viscosity$=1$, and the following property: If $0<\tau<1$ and*

$$T=\{(x, t) \in R^3 \times R: t \geqq \tau^2 \text{ and distance } (x, R^3 \sim U) \geqq \tau\},$$

*then there exist $(x_1, t_1), (x_2, t_2), \ldots, (x_M, t_M)$ where $(x_j, t_j) \in T$, $M \leqq N\tau^{-5/3}$ where $N$ is a constant that depends only on $\|w^0\|_2$, and the inequality $|(\operatorname{curl}(u))(x, t)| \leqq C\tau^{-2}$ holds whenever*

$$(x, t) \in T \sim \bigcup_{j=1}^{M} \{(y, s): |y-x_j| \leqq \tau,\ t_j-\tau^2 \leqq s \leqq t_j\}.$$

We use the notation introduced in Sect. 1 of [1]. The constant $C$ in the theorem does not depend on any of the parameters. Saying that $u$ is a weak solution means that $u$ satisfies the properties listed in Theorem 1.2 of [1] when the domain of $u$ is extended to all of $R^3 \times (0, \infty)$, using the definition $u(x, t)=0$ for $x \notin U$. The condition $\operatorname{div}(w^0)=0$ means that $\int_{R^3} w^0(x) \cdot \nabla f(x)\, dx=0$ for any $C^\infty$ function $f: R^3 \to R$ with compact support.

In later sections we will use the definition

$$(f * g)(x, t)=\int_{R^3} f(y, t)\, g(x-y)\, dy$$

when $A \subset R$, $f$ is a function defined on $R^3 \times A$, and $g$ is a function defined on $R^3$.

## 2. Preliminaries

The following definition is consistent with Definition 2.1 of [1].

*Definition 2.1.* If $A \subset R^3 \times R$ and $f$ is a function defined on $A$, then we abbreviate $I(f, A)=\int_A f$ and $M(f, A)=$the supremum of $\{|f(x, t)|: (x, t) \in A\}$. If $x \in R^3$, $t \in R$, $r>0$, $s>0$, $h>0$, and $k \in \{1, 2, 3, \ldots\}$, then

$$B(x, r)=\{y \in R^3: |y-x| \leqq r\},$$
$$K(x, t, r, s)=\{(y, w) \in R^3 \times R: |y-x| \leqq r \text{ and } t-s \leqq w \leqq t\},$$

$$T(x,t,r,s,h)=\{(y,w)\in R^3\times R: r-h\leqq|y-x|\leqq r+h \text{ and } t-s\leqq w\leqq t\},$$
$$D(t)=\{(y,w)\in R^3\times R: w\leqq t\},$$
$$G(x,t,r,k)=K(x,t,r(1-2^{-k}),r^2(1-2^{-2k})).$$

**Lemma 2.2.** *There exists an absolute constant $C_1$ with the following property: Suppose $u: R^3\times R^+\to R^3$ is a continuous function such that $\left\{\int_{R^3}|u(x,t)|^2\,dx: t>0\right\}$ is a bounded set of real numbers and $Du$ is an $L^2$ function. Suppose also that $\{a,c\}\subset R^3$, $\{b,d\}\subset R$, $\{m,p,n,q\}$ is a set of integers, $b>2^{-2m}$, $|a-c|<2^{-m}$, $b-2^{-2m}<d\leqq b$, $2^{-(n+1)}<2^{-m}-|a-c|\leqq 2^{-n}$, $2^{-2(p+1)}<d-(b-2^{-2m})\leqq 2^{-2p}$, $q=\max\{n,p\}$, and $K=K(c,d,2^{-m-2},2^{-2(p+2)})$. Then*

$$p\geqq m,\qquad n\geqq m \tag{2.1}$$

*and*

$$\begin{aligned}\int_K &|u(x,t)|^2(|x-c|+(d-t)^{1/2})^{-4}\,dx\,dt\\ &\leqq\sum_{k=m+1}^{q+1}C_1 2^{-k}M(|u|^2,D(d)\cap G(a,b,2^{-m},k-m+1))\\ &\quad+\sum_{k=m+1}^{q+1}C_1 2^{2k}I(|Du|^2,T(a,b,2^{-m},2^{-2m},2^{-k+2})).\end{aligned}$$

*Proof.* The hypotheses of Sect. 2 of [1] are all satisfied. Hence Lemmas 2.2, 2.3, 2.4 of [1] are valid in this case. The second of these lemmas yields (2.1). Now we will use a portion of the proof of Lemma 2.5 of [1]. Using the definition of $E(k)$ given in (2.30), (2.31) of [1] and (2.1), we obtain

$$K\sim\{(c,d)\}=\bigcup_{k=m+2}^{\infty}E(k).$$

From (2.33) of [1] we get

$$(|x-c|+(d-t)^{1/2})^{-4}\leqq 2^{4(k+1)}\qquad \text{if } (x,t)\in E(k).$$

Combining (2.9) of [1], Definition 2.1, and the proof of Lemma 2.5 of [1], we find the following: If $|a-c|\leqq 2^{-(m+1)}$ then

$$\sum_{k=m+2}^{\infty}2^{4k}I(|u|^2,E(k))\leqq C2^{-p}M(|u|^2,D(d)\cap G(a,b,2^{-m},p+2-m));$$

if $2^{-(m+1)}<|a-c|$ and $n\leqq p$ then

$$\begin{aligned}\sum_{k=m+1}^{\infty}&2^{4k}I(|u|^2,E(k))\\ &\leqq C2^{-q}M(|u|^2,D(d)\cap G(a,b,2^{-m},q+2-m))\\ &\quad+\sum_{k=m+1}^{q+1}C2^{2k}I(|Du|^2,T(a,b,2^{-m},2^{-2m},2^{-k+2}));\end{aligned}$$

if $2^{-(m+1)}<|a-c|$ and $n>p$ then

$$\sum_{k=m+1}^{\infty} 2^{4k} I(|u|^2, E(k))$$
$$\leqq \sum_{k=m+1}^{q+1} C 2^{-k} M(|u|^2, D(d) \cap G(a, b, 2^{-m}, k+1-m))$$
$$+\sum_{k=m+1}^{q+1} C 2^{2k} I(|Du|^2, T(a, b, 2^{-m}, 2^{-2m}, 2^{-k+2})).$$

The conclusion follows from all of the above, $q=\max\{n, p\}$, and (2.1).

**Lemma 2.3.** *Suppose $u: R^3 \times R^+ \to R^3$ is a continuous function of the type considered in Lemma 2.2. Suppose also that $\{a, c\} \subset R^3$, $\{b, d\} \subset R$, $\tau>0$, $b>\tau^2$, $|a-c|<\tau$, $b-\tau^2<d<b$, n and p are the integers determined uniquely by*

$$2^{-(n+1)}<\tau^{-1}(\tau-|a-c|) \leqq 2^{-n}, \qquad 2^{-2(p+1)}<\tau^{-2}(d-(b-\tau^2)) \leqq 2^{-2p}, \tag{2.2}$$

*and $q=\max\{n, p\}$, $K=K(c, d, \tau/4, \tau^2 2^{-2(p+2)})$. Then*

$$p \geqq 0, \qquad n \geqq 0 \tag{2.3}$$

*and*

$$\int_K |u(x, t)|^2 (|x-c|+(d-t)^{1/2})^{-4} \, dx \, dt$$
$$\leqq \sum_{k=1}^{q+1} C_1 2^{-k} \tau M(|u|^2, D(d) \cap G(a, b, \tau, k+1))$$
$$+\sum_{k=1}^{q+1} C_1 2^{2k} \tau^{-2} I(|Du|^2, T(a, b, \tau, \tau^2, \tau 2^{-k+2})).$$

*Proof.* Set $a'=\tau^{-1} a$, $c'=\tau^{-1} c$, $b'=\tau^{-2} b$, $d'=\tau^{-2} d$, and $u'(x, t)=u(\tau x, \tau^2 t)$. The conclusion follows when $u, a, b, c, d$ in Lemma 2.2 are replaced by $u', a', b', c', d'$ and the $m$ in that lemma is set equal to zero.

## 3. An Integral Inequality for Approximate Solutions

This section is a continuation of Sect. 5 of [1]. In particular, we have the choices of $\Phi, \Omega, \Psi$ made at the start of that section and we also have the function $u: R^3 \times [0, \infty) \to R^3$ of Definition 5.14 of [1]. Lemma 5.6 and Definition 5.10 of [1] imply that $u$ is a function of the type considered in Lemma 2.2. Lemma 5.17 of [1] shows that $u$ is locally in $L^3$. The number $\varepsilon$ is introduced immediately after Definition 5.1 of [1]. We may assume that $\|\Delta(\Phi)-\Delta(\Phi * \Omega * \Omega)\|_2 \leqq \varepsilon$ holds instead of (5.1) of [1].

**Lemma 3.1.** *There exists an absolute constant $C_2$ satisfying the following: If $\tau>0$ then there exists a continuous function $v: R^3 \times R^+ \to R^3$ such that $\{\int |v(x, t)|^2 \, dx : t>0\}$ is a bounded set of real numbers, $Dv$ is an $L^2$ function,*

$$|v(x, t)| \leqq C_2 \tau^{-3/2} \Big( \int_{B(a, 5\tau/4)} |u(y, t)|^2 \, dy \Big)^{1/2} \quad \textit{if } x \in B(a, \tau),$$

$$I(|Dv|^2, T(a, b, \tau, \tau^2, r)) \leqq C_2 \tau^{-4/3} r (I(|u|^3, K(a, b, 4\tau, \tau^2)))^{2/3}$$

*if* $0<r\leqq 2\tau$ *and the following is true: If* $a\in R^3$, $b\in R$, $\varepsilon<\tau/64$, $B(a,5\tau/4)\subset U$, $b>\tau^2$, $c\in R^3$, $d\in R$, $|a-c|<\tau$, $b-\tau^2<d<b$, *p and n are integers defined by* (2.2), $q=\max\{p,n\}$, *and* $A=K(a,b,5\tau/4,\tau^2)$ *then*

$$\begin{aligned}
&|(u-v)(c,d)|\\
&\leqq C_2\left(\int_{d-\tau^2 2^{-2(p+2)}}^{d}\int_{B(c,\tau/4)}|(u-v)(x,t)|^2(|x-c|+(d-t)^{1/2})^{-4}\,dx\,dt\right)\\
&\quad+C_2\tau^{-2/3}M(|u-v|,D(d)\cap G(a,b,\tau,q+2))(\int_A|u(x,t)|^3\,dx\,dt)^{1/3}\\
&\quad+C_2\tau^{-7/3}2^q(\int_A|u(x,t)|^3\,dx\,dt)^{2/3}\\
&\quad+C_2\tau^{-5/3}2^{5p/3}\left(\int_{d-\tau^2 2^{-2(p+2)}}^{d}\int_{B(a,5\tau/4)}|u(x,t)|^3\,dx\,dt\right)^{1/3}\\
&\quad+C_2\tau^2\|w^0\|_2\,\varepsilon.
\end{aligned}$$

*Proof.* Let $\lambda_1:R^3\to[0,1]$ be a $C^\infty$ function such that $\lambda_1(x)=0$ for $|x|\geqq\tau/16$, $\lambda_1(x)=1$ for $|x|\leqq\tau/32$, and $\|D^i\lambda_1\|_\infty\leqq C\tau^{-i}$ for $i=1,2,3$. Define $J:R^3\sim\{0\}\to R$ and $Q_t:R^3\to R$ for $t>0$ by

$$J(x)=-(4\pi|x|)^{-1},\qquad Q_t(x)=(2\sqrt{\pi})^{-3}t^{-3/2}\exp(-|x|^2/(4t)). \tag{3.1}$$

This is consistent with (5.6), (5.7) of [1]. The function $L:R^3\to R$ is given by

$$L(x)=-[\Delta\lambda_1(-x)J(x)-2D_j\lambda_1(-x)D_jJ(x)]\quad\text{for } x\neq 0, \tag{3.2}$$

and $L(x)=0$ for $|x|<\tau/32$. We define

$$v=u*L \tag{3.3}$$

and observe

$$\begin{aligned}
|v(x,t)|&\leqq C\tau^{-3/2}(\int_{B(a,5\tau/4)}|u(y,t)|^2\,dy)^{1/2},\quad\text{if } x\in B(a,9\tau/8),\\
|Dv(x,t)|&\leqq C\tau^{-5/2}(\int_{B(a,5\tau/4)}|u(y,t)|^2\,dy)^{1/2},\quad\text{if } x\in B(a,9\tau/8).
\end{aligned} \tag{3.4}$$

We abbreviate

$$K_1=K(c,d,\tau/8,\tau^2 2^{-2(p+2)}),\qquad K_2=K(c,d,\tau 2^{-(q+2)},\tau^2 2^{-2(q+2)}), \tag{3.5}$$

$$h=\tau 2^{-(p+2)},\qquad H=\tau 2^{-(q+2)},\qquad B=B(a,5\tau/4). \tag{3.6}$$

From $|c-a|<\tau$, $b>\tau^2$, (2.2), and $d<b$ we conclude

$$B(c,\tau/8)\subset B(a,9\tau/8),\qquad 0<b-\tau^2<d-h^2<d<b. \tag{3.7}$$

If $x\in B(c,\tau 2^{-(q+2)})$ then (2.2) and $q\geqq n$ imply

$$\begin{aligned}
|x-a|&\leqq|x-c|+|c-a|\leqq\tau 2^{-(q+2)}+\tau-\tau 2^{-(n+1)}\\
&\leqq\tau 2^{-(q+2)}+\tau-\tau 2^{-(q+1)}=\tau(1-2^{-(q+2)}).
\end{aligned}$$

If $d-\tau^2 2^{-2(q+2)} \leqq t \leqq d$ then $q \geqq p$, (2.2), and $d \leqq b$ imply

$$b-\tau^2(1-2^{-2(q+2)}) \leqq b-\tau^2(1-2^{-2(p+2)}) \leqq d-\tau^2 2^{-2(p+2)}$$
$$\leqq d-\tau^2 2^{-2(q+2)} \leqq t \leqq d \leqq b.$$

All this implies

$$K(c, d, \tau 2^{-(q+2)}, \tau^2 2^{-2(q+2)}) \subset D(d) \cap G(a, b, \tau, q+2). \tag{3.8}$$

In particular, we obtain

$$B(c, \tau 2^{-(q+2)}) \subset B(a, \tau) \subset B(a, 9\tau/8). \tag{3.9}$$

We select a $C^\infty$ function $\lambda_2: R \to [0, 1]$ such that $\lambda_2(t)=0$ for $t \leqq d-h^2$, $\lambda_2(t)=1$ for $t \geqq d-h^2/2$, and $\|(d/dt)\lambda_2\|_\infty \leqq C\tau^{-2}2^{2p}$. Then we set $\lambda(x, t)=\lambda_1(x)\lambda_2(t)$.

Now we fix $i \in \{1, 2, 3\}$, recall (3.1), and define

$$f: R^3 \times(-\infty, d) \to R, \quad F: R^3 \times(-\infty, d) \to R^3, \quad G: R^3 \times(-\infty, d) \to R^3$$

by

$$\begin{aligned} &f(x, s)=\lambda(x-c, s)(Q_{d-s} * J)(x-c), \\ &F_i(x, s)=f(x, s), \quad F_j(x, s)=0 \quad \text{if } j \in \{1, 2, 3\} \text{ and } j \neq i, \\ &\quad G=\operatorname{curl}(\operatorname{curl}(F)). \end{aligned} \tag{3.10}$$

Using Definitions 5.1, 5.2 of [1] and $\varepsilon<\tau/64$ we obtain

$$(G * \Psi * \Psi)(x, t)=0 \quad \text{if } x \notin B(c, \tau/8). \tag{3.11}$$

The function $G$ satisfies

$$|G(x, t)| \leqq C(|x-c|+(d-t)^{1/2})^{-3},$$
$$|DG(x, t)| \leqq C(|x-c|+(d-t)^{1/2})^{-4}.$$

Using the proof of Lemma 5.7 of [1] we conclude

$$\begin{aligned} |(G * \Psi * \Psi)(x, t)| &\leqq C(|x-c|+(d-t)^{1/2})^{-3}, \\ |(DG * \Psi * \Psi)(x, t)| &\leqq C(|x-c|+(d-t)^{1/2})^{-4}. \end{aligned} \tag{3.12}$$

We also have

$$\int_{d-h^2}^{d} \Big( \int_{B(a, 5\tau/4)} |u(x, t)|^2 \, dx\Big)^{3/2} dt$$
$$\leqq C\tau^{3/2} \left( \int_{d-h^2}^{d} \int_{B(a, 5\tau/4)} |u(x, t)|^3 \, dx \, dt \right). \tag{3.13}$$

If $t<d$, then (3.12) yields

$$\int_{B(a, 9\tau/8)} |(G * \Psi * \Psi)(x,t)| \, dx$$
$$\leqq C\tau^{3/5}\Big( \int_{B(a, 9\tau/8)} |(G * \Psi * \Psi)(x, t)|^{5/4} \, dx\Big)^{4/5}$$
$$\leqq C\tau^{3/5}\Big(\int_{R^3} (|x-c|+(d-t)^{1/2})^{-15/4} \, dx\Big)^{4/5} \leqq C\tau^{3/5}(d-t)^{-3/10}. \tag{3.14}$$

Using (3.11), (3.7), (3.4), (3.6), (3.14), (3.13), and (3.7) we obtain

$$\begin{aligned}
&\int_{d-h^2}^{d} \int_{R^3} |v(x,t)|\,|Dv(x,t)|\,|(G*\Psi*\Psi)(x,t)|\,dx\,dt \\
&\quad = \int_{d-h^2}^{d} \int_{B(a,9\tau/8)} |v(x,t)|\,|Dv(x,t)|\,|(G*\Psi*\Psi)(x,t)|\,dx\,dt \\
&\quad \leqq C\left(\int_{d-h^2}^{d} \tau^{-4}\Big(\int_B |u(x,t)|^2\,dx\Big)\Big(\int_{B(a,9\tau/8)} |(G*\Psi*\Psi)(x,t)|\,dx\Big)dt\right) \\
&\quad \leqq C\tau^{-17/5}\left(\int_{d-h^2}^{d} \Big(\int_B |u(x,t)|^2\,dx\Big)(d-t)^{-3/10}\,dt\right) \\
&\quad \leqq C\tau^{-17/5}\left(\int_{d-h^2}^{d} \Big(\int_B |u(x,t)|^2\,dx\Big)^{3/2}dt\right)^{2/3}\left(\int_{d-h^2}^{d} (d-t)^{-9/10}\,dt\right)^{1/3} \\
&\quad \leqq C\tau^{-12/5}h^{1/15}\left(\int_{d-h^2}^{d} \int_{B(a,5\tau/4)} |u(x,t)|^3\,dx\,dt\right)^{2/3} \\
&\quad \leqq C\tau^{-7/3}2^{-p/15}\left(\int_{b-\tau^2}^{b} \int_{B(a,5\tau/4)} |u(x,t)|^3\,dx\,dt\right)^{2/3}. \qquad (3.15)
\end{aligned}$$

From (3.5), (3.6), (3.12), (3.9), (3.4), $q \geqq p$, (3.13), and (3.7) we conclude

$$\begin{aligned}
&\int_{K_2} |v(x,t)|\,|(DG*\Psi*\Psi)(x,t)|\,dx\,dt \\
&\quad \leqq \int_{d-H^2}^{d} \int_{B(c,H)} C\,|v(x,t)|(|x-c|+(d-t)^{1/2})^{-4}\,dx\,dt \\
&\quad \leqq \int_{d-H^2}^{d} C\tau^{-3/2}\Big(\int_B |u(x,t)|^2\,dx\Big)^{1/2}\Big(\int_{B(c,H)} (|x-c|+(d-t)^{1/2})^{-4}\,dx\Big)dt \\
&\quad \leqq \int_{d-H^2}^{d} C\tau^{-3/2}\Big(\int_B |u(x,t)|^2\,dx\Big)^{1/2}(d-t)^{-1/2}\,dt \\
&\quad \leqq C\tau^{-3/2}\left(\int_{d-H^2}^{d} \Big(\int_B |u(x,t)|^2\,dx\Big)^{3/2}dt\right)^{1/3}\left(\int_{d-H^2}^{d} (d-t)^{-3/4}\,dt\right)^{2/3} \\
&\quad \leqq C\tau^{-3/2}\tau^{1/2}\left(\int_{d-h^2}^{d} \int_B |u(x,t)|^3\,dx\,dt\right)^{1/3} H^{1/3} \\
&\quad \leqq C\tau^{-2/3}2^{-q/3}\left(\int_{b-\tau^2}^{b} \int_{B(a,5\tau/4)} |u(x,t)|^3\,dx\,dt\right)^{1/3}. \qquad (3.16)
\end{aligned}$$

Using (3.11), (3.7), (3.4), (3.12), (3.6), (3.5), $q \geqq p$, and (3.7) we find

$$\begin{aligned}
&\int_{K_1 \sim K_2} |v(x,t)|^2\,|(DG*\Psi*\Psi)(x,t)|\,dx\,dt \\
&\quad \leqq \int_{K_1 \sim K_2} C\tau^{-3}\Big(\int_B |u(y,t)|^2\,dy\Big)(|x-c|+(d-t)^{1/2})^{-4}\,dx\,dt \\
&\quad \leqq \int_{d-h^2}^{d-H^2} C\tau^{-3}\Big(\int_B |u(y,t)|^2\,dy\Big)\Big(\int_{R^3} (|x-c|+(d-t)^{1/2})^{-4}\,dx\Big)dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_{d-H^2}^{d} C\tau^{-3}\Big(\int_B |u(y,t)|^2\,dy\Big)\Big(\int_{R^3\sim B(c,H)} (|x-c|+(d-t)^{1/2})^{-4}\,dx\Big)\,dt \\
&\leqq \int_{d-h^2}^{d-H^2} C\tau^{-3}\Big(\int_B |u(y,t)|^2\,dy\Big)(d-t)^{-1/2}\,dt \\
&\quad+\int_{d-H^2}^{d} C\tau^{-3}\Big(\int_B |u(y,t)|^2\,dy\Big)H^{-1}\,dt \\
&\leqq \int_{d-h^2}^{d} C\tau^{-3}\Big(\int_B |u(y,t)|^2\,dy\Big)H^{-1}\,dt \\
&\leqq C\tau^{-4}2^q\left(\int_{b-\tau^2}^{b}\int_{B(a,5\tau/4)} |u(y,t)|^2\,dy\,dt\right) \\
&\leqq C\tau^{-7/3}2^q\left(\int_{b-\tau^2}^{b}\int_{B(a,5\tau/4)} |u(x,t)|^3\,dx\,dt\right)^{2/3}. \qquad (3.17)
\end{aligned}
$$

From (3.5), (3.6), (3.11), (3.8), (3.16), (3.12), and (3.17) we conclude

$$
\begin{aligned}
&\int_{d-h^2}^{d}\int_{R^3} |(u-v)(x,t)|\,|v(x,t)|\,|(DG*\Psi*\Psi)(x,t)|\,dx\,dt \\
&\quad\leqq \int_{K_2} |(u-v)(x,t)|\,|v(x,t)|\,|(DG*\Psi*\Psi)(x,t)|\,dx\,dt \\
&\qquad+\Big(\int_{K_1\sim K_2} |(u-v)(x,t)|^2\,|(DG*\Psi*\Psi)(x,t)|\,dx\,dt\Big) \\
&\qquad+\Big(\int_{K_1\sim K_2} |v(x,t)|^2\,|(DG*\Psi*\Psi)(x,t)|\,dx\,dt\Big) \\
&\quad\leqq M(|u-v|, D(d)\cap G(a,b,\tau,q+2))\,C\tau^{-2/3}2^{-q/3}\left(\int_{b-\tau^2}^{b}\int_B |u(x,t)|^3\,dx\,dt\right)^{1/3} \\
&\qquad+C\left(\int_{d-h^2}^{d}\int_{B(c,\tau/4)} |(u-v)(x,t)|^2(|x-c|+(d-t)^{1/2})^{-4}\,dx\,dt\right) \\
&\qquad+C\tau^{-7/3}2^q\left(\int_{b-\tau^2}^{b}\int_{B(a,5\tau/4)} |u(x,t)|^3\,dx\,dt\right)^{2/3}. \qquad (3.18)
\end{aligned}
$$

Finally, (3.11) and (3.12) yield

$$
\begin{aligned}
&\int_{d-h^2}^{d}\int_{R^3} |(u-v)(x,t)|^2\,|(DG*\Psi*\Psi)(x,t)|\,dx\,dt \\
&\qquad\leqq C\left(\int_{d-h^2}^{d}\int_{B(c,\tau/4)} |(u-v)(x,t)|^2(|x-c|+(d-t)^{1/2})^{-4}\,dx\,dt\right). \qquad (3.19)
\end{aligned}
$$

From Definitions 5.1, 5.2, 5.10, 5.14 and Lemma 5.6 of [1] we conclude

$$u(x,t)=(w'(t))(x)=(w(t)*\Psi)(x),\ \operatorname{div}(w(t))=0, \qquad \Psi(x)=\Psi(-x). \qquad (3.20)$$

If $d-h^2/2<s<d$ then (3.10), (3.20), the definition of $\lambda$ and Definition 5.5 of [1] yield

$$
\begin{aligned}
\int_{R^3} [u_i(x,s)][\Delta f(x,s)]\,dx &= \int_{R^3} [u_k(x,s)][\Delta F_k(x,s)]\,dx \\
&= -\int_{R^3} [u_k(x,s)][G_k(x,s)]\,dx + \int_{R^3} [u_k(x,s)][D_k(\operatorname{div}(F))(x,s)]\,dx \\
&= -\int_{R^3} [u_k(x,s)][G_k(x,s)]\,dx = -\int_{R^3} [(w_k(s))(x)][(G_k * \Psi)(x,s)]\,dx \\
&= -\int_{d-h^2}^{s} \int_{R^3} [((D_t w_k)(t))(x)][(G_k * \Psi)(x,t)]\,dx\,dt \\
&\quad - \int_{d-h^2}^{s} \int_{R^3} [(w_k(t))(x)][(D_t G_k * \Psi)(x,t)]\,dx\,dt. \qquad (3.21)
\end{aligned}
$$

Using Lemma 5.6 and Definition 5.2 of [1], support$(G * \Psi) \subset U$, $\operatorname{div}(G * \Psi) = 0$, and (3.20) we obtain

$$
\begin{aligned}
&-\int_{R^3} [(D_t w_k(t))(x)][(G_k * \Psi)(x,t)]\,dx \\
&\quad = -\int_{R^3} [(P(S(w(t))))_k(x)][(G_k * \Psi)(x,t)]\,dx \\
&\quad = -\int_{R^3} [(S(w(t)))_k(x)][(G_k * \Psi)(x,t)]\,dx \\
&\quad = \int_{R^3} [(((w_j'(t))(D_j(w_k'(t)))) * \Psi)(x)][(G_k * \Psi)(x,t)]\,dx \\
&\qquad - \int_{R^3} [\Delta(w_k(t) * \Omega * \Omega)(x)][(G_k * \Psi)(x,t)]\,dx. \qquad (3.22)
\end{aligned}
$$

Properties (3.20) and (3.10) give us

$$
\begin{aligned}
&-\int_{R^3} [(w_k(t))(x)][(D_t G_k * \Psi)(x,t)]\,dx \\
&-\int_{R^3} [\Delta(w_k(t) * \Omega * \Omega)(x)][(G_k * \Psi)(x,t)]\,dx \\
&\quad = -\int_{R^3} [(w_k(t))(x)][(D_t G_k * \Psi)(x,t)]\,dx \\
&\qquad + \int_{R^3} [(w_k(t))(x)][(D_t D_k(\operatorname{div}(F)) * \Psi)(x,t)]\,dx \\
&\qquad - \int_{R^3} [\Delta(w_k(t) * \Omega * \Omega)(x)][(G_k * \Psi)(x,t)]\,dx \\
&\qquad + \int_{R^3} [\Delta(w_k(t) * \Omega * \Omega)(x)][(D_k(\operatorname{div}(F)) * \Psi)(x,t)]\,dx \\
&\quad = \int_{R^3} [(w_k(t))(x)][(D_t(\Delta(F_k)) * \Psi)(x,t)]\,dx \\
&\qquad + \int_{R^3} [\Delta(w_k(t) * \Omega * \Omega)(x)][(\Delta(F_k) * \Psi)(x,t)]\,dx \\
&\quad = \int_{R^3} [(w_i'(t))(x)][((D_t + \Delta)(\Delta f))(x,t)]\,dx \\
&\qquad + \int_{R^3} [(w_i(t) * (\Delta(\Psi * \Omega * \Omega) - \Delta(\Psi)))(x)][(\Delta f)(x,t)]\,dx. \qquad (3.23)
\end{aligned}
$$

Observe that (3.10) and $\Delta(Q_{d-s} * J) = Q_{d-s}$ imply

$$((D_t + \Delta)(\Delta f))(x,t) = 0 \text{ if } |x-c| \leqq \tau/32 \quad \text{and} \quad d - h^2/2 \leqq t < d.$$

Using this and the fact $h < \tau$ (recall (3.6), (2.3)) we find

$$\|(D_t + \Delta)(\Delta f)\|_{3/2} \leqq C h^{-5/3} = C \tau^{-5/3} 2^{5p/3}. \tag{3.24}$$

The identity $\Delta(Q_{d-s} * J) = Q_{d-s}$ also yields

$$\int_{R^3} |\Delta f(x,t)|\, dx \leqq C. \tag{3.25}$$

From the assumptions made at the start of this section we conclude

$$\begin{aligned} \|\Delta(\Psi * \Omega * \Omega) - \Delta \Psi\|_2 &= \|(\Delta(\Phi * \Omega * \Omega) - \Delta \Phi) * \Omega\|_2 \\ &\leqq \|\Delta(\Phi * \Omega * \Omega) - \Delta \Phi\|_2 \, \|\Omega\|_1 \leqq \varepsilon. \end{aligned} \tag{3.26}$$

Using (3.2), (3.3), the continuity of $u$, $\Delta(Q_t * J) = Q_t$, (3.21)-(3.23), (3.20), (3.7), (3.24), (3.26), (3.25), Lemma 5.6 of [1], and $h < \tau$ we find

$$\begin{aligned} &|u_i(c,d) - v_i(c,d)| \\ &= |u_i(c,d) + \int_{R^3} [u_i(x,d)][\Delta \lambda_1(x-c) J(x-c) + 2 D_j \lambda_1(x-c) D_j J(x-c)]\, dx| \\ &= \lim_{s \to d^-} |\int_{R^3} [u_i(x,s)][\Delta f(x,s)]\, dx| \\ &\leqq \left| \int_{d-h^2}^{d} \int_{R^3} [(((w_j'(t))(D_j(w_k'(t)))) * \Psi)(x)][(G_k * \Psi)(x,t)]\, dx\, dt \right| \\ &\quad + \int_{d-h^2}^{d} \int_{R^3} |(w_i'(t))(x)|\, |((D_t + \Delta)(\Delta f))(x,t)|\, dx\, dt \\ &\quad + \int_{d-h^2}^{d} \int_{R^3} |(w_i(t) * (\Delta(\Psi * \Omega * \Omega) - \Delta \Psi))(x)|\, |(\Delta f)(x,t)|\, dx\, dt \\ &\leqq \left| \int_{d-h^2}^{d} \int_{R^3} [(w_j'(t))(x)][D_j(w_k'(t))(x)][(G_k * \Psi * \Psi)(x,t)]\, dx\, dt \right| \\ &\quad + C \left( \int_{d-h^2}^{d} \int_{B(a, 5\tau/4)} |u(x,t)|^3\, dx\, dt \right)^{1/3} \tau^{-5/3} 2^{5p/3} \\ &\quad + \int_{d-h^2}^{d} \|w(t)\|_2 \, \|\Delta(\Psi * \Omega * \Omega) - \Delta \Psi\|_2 \left( \int_{R^3} |\Delta f(x,t)|\, dx \right) dt \\ &\leqq \left| \int_{d-h^2}^{d} \int_{R^3} [u_j(x,t)][D_j u_k(x,t)][(G_k * \Psi * \Psi)(x,t)]\, dx\, dt \right| \\ &\quad + C \left( \int_{d-h^2}^{d} \int_{B(a, 5\tau/4)} |u(x,t)|^3\, dx\, dt \right)^{1/3} \tau^{-5/3} 2^{5p/3} + C \tau^2 \|w^0\|_2\, \varepsilon. \end{aligned} \tag{3.27}$$

From (3.20) and (3.3) we conclude $\operatorname{div}(u) = 0$, $\operatorname{div}(v) = 0$. Hence (3.19), (3.18), and (3.15) yield

$$\left|\int_{d-h^2}^{d}\int_{R^3}[u_j(x,t)][D_j u_k(x,t)][(G_k*\Psi*\Psi)(x,t)]\,dx\,dt\right|$$

$$\leqq \int_{d-h^2}^{d}\left|\int_{R^3}[(u_j-v_j)(x,t)][D_j(u_k-v_k)(x,t)][(G_k*\Psi*\Psi)(x,t)]\,dx\right|dt$$

$$+\int_{d-h^2}^{d}\left|\int_{R^3}[v_j(x,t)][D_j(u_k-v_k)(x,t)][(G_k*\Psi*\Psi)(x,t)]\,dx\right|dt$$

$$+\int_{d-h^2}^{d}\left|\int_{R^3}[(u_j-v_j)(x,t)][D_j v_k(x,t)][(G_k*\Psi*\Psi)(x,t)]\,dx\right|dt$$

$$+\int_{d-h^2}^{d}\left|\int_{R^3}[v_j(x,t)][D_j v_k(x,t)][(G_k*\Psi*\Psi)(x,t)]\,dx\right|dt$$

$$\leqq \int_{d-h^2}^{d}\left|\int_{R^3}[(u_j-v_j)(x,t)][(u_k-v_k)(x,t)][(D_j G_k*\Psi*\Psi)(x,t)]\,dx\right|dt$$

$$+\int_{d-h^2}^{d}\left|\int_{R^3}[v_j(x,t)][(u_k-v_k)(x,t)][(D_j G_k*\Psi*\Psi)(x,t)]\,dx\right|dt$$

$$+\int_{d-h^2}^{d}\left|\int_{R^3}[(u_j-v_j)(x,t)][v_k(x,t)][(D_j G_k*\Psi*\Psi)(x,t)]\,dx\right|dt$$

$$+\int_{d-h^2}^{d}\left|\int_{R^3}[v_j(x,t)][D_j v_k(x,t)][(G_k*\Psi*\Psi)(x,t)]\,dx\right|dt$$

$$\leqq C\left(\int_{d-h^2}^{d}\int_{B(c,\tau/4)}|(u-v)(x,t)|^2(|x-c|+(d-t)^{1/2})^{-4}\,dx\,dt\right)$$

$$+M(|u-v|,D(d)\cap G(a,b,\tau,q+2))\,C\tau^{-2/3}2^{-q/3}\left(\int_{b-\tau^2}^{b}\int_{B}|u(x,t)|^3\,dx\,dt\right)^{1/3}$$

$$+C\tau^{-7/3}2^{q}\left(\int_{b-\tau^2}^{b}\int_{B(a,5\tau/4)}|u(x,t)|^3\,dx\,dt\right)^{2/3}$$

$$+C\tau^{-7/3}2^{-p/15}\left(\int_{b-\tau^2}^{b}\int_{B(a,5\tau/4)}|u(x,t)|^3\,dx\,dt\right)^{2/3}.$$

The last inequality of the lemma follows from the above, (3.27), and $q\geqq p\geqq 0$ (see (2.3)). The estimate on $v$ follows from (3.4). The argument that gave us (3.4) yields the following for $0<r\leqq 2\tau$:

$$I(|Dv|^2,T(a,b,\tau,\tau^2,r))$$

$$\leqq \int_{T(a,b,\tau,\tau^2,r)}\left(C\tau^{-5/2}\left(\int_{B(a,4\tau)}|u(y,t)|^2\,dy\right)^{1/2}\right)^2 dx\,dt$$

$$\leqq \int_{b-\tau^2}^{b} Cr\tau^{-3}\left(\int_{B(a,4\tau)}|u(y,t)|^2\,dy\right)dt$$

$$\leqq Cr\tau^{-4/3}\left(\int_{b-\tau^2}^{b}\int_{B(a,4\tau)}|u(y,t)|^3\,dy\,dt\right)^{2/3}$$

This gives us the estimate on $Dv$.

## 4. An Estimate for Approximate Solutions

The assumptions of Sect. 3 are still in force.

**Lemma 4.1.** *There exist absolute constants $C_3$, $C_4$ with the following property: Suppose $\tau>0$, $v$ is the function of Lemma 3.1, $a\in R^3$, $b\in R$, $\varepsilon<\tau/64$, $B(a,5\tau/4)\subset U$, and $b>\tau^2$. Suppose also that*

$$I(|Du|^2, T(a,b,\tau,\tau^2,r))\leqq C_3 r \quad \text{if } 0<r\leqq 2\tau, \tag{4.1}$$

$$I(|u|^3, K(a,b-\tau^2+s,4\tau,s))\leqq C_3 s \quad \text{if } 0<s\leqq \tau^2, \tag{4.2}$$

$$\|w^0\|_2\leqq C_3\tau^{-3}\varepsilon^{-1}. \tag{4.3}$$

*Then*

$$|(u-v)(x,t)|\leqq 2C_4\tau^{-1} \quad \text{if } (x,t)\in K(a,b,\tau/2,3\tau^2/4). \tag{4.4}$$

*Proof.* We choose $C_4>0$ so that $16C_2C_1C_4^2\leqq(1/4)C_4$. Then we choose $C_3>0$ so that

$$32C_1C_2C_3+32C_1C_2^2C_3^{2/3}+4C_2C_3^{1/3}C_4+C_2C_3^{2/3}+C_2C_3^{1/3}+C_2C_3\leqq(1/4)C_4.$$

We will use the method in the proof of Lemma 3.1 of [1]. A slight modification of the construction in that lemma gives us a continuous function

$$f\colon \text{interior}(K(a,b,\tau,\tau^2))\to R^+$$

such that (recall Definition 2.1)

$$C_4 2^i\tau^{-1}\geqq f(x,t)\geqq C_4 2^{i-1}\tau^{-1} \quad \text{if } (x,t)\in G(a,b,\tau,i)\sim G(a,b,\tau,i-1) \tag{4.5}$$

for $i=1,2,3,\ldots$. Here we define $G(a,b,\tau,0)$ to be the empty set. In particular, we get

$$f(x,t)\leqq C_4 2^i\tau^{-1} \quad \text{if } (x,t)\in G(a,b,\tau,i). \tag{4.6}$$

We intend to show

$$|(u-v)(x,t)|\leqq f(x,t) \quad \text{for all } (x,t)\in\text{interior}(K(a,b,\tau,\tau^2)). \tag{4.7}$$

Assume that (4.7) is false. Then the nature of $f$ and the continuity of $f$ and $u-v$ imply the existence of $(c,d)\in\text{interior}(K(a,b,\tau,\tau^2))$ such that

$$|(u-v)(c,d)|=f(c,d), \tag{4.8}$$

$$|(u-v)(x,t)|\leqq f(x,t) \quad \text{if } (x,t)\in D(d)\cap\text{interior}(K(a,b,\tau,\tau^2)). \tag{4.9}$$

Using this $(c,d)$ we define the integers $p$, $n$, $q$ by (2.2) and $q=\max\{p,n\}$. From (2.2) and (4.2) we obtain

$$\begin{aligned} &I(|u|^3, K(a,d,5\tau/4,\tau^2 2^{-2(p+2)}))\\ &\quad\leqq I(|u|^3, K(a,d,5\tau/4,d-(b-\tau^2))\leqq C_3(d-(b-\tau^2))\leqq C_3\tau^2 2^{-2p}. \end{aligned} \tag{4.10}$$

We can use Lemma 3.1, (4.1), and (4.2) to conclude the following for $k \geqq 1$:

$$\begin{aligned} &I(|D(u-v)|^2, T(a,b,\tau,\tau^2,\tau 2^{-k+2})) \\ &\leqq 2I(|Du|^2, T(a,b,\tau,\tau^2,\tau 2^{-k+2})) + 2I(|Dv|^2, T(a,b,\tau,\tau^2,\tau 2^{-k+2})) \\ &\leqq 2C_3\tau 2^{-k+2} + 2C_2\tau^{-4/3}\tau 2^{-k+2}(I(|u|^3, K(a,b,4\tau,\tau^2)))^{2/3} \\ &\leqq 2C_3\tau 2^{-k+2} + 2C_2 C_3^{2/3}\tau 2^{-k+2}. \end{aligned} \tag{4.11}$$

Since the definitions of $p$, $n$, $q$ imply $(c,d) \notin \text{interior}(G(a,b,\tau,q))$, we find that (4.5), (4.8) yield

$$|(u-v)(c,d)| = f(c,d) \geqq C_4 2^q \tau^{-1}. \tag{4.12}$$

Combining (4.12), Lemma 3.1, Lemma 2.3 with $u-v$ in place of $u$, (4.2), (4.10), (4.9), (4.6), (4.3), (4.11), (2.3), $q \geqq p$, and the definitions of $C_3$, $C_4$ we find

$$\begin{aligned} &C_4 2^q \tau^{-1} \leqq |(u-v)(c,d)| \\ &\leqq C_2 \left( \int_{d-\tau^2 2^{-2(p+2)}}^{d} \int_{B(c,\tau/4)} |(u-v)(x,t)|^2 (|x-c| + (d-t)^{1/2})^{-4} \, dx \, dt \right) \\ &\quad + C_2\tau^{-2/3} M(|u-v|, D(d) \cap G(a,b,\tau,q+2))(C_3\tau^2)^{1/3} \\ &\quad + C_2\tau^{-7/3} 2^q (C_3\tau^2)^{2/3} + C_2\tau^{-5/3} 2^{5p/3}(C_3\tau^2 2^{-2p})^{1/3} + C_2\tau^2 \|w^0\|_2 \varepsilon \\ &\leqq \sum_{k=1}^{q+1} C_2 C_1 2^{-k}\tau M(|u-v|^2, D(d) \cap G(a,b,\tau,k+1)) \\ &\quad + \sum_{k=1}^{q+1} C_2 C_1 2^{2k}\tau^{-2} I(|D(u-v)|^2, T(a,b,\tau,\tau^2,\tau 2^{-k+2})) \\ &\quad + C_2\tau^{-2/3}(C_4 2^{q+2}\tau^{-1})(C_3\tau^2)^{1/3} + C_2 C_3^{2/3}\tau^{-1} 2^q + C_2 C_3^{1/3}\tau^{-1} 2^p + C_2 C_3 \tau^{-1} \\ &\leqq \sum_{k=1}^{q+1} C_2 C_1 2^{-k}\tau (C_4 2^{k+1}\tau^{-1})^2 + \sum_{k=1}^{q+1} C_2 C_1 2^{2k}\tau^{-2}(2C_3\tau 2^{-k+2} + 2C_2 C_3^{2/3}\tau 2^{-k+2}) \\ &\quad + 4C_2 C_4 C_3^{1/3}\tau^{-1} 2^q + C_2 C_3^{2/3}\tau^{-1} 2^q + C_2 C_3^{1/3}\tau^{-1} 2^q + C_2 C_3\tau^{-1} 2^q \\ &\leqq 16 C_2 C_1 C_4^2 \tau^{-1} 2^q + 32 C_2 C_1 C_3 \tau^{-1} 2^q + 32 C_2^2 C_1 C_3^{2/3}\tau^{-1} 2^q + 4C_2 C_4 C_3^{1/3}\tau^{-1} 2^q \\ &\qquad + C_2 C_3^{2/3}\tau^{-1} 2^q + C_2 C_3^{1/3}\tau^{-1} 2^q + C_2 C_3 \tau^{-1} 2^q \\ &\quad \leqq (1/4) C_4 \tau^{-1} 2^q + (1/4) C_4 \tau^{-1} 2^q = (1/2) C_4 \tau^{-1} 2^q. \end{aligned}$$

This is a contradiction, hence we cannot assume that (4.7) is false. The conclusion follows from (4.7), (4.6), and the definition of $G(a,b,\tau,i)$.

**Lemma 4.2.** *There exist absolute constants $C_5$, $C_6$ with the following property: Suppose $\sigma > 0$, $a \in R^3$, $b \in R$, $\varepsilon < \sigma/128$, $B(a, 5\sigma/4) \subset U$, and $b > \sigma^2$. Suppose also that*

$$I(|Du|^2, K(a,b,3\sigma,\sigma^2)) \leqq C_5 \sigma, \tag{4.13}$$

$$I(|u|^3, K(a,b,4\sigma,\sigma^2)) \leqq C_5 \sigma^2, \tag{4.14}$$

$$\|w^0\|_2 \leqq C_5 \sigma^{-3} \varepsilon^{-1}. \tag{4.15}$$

*Then*

$$\int_{b-3\sigma^2/16}^{b} (\max\{|u(x,t)|: x\in B(a,\sigma/4)\})^3\,dt \leqq C_6\sigma^{-1}. \tag{4.16}$$

*Proof.* Using the method in Sect. 4 of [1] (which is an application of the Hardy-Littlewood weak-type inequality for $L^1$), we can find $C_5 \leqq C_3$ such that (4.13), (4.14) imply that (4.1), (4.2) hold for some $\tau$ such that $\sigma/2<\tau<\sigma$. Using Lemma 4.1 we obtain (4.4) and hence

$$|(u-v)(x,t)| \leqq 4C_4\sigma^{-1} \quad \text{if } (x,t)\in K(a,b,\sigma/4, 3\sigma^2/16), \tag{4.17}$$

where $v$ is the function corresponding to $\tau$ in Lemma 3.1. From (4.17), Lemma 3.1, $\sigma/2<\tau<\sigma$, Hölder's inequality and (4.14) we get

$$\begin{aligned}
&\int_{b-3\sigma^2/16}^{b} (\max\{|u(x,t)|: x\in B(a,\sigma/4)\})^3\,dt\\
&\leqq \int_{b-3\sigma^2/16}^{b} \Big(4C_4\sigma^{-1} + C_2(\sigma/2)^{-3/2}\Big(\int_{B(a,5\sigma/4)} |u(x,t)|^2\,dx\Big)^{1/2}\Big)^3\,dt\\
&\leqq C\sigma^{-1} + C\sigma^{-3}\left(\int_{b-3\sigma^2/16}^{b}\int_{B(a,5\sigma/4)} |u(x,t)|^3\,dx\,dt\right) \leqq C\sigma^{-1}.
\end{aligned}$$

## 5. An Integral Inequality for the Vorticity of Approximate Solutions

We continue with the same assumptions made in Sect. 3. Recalling Definition 5.14 of [1], we have

$$z = \operatorname{curl}(u). \tag{5.1}$$

**Lemma 5.1.** *There exists an absolute constant $C_7$ with the following property: Suppose $a\in R^3$, $b\in R$, $\tau>0$, $B(a,5\tau/4)\subset U$, $b>\tau^2$, $\varepsilon<\tau/64$,*

$$M(t) = \max\{|u(x,t)|: x\in B(a,5\tau/4)\},$$

*$|a-c|<\tau$, $b-\tau^2<d<b$, and $n$, $p$, $q$ are defined as in Lemma 2.3. Then*

$$\begin{aligned}
|z(c,d)| &\leqq C_7\tau^{1/3} M(|z|, D(d)\cap G(a,b,\tau,q+2))\left(\int_{b-\tau^2}^{b} M(t)^3\,dt\right)^{1/3}\\
&+\sum_{k=p+2}^{q+1} C_7\tau^{-3}2^{3k} I(|Du|^2, T(a,b,\tau,\tau^2,\tau 2^{-k+2}))\\
&+C_7\tau^{-4/3}2^{2q}\left(\int_{b-\tau^2}^{b} M(t)^3\,dt\right)^{2/3}\\
&+C_7\tau^{-3}2^{3p}\left(\int_{d-\tau^2 2^{-2(p+2)}}^{d}\int_{B(a,5\tau/4)} |Du(x,t)|^2\,dx\,dt\right)\\
&+C_7\tau^{-5/2}2^{5p/2}\left(\int_{d-\tau^2 2^{-2(p+2)}}^{d}\int_{B(a,5\tau/4)} |Du(x,t)|^2\,dx\,dt\right)^{1/2}\\
&+C_7\|w^0\|_2\tau\varepsilon.
\end{aligned}$$

*Proof.* Let $h=\tau 2^{-(p+2)}$, $H=\tau 2^{-(q+2)}$,

$$K_1=K(c,d,\tau 2^{-(p+2)},\tau^2 2^{-2(p+2)}),$$

$$K_2=K(c,d,\tau 2^{-(q+2)},\tau^2 2^{-2(q+2)}),$$

$$K_3=K(c,d,\tau/4,\tau^2 2^{-2(p+2)}),$$

$$E(k)=K(c,d,\tau 2^{-k},\tau^2 2^{-2k})\sim K(c,d,\tau 2^{-(k+1)},\tau^2 2^{-2(k+1)})$$

if $k$ is an integer and $k\geqq p+2$. The argument that led to (3.8) still applies, so we have

$$K_2\subset D(d)\cap G(a,b,\tau,q+2). \tag{5.2}$$

As in Lemma 2.3, we can still say

$$p\geqq 0, \qquad n\geqq 0. \tag{5.3}$$

Now we will prove (recall Definition 2.1)

$$K(c,d,\tau 2^{-k},\tau^2 2^{-2k})\subset T(a,b,\tau,\tau^2,\tau 2^{-k+2}) \qquad \text{if } p+2\leqq k\leqq n+1 \tag{5.4}$$

and $k$ is an integer. If $x\in B(c,\tau 2^{-k})$ and $k\leqq n+1$ then

$$|x-a|\leqq|c-a|+|x-c|<\tau+\tau 2^{-k}<\tau+\tau 2^{-k+2},$$

$$|x-a|\geqq|c-a|-|x-c|\geqq\tau-\tau 2^{-n}-\tau 2^{-k}\geqq\tau-\tau 2^{-k+1}-\tau 2^{-k}>\tau-\tau 2^{-k+2}.$$

This implies

$$B(c,\tau 2^{-k})\subset\{x\in R^3:\tau-\tau 2^{-k+2}\leqq|x-a|\leqq\tau+\tau 2^{-k+2}\} \qquad \text{if } k\leqq n+1. \tag{5.5}$$

The argument that gave (3.7) is still valid, so we have

$$0<b-\tau^2<d-\tau^2 2^{-2(p+2)}<d<b. \tag{5.6}$$

Combining (5.5), (5.6) we get (5.4). In addition, we get

$$(|x-c|+(d-t)^{1/2})^{-3}\leqq\tau^{-3}2^{3(k+1)} \qquad \text{if } (x,t)\in E(k). \tag{5.7}$$

Using (5.2), (5.6), and $q\geqq p$ we find

$$\int_{K_2}|u(x,t)|\,|z(x,t)|(|x-c|+(d-t)^{1/2})^{-4}\,dx\,dt$$

$$\leqq M(|z|,K_2)\left(\int_{d-H^2}^{d}\left(M(t)\int_{B(c,H)}(|x-c|+(d-t)^{1/2})^{-4}\,dx\right)dt\right)$$

$$\leqq M(|z|,K_2)\,C\left(\int_{d-H^2}^{d}M(t)(d-t)^{-1/2}\,dt\right)$$

$$\leqq M(|z|,K_2)\,C\left(\int_{d-H^2}^{d}M(t)^3\,dt\right)^{1/3}\left(\int_{d-H^2}^{d}(d-t)^{-3/4}\,dt\right)^{2/3}$$

$$\leqq M(|z|, K_2)\, C \left( \int_{b-\tau^2}^{b} M(t)^3\, dt \right)^{1/3} H^{1/3}$$

$$\leqq C\tau^{1/3} 2^{-q/3} M(|z|, D(d) \cap G(a, b, \tau, q+2)) \left( \int_{b-\tau^2}^{b} M(t)^3\, dt \right)^{1/3}. \tag{5.8}$$

If $q > p$ then $q = n > p$, so (5.7) and (5.4) yield

$$\int_{K_1 \sim K_2} |z(x,t)|^2 (|x-c| + (d-t)^{1/2})^{-3}\, dx\, dt$$

$$\leqq \sum_{k=p+2}^{q+1} I(|z|^2, E(k))\, \tau^{-3} 2^{3(k+1)}$$

$$\leqq \sum_{k=p+2}^{q+1} C\tau^{-3} 2^{3k} I(|Du|^2, T(a, b, \tau, \tau^2, \tau 2^{-k+2})). \tag{5.9}$$

From (5.3) and $|a-c| < \tau$ we conclude $B(c, \tau 2^{-(p+2)}) \subset B(a, 5\tau/4)$. Using this and (5.6) we conclude the following when $q > p$:

$$\int_{K_1 \sim K_2} |u(x,t)|^2 (|x-c| + (d-t)^{1/2})^{-5}\, dx\, dt$$

$$\leqq \int_{d-h^2}^{d-H^2} \left( M(t)^2 \int_{R^3} (|x-c| + (d-t)^{1/2})^{-5}\, dx \right) dt$$

$$+ \int_{d-H^2}^{d} \left( M(t)^2 \int_{R^3 \sim B(c,H)} (|x-c| + (d-t)^{1/2})^{-5}\, dx \right) dt$$

$$\leqq C \left( \int_{d-h^2}^{d-H^2} M(t)^2 (d-t)^{-1}\, dt \right) + C \left( \int_{d-H^2}^{d} M(t)^2 H^{-2}\, dt \right)$$

$$\leqq C \left( \int_{d-h^2}^{d} M(t)^2 H^{-2}\, dt \right) \leqq CH^{-2} \left( \int_{b-\tau^2}^{b} M(t)^2\, dt \right)$$

$$\leqq C\tau^{-4/3} 2^{2q} \left( \int_{b-\tau^2}^{b} M(t)^3\, dt \right)^{2/3}. \tag{5.10}$$

If $q = p$ then $K_1 \sim K_2$ is the empty set. Hence we can use (5.8)–(5.10) and the inequality

$$|u(x,t)|\, |z(x,t)| (|x-c| + (d-t)^{1/2})^{-4}$$

$$\leqq (1/2)\, |u(x,t)|^2 (|x-c| + (d-t)^{1/2})^{-5} + (1/2)\, |z(x,t)|^2 (|x-c| + (d-t)^{1/2})^{-3}$$

to conclude

$$\int_{K_1} |u(x,t)|\, |z(x,t)| (|x-c| + (d-t)^{1/2})^{-4}\, dx\, dt$$

$$\leqq C\tau^{1/3} 2^{-q/3} M(|z|, D(d) \cap G(a, b, \tau, q+2)) \left( \int_{b-\tau^2}^{b} M(t)^3\, dt \right)^{1/3}$$

$$+ \sum_{k=p+2}^{q+1} C\tau^{-3} 2^{3k} I(|Du|^2, T(a, b, \tau, \tau^2, \tau 2^{-k+2}))$$

$$+ C\tau^{-4/3} 2^{2q} \left( \int_{b-\tau^2}^{b} M(t)^3\, dt \right)^{2/3}. \tag{5.11}$$

If $q=p$ then the second term in the right hand side of (5.11) is, of course, zero. Using $|c-a|<\tau$ and (5.6) we find

$$\begin{aligned}
&\int_{K_3\sim K_1} |u(x,t)|\,|z(x,t)|(|x-c|+(d-t)^{1/2})^{-4}\,dx\,dt \\
&\quad\leqq \int_{K_3\sim K_1} |u(x,t)|^2(|x-c|+(d-t)^{1/2})^{-5}\,dx\,dt \\
&\qquad+\int_{K_3\sim K_1} |z(x,t)|^2(|x-c|+(d-t)^{1/2})^{-3}\,dx\,dt \\
&\quad\leqq \int_{d-h^2}^{d} \Big(M(t)^2 \int_{R^3\sim B(c,h)} (|x-c|+(d-t)^{1/2})^{-5}\,dx\Big)dt \\
&\qquad+C\Big(\int_{K_3} |Du(x,t)|^2 h^{-3}\,dx\,dt\Big) \\
&\quad\leqq C\left(\int_{d-h^2}^{d} M(t)^2 h^{-2}\,dt\right)+Ch^{-3}\Big(\int_{K_3}|Du(x,t)|^2\,dx\,dt\Big) \\
&\quad\leqq C\tau^{-4/3}2^{4p/3}\left(\int_{d-h^2}^{d} M(t)^3\,dt\right)^{2/3}+C\tau^{-3}2^{3p}\Big(\int_{K_3}|Du(x,t)|^2\,dx\,dt\Big) \\
&\quad\leqq C\tau^{-4/3}2^{4p/3}\left(\int_{b-\tau^2}^{b} M(t)^3\,dt\right)^{2/3} \\
&\qquad+C\tau^{-3}2^{3p}\left(\int_{d-h^2}^{d}\int_{B(a,5\tau/4)} |Du(x,t)|^2\,dx\,dt\right).
\end{aligned}\tag{5.12}$$

Now let $\lambda_1$: $R^3\to[0,1]$ be a $C^\infty$ function such that $\lambda_1(x)=0$ for $|x-c|\geqq\tau/16$, $\lambda_1(x)=1$ for $|x-c|\leqq\tau/32$, and $\|D^i\lambda_1\|_\infty\leqq C\tau^{-i}$ for $i=1,2$. Let $\lambda_2$: $R\to[0,1]$ be a $C^\infty$ function such that $\lambda_2(t)=0$ for $t\leqq d-h^2$, $\lambda_2(t)=1$ for $t\geqq d-h^2/2$, and $\|(d/dt)\lambda_2\|_\infty\leqq Ch^{-2}$. We set $\lambda(x,t)=\lambda_1(x)\lambda_2(t)$, fix $i\in\{1,2,3\}$, and define $f$: $R^3\times(-\infty,d)\to R$, $g$: $R^3\times(-\infty,d)\to R^3$ by (recall (3.1))

$$f(x,s)=\lambda(x,s)\,Q_{d-s}(x-c),\ g_i(x,s)=f(x,s),\ g_j(x,s)=0 \quad \text{if } j\in\{1,2,3\} \text{ and } j\neq i.$$

We have

$$\begin{aligned}
|f(x,t)|&\leqq C(|x-c|+(d-t)^{1/2})^{-3},\\
|Df(x,t)|&\leqq C(|x-c|+(d-t)^{1/2})^{-4},
\end{aligned}\tag{5.13}$$

and hence the method of the proof of Lemma 5.7 of [1] yields

$$|(Df*\Psi*\Psi)(x,t)|\leqq C(|x-c|+(d-t)^{1/2})^{-4}.$$

From $\varepsilon<\tau/64$ we obtain

$$(Df*\Psi*\Psi)(x,t)=0 \quad \text{if } x\notin B(c,\tau/8).$$

If $d-h^2\leqq t<d$ then the above, $|c-a|<\tau$, and Lemmas 5.13, 5.6 of [1] yield

$$\begin{aligned}
&\Big|\int_{R^3} D_t z_i(x,t) f(x,t)+z_i(x,t)\,D_t f(x,t)\,dx\Big| \\
&\quad=\Big|\int_{R^3} u_j(x,t)\,z_i(x,t)(D_j f*\Psi*\Psi)(x,t)\,dx
\end{aligned}$$

$$
\begin{aligned}
&-\int_{R^3} z_j(x,t)\,u_i(x,t)(D_j f*\Psi*\Psi)(x,t)\,dx \\
&+\int_{R^3} z_i(x,t)(\Delta f+D_t f)(x,t)\,dx \\
&+\int_{R^3} [(w_k(t)*(\Delta(\Psi*\Omega*\Omega)-\Delta\Psi))(x)][\operatorname{curl}(g)(x,t)]_k\,dx\Big| \\
\leqq{}& C\Big(\int_{B(c,\tau/8)} |u(x,t)|\,|z(x,t)|(|x-c|+(d-t)^{1/2})^{-4}\,dx\Big) \\
&+\int_{B(a,5\tau/4)} |z(x,t)|\,|(\Delta f+D_t f)(x,t)|\,dx \\
&+C\,\|w^0\|_2\,\|\Delta(\Psi*\Omega*\Omega)-\Delta\Psi\|_2\Big(\int_{B(a,5\tau/4)} |Dg(x,t)|\,dx\Big).
\end{aligned}
\tag{5.14}
$$

Since $h<\tau$ [see (5.3)] and $(\Delta f+D_t f)(x,t)=0$ if the two conditions $|x-c|\leqq\tau/32$, $d-h^2/2\leqq t<d$ are satisfied, we conclude $\|\Delta f+D_t f\|_2\leqq Ch^{-5/2}$. Using this, (5.14), (3.26), (5.11), (5.12), $q\geqq p$, and (5.13) we conclude

$$
\begin{aligned}
|z_i(c,d)| &= \Big|\lim_{s\to d^-}\int_{R^3} z_i(x,s)f(x,s)\,dx\Big| \\
&= \left|\lim_{s\to d^-}\int_{d-h^2}^{s}\int_{R^3} D_t z_i(x,t)f(x,t)+z_i(x,t)D_t f(x,t)\,dx\,dt\right| \\
&\leqq C\left(\int_{d-h^2}^{d}\int_{B(c,\tau/8)} |u(x,t)|\,|z(x,t)|(|x-c|+(d-t)^{1/2})^{-4}\,dx\,dt\right) \\
&\quad+\left(\int_{d-h^2}^{d}\int_{B(a,5\tau/4)} |z(x,t)|^2\,dx\,dt\right)^{1/2}\|\Delta f+D_t f\|_2 \\
&\quad+C\,\|w^0\|_2\left(\int_{d-h^2}^{d}\int_{B(a,5\tau/4)} |Dg(x,t)|\,dx\,dt\right)\varepsilon \\
&\leqq C\tau^{1/3}2^{-q/3}M(|z|,D(d)\cap G(a,b,\tau,q+2))\left(\int_{b-\tau^2}^{b} M(t)^3\,dt\right)^{1/3} \\
&\quad+\sum_{k=p+2}^{q+1} C\tau^{-3}2^{3k}I(|Du|^2,T(a,b,\tau,\tau^2,\tau 2^{-k+2})) \\
&\quad+C\tau^{-4/3}2^{2q}\left(\int_{b-\tau^2}^{b} M(t)^3\,dt\right)^{2/3} \\
&\quad+C\tau^{-3}2^{3p}\left(\int_{d-h^2}^{d}\int_{B(a,5\tau/4)} |Du(x,t)|^2\,dx\,dt\right) \\
&\quad+C\left(\int_{d-h^2}^{d}\int_{B(a,5\tau/4)} |Du(x,t)|^2\,dx\,dt\right)^{1/2} h^{-5/2}+C\,\|w^0\|_2\,h\varepsilon.
\end{aligned}
$$

The conclusion of the lemma follows from the above, (5.3), and $q\geqq p$.

## 6. An Estimate for the Vorticity of Approximate Solutions

The assumptions of Sect. 3 are still in force.

**Lemma 6.1.** *There exist absolute constants $C_8$, $C_9$ with the following property: Suppose $\tau>0$, $a\in R^3$, $b\in R$, $B(a,5\tau/4)\subset U$, $b>\tau^2$, $\varepsilon<\tau/64$, and $M(t) =\max\{|u(x,t)|: x\in B(a,5\tau/4)\}$. Suppose also*

$$\int_{b-\tau^2}^{b} M(t)^3\,dt \leqq C_8\tau^{-1}, \tag{6.1}$$

$$I(|Du|^2, T(a,b,\tau,\tau^2,r))\leqq r \qquad \text{if } 0<r\leqq\tau, \tag{6.2}$$

$$I(|Du|^2, K(a,b-\tau^2+s, 5\tau/4, s))\leqq \tau^{-1}s \qquad \text{if } 0<s\leqq\tau^2, \tag{6.3}$$

$$\|w^0\|_2\leqq\tau^{-3}\varepsilon^{-1}. \tag{6.4}$$

*Then (recall* (5.1)*)*

$$|z(x,t)|\leqq 4C_9\tau^{-2} \qquad \text{if } (x,t)\in K(a,b,\tau/2,3\tau^2/4). \tag{6.5}$$

*Proof.* Choose $C_8>0$ so that $2^4C_7C_8^{1/3}\leqq 1/4$. Then choose $C_9>0$ so that $2^6C_7 +C_7C_8^{2/3}+3C_7\leqq C_9/4$. Once again we use the method of the proof of Lemma 3.1 of [1]. We construct a continuous function

$$f\colon \text{interior}(K(a,b,\tau,\tau^2))\to R^+$$

such that

$$C_9 2^{2i}\tau^{-2}\geqq f(x,t)\geqq C_9 2^{2(i-1)}\tau^{-2} \qquad \text{if } (x,t)\in G(a,b,\tau,i)\sim G(a,b,\tau,i-1) \tag{6.6}$$

for $i=1,2,3,\ldots$. As before, we define $G(a,b,\tau,0)$ to be the empty set. Again, we have

$$f(x,t)\leqq C_9 2^{2i}\tau^{-2} \qquad \text{if } (x,t)\in G(a,b,\tau,i). \tag{6.7}$$

We will prove

$$|z(x,t)|\leqq f(x,t) \qquad \text{for all } (x,t)\in\text{interior}(K(a,b,\tau,\tau^2)). \tag{6.8}$$

Assume that (6.8) is false. Then we can find $(c,d)\in\text{interior}(K(a,b,\tau,\tau^2))$ such that

$$|z(c,d)|=f(c,d), \tag{6.9}$$

$$|z(x,t)|\leqq f(x,t) \qquad \text{if } (x,t)\in D(d)\cap\text{interior}(K(a,b,\tau,\tau^2)). \tag{6.10}$$

Using this $(c,d)$ we define the integers $p$, $n$, $q$ by means of (2.2) and $q =\max\{n,p\}$. Since $(c,d)\notin\text{interior}(G(a,b,\tau,q))$, (6.6) and (6.9) yield

$$|z(c,d)|=f(c,d)\geqq C_9 2^{2q}\tau^{-2}. \tag{6.11}$$

Using (6.3) and the definition of $p$ we find

$$\begin{aligned}&\int_{d-\tau^2 2^{-2(p+2)}}^{d}\int_{B(a,5\tau/4)}|Du(x,t)|^2\,dx\,dt\\ &\leqq\int_{b-\tau^2}^{d}\int_{B(a,5\tau/4)}|Du(x,t)|^2\,dx\,dt\leqq\tau^{-1}(d-(b-\tau^2))\leqq\tau 2^{-2p}.\end{aligned} \tag{6.12}$$

Using (6.11), Lemma 5.1, (6.10), (6.7), (6.1), (6.2), $q \geqq p \geqq 0$, (6.1), (6.12), and (6.4) we obtain

$$\begin{aligned}
\tau^{-2} 2^{2q} C_9 &\leqq |z(c,d)| \\
&\leqq C_7 \tau^{1/3} C_9 2^{2(q+2)} \tau^{-2} C_8^{1/3} \tau^{-1/3} \\
&\quad + \sum_{k=p+2}^{q+1} C_7 \tau^{-3} 2^{3k} \tau 2^{-k+2} + C_7 \tau^{-4/3} 2^{2q} (C_8 \tau^{-1})^{2/3} \\
&\quad + C_7 \tau^{-3} 2^{3p} (\tau 2^{-2p}) + C_7 \tau^{-5/2} 2^{5p/2} (\tau 2^{-2p})^{1/2} + C_7 \tau^{-3} \tau \\
&\leqq \tau^{-2} 2^{2q} (2^4 C_7 C_9 C_8^{1/3}) + \tau^{-2} 2^{2q} (2^6 C_7) + \tau^{-2} 2^{2q} (C_7 C_8^{2/3}) + \tau^{-2} 2^p C_7 \\
&\quad + \tau^{-2} 2^{3p/2} C_7 + \tau^{-2} C_7 \\
&\leqq (1/4) \tau^{-2} 2^{2q} C_9 + \tau^{-2} 2^{2q} (2^6 C_7 + C_7 C_8^{2/3} + C_7 + C_7 + C_7) \\
&\leqq (1/4) \tau^{-2} 2^{2q} C_9 + (1/4) \tau^{-2} 2^{2q} C_9 = (1/2) \tau^{-2} 2^{2q} C_9.
\end{aligned}$$

This is a contradiction. Hence (6.8) must be true. Setting $i=1$ in (6.7) yields (6.5).

**Lemma 6.2.** *There exist absolute constants $C_{10}$, $C_{11}$ with the following property: Suppose $\sigma > 0$, $a \in R^3$, $b \in R$, $B(a, 5\sigma/4) \subset U$, $b > \sigma^2$, and $\varepsilon < \sigma/128$. Suppose also*

$$\int_{b-\sigma^2}^{b} (\max\{|u(x,t)| : x \in B(a, 5\sigma/4)\})^3 \, dt \leqq C_{10} \sigma^{-1}, \tag{6.13}$$

$$I(|Du|^2, K(a, b, 2\sigma, \sigma^2)) \leqq C_{10} \sigma, \tag{6.14}$$

$$\|w^0\|_2 \leqq \sigma^{-3} \varepsilon^{-1}. \tag{6.15}$$

*Then*

$$|z(x,t)| \leqq C_{11} \sigma^{-2} \qquad \text{if } (x,t) \in K(a, b, \sigma/4, 3\sigma^2/16). \tag{6.16}$$

*Proof.* As in the proof of Lemma 4.2, we find $C_{10} \leqq C_8$ such that (6.14) implies that (6.2), (6.3) hold for some $\tau$ such that $\sigma/2 < \tau < \sigma$. Then (6.1) is a consequence of (6.13) and (6.4) is a consequence of (6.15). Using Lemma 6.1 we conclude (6.16).

**Lemma 6.3.** *There exist absolute constants $\eta < 1$, $C_{12}$, $C_{13}$ with the following property: Suppose $\sigma > 0$, $a \in R^3$, $b \in R$, $\varepsilon < \eta\sigma/32$, $B(a, 5\sigma/4) \subset U$, and $b > \sigma^2$. Suppose also that*

$$I(|Du|^2, K(a, b, 3\sigma, \sigma^2)) \leqq C_{12} \sigma, \tag{6.17}$$

$$I(|u|^3, K(a, b, 4\sigma, \sigma^2)) \leqq C_{12} \sigma^2, \tag{6.18}$$

$$\|w^0\|_2 \leqq C_{12} \sigma^{-3} \varepsilon^{-1}. \tag{6.19}$$

*Then*

$$|z(x,t)| \leqq C_{13} (\eta\sigma)^{-2} \qquad \text{if } (x,t) \in K(a, b, \eta\sigma, (\eta\sigma)^2). \tag{6.20}$$

*Proof.* Let $\eta > 0$ be small enough so that $(4\eta)^2 \leqq 3/16$, $5\eta \leqq 1/4$, $C_6 \leqq C_{10} (4\eta)^{-1}$. Then choose $C_{12} > 0$ small enough so that $C_{12} \leqq C_{10} (4\eta)$, $C_{12} \leqq C_5$,

$C_{12} \leqq (4\eta)^{-3}$. Then Lemma 4.2 yields

$$\int_{b-(4\eta\sigma)^2}^{b} (\max\{|u(x,t)| : x \in B(a, 5(4\eta\sigma)/4)\})^3 \, dt \leqq C_6 \sigma^{-1} \leqq C_{10}(4\eta\sigma)^{-1}.$$

We also have

$$I(|Du|^2, K(a,b,2(4\eta\sigma),(4\eta\sigma)^2)) \leqq I(|Du|^2, K(a,b,2\sigma,\sigma^2)) \leqq C_{12}\sigma \leqq C_{10}(4\eta\sigma),$$

$$\|w^0\|_2 \leqq C_{12}\sigma^{-3}\varepsilon^{-1} \leqq (4\eta\sigma)^{-3}\varepsilon^{-1}.$$

Lemma 6.2 with $\sigma$ replaced by $4\eta\sigma$ yields $|z(x,t)| \leqq C_{11}(4\eta\sigma)^{-2}$ if $(x,t) \in K(a,b,\eta\sigma,(\eta\sigma)^2)$.

**Lemma 6.4.** *There exist an absolute constant* $C_{14}$ *and a number* $N$, *where* $N$ *depends only on* $\|w^0\|_2$, *such that the following is true. Suppose* $0 < \tau < 1$,

$$\varepsilon < \eta(4\tau/5)/32, \qquad \|w^0\|_2 \leqq C_{12}(4\tau/5)^{-3}\varepsilon^{-1},$$

*and*

$$T = \{(x,t) \in R^3 \times R : t \geqq \tau^2 \text{ and distance } (x, R^3 \sim U) \geqq \tau\}.$$

*Then there exist* $(x_1,t_1), (x_2,t_2), \ldots, (x_M,t_M)$ *where* $(x_j,t_j) \in T$, $M \leqq N\tau^{-5/3}$, *and*

$$|(\operatorname{curl}(u))(x,t)| \leqq C_{14}\tau^{-2} \qquad \text{if } (x,t) \in T \sim \bigcup_{j=1}^{M} K(x_j,t_j,\tau,\tau^2).$$

*Proof.* We define $\sigma = 4\tau/5$. Observe that there exist $(a_1,b_1)$, $(a_2,b_2)$, ... and an absolute constant $C_{15}$ such that $(a_i,b_i) \in R^3 \times R$, $B(a_i,\tau) \subset U$, $b_i > \tau^2$,

$$T \subset \bigcup_{i=1}^{\infty} K(a_i,b_i,\eta\sigma,(\eta\sigma)^2), \tag{6.21}$$

and for every $(x,t) \in R^3 \times R$ there exist at most $C_{15}$ integers $i$ such that $(x,t) \in K(a_i,b_i,4\sigma,\sigma^2)$. Hölder's inequality yields

$$I(|u|^3, K(a_i,b_i,4\sigma,\sigma^2)) \leqq C_{16}\sigma^{1/2}(I(|u|^{10/3}, K(a_i,b_i,4\sigma,\sigma^2)))^{9/10}. \tag{6.22}$$

From Lemma 5.6, Definition 5.10, Definition 5.14, and Lemma 5.17 of [1] we conclude

$$\int_0^\infty \int_{R^3} |Du(x,t)|^2 \, dx \, dt \leqq (1/2)\|w^0\|_2^2,$$

$$\int_0^\infty \int_{R^3} |u(x,t)|^{10/3} \, dx \, dt \leqq C\|w^0\|_2^{10/3}.$$

Hence the number of integers $i$ for which the inequality

$$I(|Du|^2, K(a_i,b_i,3\sigma,\sigma^2)) \leqq C_{12}\sigma$$

fails to hold is at most $C\|w^0\|_2^2\sigma^{-1}$. Similarly, the number of integers $i$ for which the inequality

$$I(|u|^{10/3}, K(a_i,b_i,4\sigma,\sigma^2)) \leqq (C_{12}C_{16}^{-1})^{10/9}\sigma^{5/3}$$

fails to hold is at most $C\|w^0\|_2^{10/3}\sigma^{-5/3}$. Combining this with (6.22) we find that the number of integers $i$ or which the inequality

$$I(|u|^3, K(a_i, b_i, 4\sigma, \sigma^2)) \leqq C_{12}\sigma^2$$

fails to hold is at most $C\|w^0\|_2^{10/3}\sigma^{-5/3}$. Since $\tau<1$ we conclude that there exist integers $i_1, i_2, \ldots, i_M$ such that

$$M \leqq N\tau^{-5/3} \text{ where } N \text{ depends only on } \|w^0\|_2,$$

$$I(|Du|^2, K(a_i, b_i, 3\sigma, \sigma^2)) \leqq C_{12}\sigma \quad \text{if } i \notin \{i_1, \ldots, i_M\},$$

$$I(|u|^3, K(a_i, b_i, 4\sigma, \sigma^2)) \leqq C_{12}\sigma^2 \quad \text{if } i \notin \{i_1, \ldots, i_M\}.$$

From Lemma 6.3 we conclude that $|z(x,t)| \leqq C_{13}(\eta\sigma)^{-2}$ if $(x,t) \in K(a_i, b_i, \eta\sigma, (\eta\sigma)^2)$ and $i \notin \{i_1, \ldots, i_M\}$. Now (6.21) implies that $|z(x,t)|$ is bounded by $C_{13}(\eta\sigma)^{-2}$ on the set

$$T \sim \bigcup_{j=1}^{M} K(a_{i_j}, b_{i_j}, \eta\sigma, (\eta\sigma)^2).$$

The conclusion follows from $\eta<1$ and (5.1).

The theorem in Sect. 1 now follows easily. The function $u$ was constructed using fixed $0<\alpha<\varepsilon$. In Sect. 6 of [1] it is shown that a sequence of such functions $u$ with $\varepsilon \to 0$ converges weakly to a weak solution to the Navier-Stokes equations with initial condition $w^0$ and the adherence condition at the boundary of $U$. The conclusion follows from Lemma 6.4 and a compactness argument.

## References

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