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How Conclusive is the Scaling Argument? The Connection Between Local and Global Scale Variations of Finite Action Solutions of Classical Euler–Lagrange Equations

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Abstract. We analyze the argument that a critical point of the action is stationary under a global scale transformation. We establish a general criterion which allows one to prove rigorously the validity or nonvalidity of the argument in the various relevant classes of Euler–Lagrange equations. Furthermore, we give a priori estimates on solutions at infinity.

1. Introduction

In the physical literature on finite energy solutions (respectively Gibbs-, free energy-, action- etc.) of nonlinear partial differential equations one finds frequently the argument that non-stationarity of the action under global dilations entails the nonexistence of finite action solutions for a wide class of model Lagrangians and for the interesting space dimensions. In other cases global stationarity serves as a means to establish a priori constraints on the solutions. The former point was for the first time emphasized by Derrick [1]. Very readable accounts of the whole subject are [2], [3]. The latter point was exploited in [4] to show the existence of solutions for a wide class of scalar models.

That the so-called "Derrick argument" might perhaps not be fully satisfactory was, as far as we know, for the first time emphasized in [5] for the nonlinear σ-model. In this paper we want to discuss, among other things, the limits respectively validity of this argument in full generality. We then apply the results to several classes of model Lagrangians, including the ones discussed in [4] and some of the models of classical nonabelian gauge theory, e.g. the Prasad–Sommerfield(PS) monopole solution etc.

[1]The critical point in the usual argument is the following. The Euler–Lagrange

1 The notion critical point for a stationary point of the action was a suggestion of Prof. A. Jaffe

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(E–L) equations are derived from a local variational principle, that is, $S := \int \mathscr{L} d^k x$ is to be stationary under variations of the fields which occur in $\mathscr{L}$, the variations vanishing outside an arbitrary but bounded domain of $\mathbb{R}^k$. Global dilations, on the other hand, do not vanish at infinity and it is not clear whether variations under the integral are allowed.

We will solve this problem on a general level, deriving exact formulas and criteria which will appear quite useful in another context, namely they allow us to establish several independent constraint equations which the solutions have to satisfy. As a result nontrivial finite energy solutions have a very narrow range of allowed decay rates at infinity. Furthermore we will discuss Lagrangians with solutions of the "hedgehog" type. We will show that in this case local stationarity does *not* imply global stationarity.

2. The Connection Between Local and Global Dilations

The following analysis appears independent of the precise form of the model Lagrangian $\mathscr{L}$ or Hamiltonian density $\mathscr{H}$. In particular the actual scale dimensions of the fields under discussion are not relevant for most of the arguments. Local stationarity means stationarity under completely arbitrary variations in every finite domain of space. Hence dilations with arbitrary scale dimensions have to be admitted. In order to limit the amount of complicated notation, we find it more advantageous to carry through the discussion with the help of a concrete model, and then prove the immediate generalizations for the more complicated models.

As a simple example we take the nonlinear σ-model in k dimensions (see e.g. [6], [5]) with Lagrangian

$$\mathscr{L}(\mathbf{n})(x) := \sum_{i=1}^{n} \sum_{j=1}^{k} (\partial_j n_i(x))^2 \tag{1}$$

$|\mathbf{n}|(x) = 1, x \in \mathbb{R}^k, \mathbf{n}(x) \in \mathbb{R}^n$.

The action is $S(n) := \int \mathscr{L}(\mathbf{n})(x) d^k x$, and one is interested in finite action solutions of the related E–L equations. These critical points are stationary under variations with compact support. The usual arguments however use a global dilation (with noncompact support) from the outset. To compare local and global variations we approximate the global dilation by a sequence of local ones which are confined to spheres $K_R \subset \mathbb{R}^k$ with radius R, centred at the origin. For the local dilations one exploits the E–L equations, the surviving terms have to be estimated in the limit $R \to \infty$.

Hence our first task is to define a sequence of "dilations" confined to spheres K_R. The localized dilation $d_R(\cdot, \lambda)$, $\lambda \in \mathbb{R}$, is defined by a bijective map: $\mathbb{R}^k \to \mathbb{R}^k$ as follows:

$$d_R(x, \lambda) := \begin{cases} \lambda \cdot x & \text{for } |x| \leqq R \\ \tilde{d}_R(|x|, \lambda) \cdot \hat{x} & \text{for } R \leqq |x| \leqq \lambda \cdot R + \varepsilon \\ x & \text{for } |x| \geqq \lambda \cdot R + \varepsilon \end{cases} \tag{2}$$

where $\varepsilon > 0$ is fixed during the course of manipulations, $\tilde{d}_R(\cdot, \lambda)$ maps the interval

$[R, \lambda R + \varepsilon]$ bijectively on $[\lambda R, \lambda R + \varepsilon]$, $\hat{x}$ denotes the unit vector in x-direction. The map is assumed to be sufficiently smooth in (x, λ) so that the differentiability properties hold to the order we need them. Furthermore $d_R(x, 1) \equiv x$. It is well known how to achieve these properties by gluing together the different pieces of the map. Smoothness means in particular:

$$\tilde{d}_R(R, \lambda) = \lambda \cdot R, \tilde{d}_R(\lambda R + \varepsilon, \lambda) = \lambda R + \varepsilon. \tag{3}$$

In other words, $d_R(\cdot, \lambda)$ is a true dilation for $|x| \leqq R$ and the identity for $x| \geqq \lambda R + \varepsilon$.

The action S is stationary under global dilations if:

$$\left.\frac{d}{d\lambda}\right|_{\lambda=1} \int \mathscr{L}(\mathbf{n}(\lambda x)) d^k x = 0. \tag{4}$$

Remarks. (i) At the moment we assume the scale dimension of $\mathbf{n}$ to be zero.

(*ii*) For the sake of notational clarity we distinguish $\mathscr{L}(\mathbf{n}(\lambda x))$ from $\mathscr{L}(\mathbf{n})(\lambda x)$. The latter one is the original $\mathscr{L}$, taken at the point λx; in the former, differentiation is meant with respect to x.

For the σ-model $\mathscr{L} \geqq 0$, so that finite action and integrability of $\mathscr{L}$ is equivalent. In more complicated models where not all integrands are positive we need a slightly stronger property.

It seems quite hopeless to give an a priori argument that it is allowed to perform the differentiation with respect to λ under the integral in (4). To compute (4) is quite simple, but only when ∂_{λ_1} is directly applied to $\mathscr{L}$ the E–L equations come into play (henceforth ∂_{λ_1} denotes exclusively differentiation at $\lambda = 1$). In other words, the whole argument rests on this assumption of the allowed interchangeability of the two operations. It seems a more advantageous strategy to approximate the global dilation by the above defined local ones and control the limit $R \to \infty$, thereby interchanging differentiation and integration only where it is strictly allowed. We shall see that boundary terms will survive which have to be discussed separately in the different classes of models. We have:

$$\begin{aligned}\partial_{\lambda_1} \int \mathscr{L}(\mathbf{n}(\lambda x)) d^k x &= \partial_{\lambda_1} \int \mathscr{L}(\mathbf{n}(d_R(x, \lambda))) d^k x \\ &+ \partial_{\lambda_1} \int \{\mathscr{L}(\mathbf{n}(\lambda x)) - \mathscr{L}(\mathbf{n}(d_R(x, \lambda)))\} d^k x\end{aligned} \tag{5}$$

With $\mathbf{n}(x)$ fulfilling the E–L equations the first term on the right side vanishes (note that differentiation of the integrand with respect to λ is now allowed since the variations extend only over a finite domain for $\lambda \to 1$), that is, we have only to deal with the second term in the sequel.

We divide the domain of integration into the three domains:

$$\text{(i) } |x| \leqq R \quad \text{(ii) } R < |x| \leqq \lambda R + \varepsilon \quad \text{(ii) } |x| > \lambda R + \varepsilon.$$

For $|x| = : r \leqq R$ the global and local dilation are identical, hence we are left with:

$$\begin{aligned}&\partial_{\lambda_1} \int_{R < r \leqq \lambda R + \varepsilon} \{\mathscr{L}(\mathbf{n}(\lambda x)) - \mathscr{L}(\mathbf{n}(d_R(x, \lambda)))\} d^k x \\ &+ \partial_{\lambda_1} \int_{r > \lambda R + \varepsilon} \{\mathscr{L}(\mathbf{n}(\lambda x)) - \mathscr{L}(\mathbf{n}(d_R(x, \lambda)))\} d^k x.\end{aligned} \tag{6}$$

For $r > \lambda R + \varepsilon$ we have with $d_R(x. \lambda) = x$:

$$\partial_{\lambda_1}(\lambda^{2-k} \int_{r > \lambda(\lambda R + \varepsilon)} \mathscr{L}(\mathbf{n}(x))d^k x - \int_{r > \lambda R + \varepsilon} \mathscr{L}(\mathbf{n}(x))d^k x). \tag{7}$$

With $dS(r)$ the canonical measure on the sphere $S_{k-1}(r)$ of radius r, this yields:

$$(2-k)\cdot \int_{r > R+\varepsilon} dr \int dS(r)\mathscr{L}(\mathbf{n}) - (2\dot{R} + \varepsilon)\cdot \int dS(R+\varepsilon)\mathscr{L}(\mathbf{n}) + R\cdot \int dS(R+\varepsilon)\mathscr{L}(\mathbf{n}). \tag{8}$$

We have to show that for R sufficiently large the various terms can be made arbitrarily small. The first term in (8) goes to zero for $R \to \infty$ because of the summability of $\mathscr{L}(\mathbf{n})$. For the remaining terms one needs a property of the calibre:

$$R \int dS(R)\mathscr{L}(\mathbf{n}) \to 0 \text{ for } R \to \infty .$$

A slightly weaker property is however already sufficient, namely the existence of a sequence $\{R_i\}$ with

$$(R_i + \varepsilon)\int dS(R_i + \varepsilon)\mathscr{L}(\mathbf{n}) \to 0 \text{ for } R_i \to \infty. \tag{9}$$

This is eventually of some use if there are solutions with a complicated oscillatory behaviour at infinity. (It is easy to construct positive functions with oscillating behavior at infinity which are nevertheless integrable).

Lemma. *With* $\mathscr{L}(\mathbf{n}) \geqq 0$ *and* $\int \mathscr{L}(\mathbf{n})d^k x < \infty$ *there always exists a sequence* $\{R_i\}$ *with the property* (9).

Proof. There is at least one accumulation point of the directed system $R\cdot \int dS(R)\mathscr{L}(\mathbf{n})$, $R \to \infty$, (note $\mathscr{L} \geqq 0$!), the value ∞ included. Hence lim inferior, the smallest of these accumulation points, does exist. We choose a sequence which converges toward lim. inf. (indexed by $\{R_i + \varepsilon\}$). Assuming

$$\text{lim. inf. } R\int dS(R)\mathscr{L}(\mathbf{n}) = a > 0$$

there is a R_a s.t. $R\int dS(R)\mathscr{L}(\mathbf{n}) \geqq \frac{a}{2}$ for $R > R_a$, otherwise there would exist a smaller accumulation point. This however entails:

$$S(\mathbf{n}) = \int_0^\infty dr \int dS(r)\mathscr{L}(\mathbf{n}) \geqq \int_{R_a}^\infty dr \frac{1}{r} . r\int dS(r)\mathscr{L}(\mathbf{n}) \geqq \frac{a}{2}\int_{R_a}^\infty dr \frac{1}{r} = \infty, \tag{10}$$

hence a contradiction. In other words,
lim. inf. $R\int dS(R)\mathscr{L}(\mathbf{n}) = 0$, hence $\lim_{R_i} (R_i + \varepsilon)\int dS(R_i + \varepsilon)\mathscr{L}(\mathbf{n}) = 0$. q.e.d.

With the help of the lemma, choosing the special sequence $\{R_i\}$, the remaining two terms can be made arbitrarily small for $R_i \to \infty$. It remains to discuss the first term in (6).

Remark. For the sake of greater generality, we discuss at this stage the general case of an $\mathscr{L}$ depending on an arbitrary number of fields combined in an n-vector $\boldsymbol{\phi}$

and their derivations, leaving their special transformation properties open at the moment.

For $\partial_{\lambda_1} \int_{R<r<\lambda R+\varepsilon} \mathscr{L}(\boldsymbol{\phi}, \partial\boldsymbol{\phi})d^k x$ the calculation is completely equivalent to the discussion of the second term of (6). The remaining integral we treat as follows:

$$\partial_{\lambda_1} \int_{R<r<\lambda R+\varepsilon} \mathscr{L}(\boldsymbol{\phi}(d_R(x,\lambda)), \partial\boldsymbol{\phi}(d_R(x,\lambda)))d^k x = \int_{R<r<\lambda R+\varepsilon} \left(\frac{\partial\mathscr{L}}{\partial\phi_i}\partial_{\lambda_1}\phi_i(d_R(x,\lambda))\right.$$
$$\left. + \frac{\partial\mathscr{L}}{\partial(\partial_\nu\phi_i)}\cdot\partial_{\lambda_1}\partial_\nu\phi_i\right)d^k x + R\int dS(R+\varepsilon)\mathscr{L}(\phi,\partial\phi). \tag{11}$$

The second term converges toward zero for $R_i \to \infty$. In the first integral $\partial_{\lambda_1}, \partial_\nu$ commute. A partial integration yields immediately (remember the boundary condition (3) for $\lambda = 1$ at $r = R$, $r = R + \varepsilon$):

$$\int_{R<r<R+\varepsilon} \frac{\partial\mathscr{L}}{\partial\phi_i} - \partial_\nu\frac{\partial\mathscr{L}}{\partial(\partial_\nu\phi_i)}\partial_{\lambda_1}\phi_i d^k x + \int dS(R)\frac{\partial\mathscr{L}}{\partial(\partial_\nu\phi_i)}\partial_{\lambda_1}\phi_i\cdot\frac{x_\nu}{R}. \tag{12}$$

(Note that $d\mathbf{o} = -dS(R).\dfrac{x}{R}$ for the inner sphere $r = R$!)

The first integral vanishes because of the E–L equations, yielding the final result:

$$\partial_{\lambda_1}\int\mathscr{L}(\boldsymbol{\phi}.\,\partial\boldsymbol{\phi})d^k x = \lim_{R\to\infty}\int dS(R_i)\frac{\partial\mathscr{L}}{\partial(\partial_\nu\phi_i)}\partial_{\lambda_1}\phi_i\cdot\frac{x_\nu}{R_i}. \tag{13}$$

Remembering $d_R(x,\lambda) = \lambda\cdot x$ for $|x| = R$, we can make the structure of this expression more explicit arriving at:

$$\lim_{R_i\to\infty}\int dS(R_i)\frac{x_\nu}{R_i}\cdot\left(\frac{\partial\mathscr{L}}{\partial(\partial_\nu\phi_i)}\cdot\partial_\mu\phi_i\cdot x_\mu\right)$$
$$= \lim_{R_i\to\infty}\int dS(R_i)x_\nu\cdot\left(\frac{\partial\mathscr{L}}{\partial(\partial_\nu\phi_i)}\partial_r\phi_i\right) \tag{14}$$

with ∂_r denoting the radial derivative.

Before putting the result into final shape we convince ourselves that the generalizations to the more complicated models are straightforward. Up to now we have used as special properties of the σ-model only $\mathscr{L} \geqq 0$ and the appearance of the prefactor λ^{2-k} in front of the integrals (scale dimension zero). Both properties are not essential for the discussion. Instead of $\mathscr{L} \geqq 0$ it is enough to assume integrability of the terms $\mathscr{L}_i$ in $\mathscr{L}$ which have the same transformation properties under dilations. Furthermore we can assume arbitrary scale dimensions for the various fields occurring in $\mathscr{L}$.

So let us define:

$$\phi_{\lambda,i}(x) := \lambda^{\alpha_i}\phi_i(\lambda x) \quad \text{with } \alpha_i \in \mathbb{N} \tag{15}$$

The bijective map $d_R(x,\lambda)$ defined in (2) has to be supplemented by an additional prefactor $f_i^{(R)}(\lambda, r)$ in front of the fields, which is smooth, taking the values:

$$f_i^{(R)}(\lambda, r) := \begin{cases} \lambda^{\alpha_i} \text{ for } r \leqq R \\ \text{interpolating in between} \\ 1 \text{ for } r > \lambda R + \varepsilon \end{cases} \tag{16}$$

with $f_i^{(R)}(1, r) \equiv 1$.

In most of the terms this changes the results only by a trivial constant prefactor arising from differentiation of products of various f_i's at $\lambda = 1$. This is the case for the domains of integration (i) and (iii). For the domain (ii) $R < r \leqq \lambda R + \varepsilon$ we have:

$$\partial_{\lambda_1}\left(P_i(\lambda) \int_{R < r \leqq \lambda R + \varepsilon} \mathscr{L}_i(\phi(\lambda x) d^k x - \int \mathscr{L}(f_i^{(R)}(\lambda, r) \cdot \phi(d_R(x, y)) d^k x \right) \tag{17}$$

with $P_i(\lambda)$ a monomial in the various λ^{α_i}'s. (We dropped in $\mathscr{L}$ the variable $\partial\phi$). But the additional term $(\partial_{\lambda_1} P_i(\lambda)) \cdot \int_{R < r \leqq R + \varepsilon} \mathscr{L}_i(\phi(x)) d^k x$ converges to zero for $R \to \infty$. As to the second second term we get:

$$\int_{R < r \leqq R + \varepsilon} \frac{\partial \mathscr{L}}{\partial \phi_i} \partial_{\lambda_1}(f_i^{(R)}(\lambda, r)\phi_i(D_R(x, \lambda))) + \frac{\partial \mathscr{L}}{\partial(\partial_\nu \phi_i)} \partial_{\lambda_1} \partial_\nu (f_i^{(R)}(\lambda, r) \cdot \phi_i(D_R(x, \lambda))) d^k x$$
$$+ R \cdot \int dS(R + \varepsilon) \mathscr{L}(\phi, \partial\phi) \tag{18}$$

(note that $f_i(1, r) \equiv 1$).

In the limit $R_i \to \infty$ this yields:

$$\lim_{R_i} \int dS(R_i) \frac{\partial \mathscr{L}}{\partial(\partial_\nu \phi_i)} \partial_{\lambda_1}(f_i^{(R)}(\lambda, r) \cdot \phi_i(d_R(x, \lambda))) \cdot \frac{x_\nu \cdot}{R_i} \cdot \tag{19}$$

But $r = R$ we can again take the derivative with respect to λ for $r < R$ and afterwards $r \to R$ because of the smoothness property of $f_i^{(R)}$ yielding:

$$\lim_{R_i} \int dS(R_i) \frac{x_\nu}{R_i} \frac{\partial \mathscr{L}}{\partial(\partial_\nu \phi_i)} (\partial_\mu \phi_i \cdot x_\mu + \alpha_i \phi_i). \tag{20}$$

So we arrive at the main theorem of this section:

Theorem. *Let $\phi(x)$ be a solution of the* E–L *equations to the Lagrange density $\mathscr{L}(\phi, \partial\phi)$ with ϕ the n-tupel of fields with arbitrary scale dimensions α_i. It is assumed that the action is finite in the sense that for all terms of $\mathscr{L}$, having the same transformation properties under dilations and being combined in an $\mathscr{L}_i$, we have $\int |\mathscr{L}_i| d^k x < \infty$. Then we have*

$$\partial_{\lambda_1} \int \mathscr{L}(\phi_\lambda, \partial\phi_\lambda) d^k x = \lim_{R_i \to \infty} \int dS(R_i) \frac{x_\nu}{R_i} \cdot \cdot \frac{\partial \mathscr{L}}{\partial(\partial_\nu \phi_i)} (R_i \cdot \partial_r \phi_i + \alpha_i \phi_i) \tag{21}$$

with $\phi_{\lambda, i}(x) := \lambda^{\alpha_i} \phi_i(\lambda x)$.

In other words, a solution of the E–L equations is stationary under global dilations if and only if the above expression vanishes for $R_i \to \infty$.

The usefulness of this result will become apparent in the next section, particularly when it is combined with other results (e.g. [4]). For many model Lagrangians one can explicitly show the vanishing of the above term for $R \to \infty$ for a wide classes of scale factor $\{\alpha_i\}$. Hence we usually get several independent constraint equations a finite energy solution has to fulfill. Furthermore the assumed finiteness of the various parts $\int \mathscr{L}_i d^k x$ of $\int \mathscr{L} d^k x$, consisting of the terms with analogous transformation properties is a sensible tool to test the behavior at infinity of the finite energy solutions.

3. Applications

(i) σ-Model

It is shown that the usual Derrick argument is applicable to the σ-model, that is, the above expression (21) vanishes for $R \to \infty$ for a finite energy solution. With scale dimension $\alpha = 0$ we have:

$$\begin{aligned}\lim_R \int dS(R) x_\nu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\nu n_i)} \cdot \partial_r n_i \right) &= \lim_R R \cdot \int dS(R)(\partial_r n_i)^2 \\ &\leqq \lim_R R \cdot \int dS(R)(\partial_\nu \phi_i)^2 = \lim_R R \int dS(R) \mathscr{L}(\partial_\nu n_i) \\ &= 0. \end{aligned} \tag{22}$$

The last equality follows from the argument given in (10).

That is, the E–L equations imply stationarity under global dilations. The Derrick argument then shows the nonexistence of finite energy solutions for $k > 2$ with the help of:

$$\partial_{\lambda_1}(\lambda^{2-k} \cdot \int \mathscr{L}(\mathbf{n}(x)) d^k x) \neq 0 \quad \text{for } k > 2.$$

(ii) Ginzburg–Landau Type Lagrangians

A slightly more complicated class of model Lagrangians consists of models having the time-independent action (vector potential $\mathbf{A} = 0$ at the moment):

$$S(\boldsymbol{\phi}) = \int \{ \tfrac{1}{2} (\nabla \boldsymbol{\phi})^2 + U(\boldsymbol{\phi}) \} d^k x \tag{23}$$

with $\boldsymbol{\phi}$ an n-tupel of real fields, U a polynomial. For simplicity we take U as:

$$U(\boldsymbol{\phi}) = -a|\boldsymbol{\phi}|^2 + b|\boldsymbol{\phi}|^4, \quad a, b \geqq 0. \tag{24}$$

Remark. (*i*) For U positive everywhere we can again exploit the Derrick argument.

(*ii*). Note that this implies that if one normalizes U so that the minima are at $U = 0$ there are no solutions of finite energy for $k > 1$, while the shifted U of (24) allows them.

It was shown in ([4]) for a wide class of U's and scalar ϕ that under mild restrictions there are always nontrivial finite energy solutions vanishing at infinity (U normalized to $U(0) = 0$), and that the solution of minimal action is spherically symmetric and monotone decreasing.

For scale dimension $\alpha = 0$ we have the usual constraint:

$$(2 - k)T - kV = 0 \tag{25}$$

$$T := \tfrac{1}{2} \int (\nabla \boldsymbol{\phi})^2 d^k x, \quad V := \int U(\boldsymbol{\phi}) d^k x.$$

We want to derive a further constraint by applying our general result to the case $\alpha = 1$. To infer global stationarity from the E–L equations we have to show the vanishing of

$$\lim_R \int dS(R) \frac{x_\nu}{R} \partial_\nu \phi_i \cdot \phi_i = \lim_R \frac{1}{2} \int dS(R) \partial_r (\phi)^2. \tag{26}$$

(This is the contribution in (21) coming from $\alpha = 1$, the other term in (21) vanishes for $R \to \infty$ since the expression converges to zero for $\alpha = 0$.)

As already stated we can assume the solution of minimal action to be a function of r only, vanishing at infinity. Assuming an asymptotic behavior at infinity $\phi(r) \lesssim r^{-\gamma}$, we have by qualitative reasoning: $\partial_r(\phi)^2 \lesssim r^{-(2\gamma+1)}$ at infinity. On the other hand, assuming T to be finite we have roughly: $\partial_r\phi \lesssim r^{-(\gamma+1)}$ with $\gamma > \frac{k-2}{2}$, that is $\partial_r(\phi)^2 \lesssim r^{-(k-1)-\varepsilon}$, which is enough to entail $\lim_R \int dS(R)\partial_r(\phi)^2 = 0$.

Hence we should be allowed to apply our general result which yields for the special case $\alpha = 1$, $U(\phi) = -a\phi^2 + b\phi^4$:

$$T \to \lambda^{(4-k)} \cdot T,\ V_1 \to \lambda^{(2-k)} \cdot V_1,\ V_2 \to \lambda^{(4-k)} \cdot V_2 \tag{27}$$

with $V_1 = \int \phi^2 d^k x$, $V_2 = \int \phi^4 d^k x$.

Remark. Note that we have to make the assumption $V_1, V_2 < \infty$ (this was the input to infer the theorem of the last section for $\alpha = 1$) for a finite action solution. For $\alpha = 0$ we have to assume only $\int U(\phi) d^k x < \infty$.

Hence assuming $V_1 < \infty$ for a moment which immediately implies $V_2 < \infty$, since we already know that the solution with $\int U(\phi) d^k x < \infty$ does exist, we have now two constraint equations, namely:

$$(i)\ (2-k)T - k(-aV_1 + bV_2) = 0; \quad \alpha = 0 \tag{28}$$

$$(ii)\ (4-k)T - (2-k)aV_1 + (4-k)bV_2 = 0; \quad \alpha = 1$$

yielding the dimension-independent constraint:

$$(iii)\ 2T - 2aV_1 + 4bV_2 = 0. \tag{29}$$

For $k = 3$ we get, combining (i) and (iii):

$$T + aV_1 + bV_2 \doteq 0 \quad \text{hence } T = V_1 = V_2 = 0. \tag{30}$$

(For $k = 4$ we get $V_1 = 0$, that is $\phi = 0$.)

On the other hand the existence of a finite energy solution was shown in ([4]). Hence our conclusion is that V_1, V_2 are not finite separately, even for a solution vanishing at infinity. With the help of $\int \phi^4 d^k x = \infty$ a rough estimate yields:

$$\phi \to 0 \text{ for } r \to \infty \text{ weaker than } r - 1/4k.$$

Combined with the above estimate in the paragraph after (26) we have, assuming $\phi(r) \lesssim r^{-\gamma}$ for $r \to \infty$:

$$\tfrac{1}{2}k - 1 < \gamma < \tfrac{1}{4}k.$$

Conclusion

A finite energy solution ϕ for the Lagrangian (23) vanishing at infinity, approaches zero for $r \to \infty$ stronger than $r^{-((1/2)k-1)}$ and weaker than $r^{-(1/4)k}$ e.g. $k = 3 \to \frac{1}{2} < \gamma < \frac{3}{4}$).

As an example of an abelian gauge theory we can take e.g. the Ginzburg–Landau hamiltonian with a nonvanishing vector potential $\mathbf{A}(x)$ and a complex scalar field ϕ. The action reads (we think mainly of an energy hence the different sign convention):

$$S = \int \{\tfrac{1}{2}|(\partial_\mu + iqA_\mu)\phi|^2 + (-a|\phi|^2 + b|\phi|^4) + \tfrac{1}{2}|\mathbf{B}|^2\} d^k x. \tag{31}$$

With the normalization as in (31) the well-known vortex solution of type II superconductors (in a wider context also discussed in ([7]), [8]) approaches a minimum of the potential $U(\phi)$, namely $|\phi| \to \left(\frac{a}{2b}\right)^{1/2}$ for $r \to \infty$ $(k = 2)$. This would become a finite energy solution only after a shift of $U\left(U(\phi) \to b\left(|\phi|^2 - \frac{1}{2}\frac{a}{b}\right)^2\right)$. In any case, $V_1 = V_2 = \infty$, but in a trivial way, and we can conclude nothing interesting from this fact. On the other hand, as above we can ask for solutions of (31) vanishing at infinity (it is not known to us whether they do exist at all, not to speak of their topological stability). Choosing *(i)* $\alpha_A = 1, \alpha_\phi = 0$ *(ii)* $\alpha_A = 1, a_\phi = 1$ we have $(V_3 := \frac{1}{2}\int |\mathbf{B}|^2 d^k x)$:

$$\begin{aligned} &(2-k)T - k(-aV_1 + bV_2) + (4-k)V_3 = 0 \\ &(4-k)T - (2-k)aV_1 + (4-k)bV_2 + (4-k)V_3 = 0 \end{aligned} \tag{32}$$

which yields again:

$$\begin{aligned} &2T - 2aV_1 + 4bV_2 = 0 \quad \text{and e.g. for } k = 3: \\ &T + aV_1 + bV_2 + V_3 = 0 \to T = V_1 = V_2 = V_3 = 0. \end{aligned}$$

That is, the trivial solution, and if a nontrivial solution exists, $V_1 = V_2 = \infty$ and we had the same poor decay with the same bounds as in the conclusion above.

The Non-Stationarity of the Bogomolny–Prasad–Sommerfield Solution

In this section we want to discuss the P–S-solution as an example of a wide class of models where the usual scaling arguments completely fail. As to the motivation and the physical significance of the various expressions occurring in the following we refer e.g. to ([2], [3], [9], [8], [10]). Here we are only interested in the connection between local and global variation in this model. For fields constant in time $\mathscr{L}$ has the form:

$$\mathscr{L} = -\mathscr{H} = -\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \tfrac{1}{2}\mathscr{D}_\mu\phi^a\mathscr{D}^\mu\phi^a =: -\mathscr{H}_1 - \mathscr{H}_2 \tag{33}$$

with $\mu, \nu, a = 1, 2, 3$ and:

$$\begin{aligned} G^{a\mu\nu} &= \partial^\mu W^{a\nu} - \partial^\nu W^{a\mu} - e\varepsilon_{abc} W^{b\mu} \cdot W^{c\nu} \\ \mathscr{D}^\mu\phi^a &= \partial^\mu\phi^a - e\varepsilon_{abc} W^{b\mu}\phi^c. \end{aligned} \tag{34}$$

A spherically symmetric solution was for the first time given by Prasad and Sommerfield with the help of two functions $K(\zeta), H(\zeta)\ \zeta := aer$, a defined by $a := \lim_{r\to\infty} |\phi|(r)$.

$$\begin{aligned} &H(\zeta) = \zeta \coth \zeta - 1, K(\zeta) = \zeta(\sinh \zeta)^{-1} \\ &\phi^a(r) = \frac{r^a}{er^2} H(aer), W^{ai} = -\varepsilon_{aij} \cdot \frac{r^j}{er^2}(1 - K(aer)). \end{aligned} \tag{35}$$

For the P–S solution the energy density is of the simple form:

$$\mathscr{H}(x) = (\mathscr{D}^i\phi)^2 = \partial^i(\phi\mathscr{D}^i\phi) = \nabla^2(\phi^2). \tag{36}$$

It is a special property of these equations that the asymptotic behavior, expressed in the free constant a is not fixed by the solution. In ([10]) this phenomenon was related to the nonvanishing of certain boundary terms occurring in the generator of the dilations. We want to show that this is only a special feature of a more general phenomenon, namely the non-stationarity of solutions under global dilations while they are true solutions of the E–L equations.

At first we want to exhibit the behavior of $\mathscr{H}(x)$ under dilations, the fields endowed with different scale dimensions. (Obviously $\mathscr{H}(x)$ does not keep its simple form (36) after a variation!)

$$(i)\ \mathbf{A}^a_\lambda(x) := \lambda\tilde{\mathbf{A}}^a(\lambda x),\ \phi^a_\lambda(x) := \tilde{\phi}^a(\lambda x) \tag{37}$$

with $\sim$ denoting the P–S solution. With $\mathscr{H}_1, \mathscr{H}_2$ as in (33) we have the scaling properties:

$$\mathscr{H}_\lambda(x) = \lambda^4\mathscr{H}_1(\lambda x) + \lambda^2\mathscr{H}_2(\lambda x) \tag{38}$$

(with the derivatives in $\mathscr{H}_i(\lambda x)$ acting at λx!). This yields:

$$\int\mathscr{H}_\lambda(x)d^3x = \lambda\int\mathscr{H}_1(x)d^3x + \lambda^{-1}\int\mathscr{H}_2(x)d^3x \tag{39}$$

with $\mathscr{H}_1 + \mathscr{H}_2$ now having the simple form (36). In other words:

$$\int\mathscr{H}_\lambda(x)d^3x = \lambda\int\tilde{\mathscr{H}}(x)d^3x - \lambda\int\tilde{\mathscr{H}}_2(x)d^3x + \lambda^{-1}\int\tilde{\mathscr{H}}_2(x)d^3x$$

hence $\partial_{\lambda 1}\int\mathscr{H}_\lambda(x)d^3x = \int\tilde{\mathscr{H}}(x)d^3x - 2\int\tilde{\mathscr{H}}_2(x)d^3x = 0$.
(Note that for the P–S solution $2\tilde{\mathscr{H}}_2 = \tilde{\mathscr{H}}$!)

That is, under this type of dilation we have global stationarity. But the solution is to be stationary locally under any type of dilation. Another choice is:

$$(ii)\ \mathbf{A}^\alpha_\lambda(x) := \lambda\tilde{\mathbf{A}}^a(\lambda x),\ \phi^\alpha_\lambda(x) := \lambda\tilde{\phi}^a(\lambda x) \tag{40}$$

yielding: $\mathscr{H}_\lambda(x) = \lambda^4\tilde{\mathscr{H}}(\lambda x)$ and $\partial_{\lambda_1}\int\mathscr{H}_\lambda(x)d^3x = \int\tilde{\mathscr{H}}(x)d^3x \neq 0$.

In other words, under this type of dilation the P–S solution is not stationary. Since it is a solution of the E–L equations, this phenomenon must have its roots in the non-vanishing of the boundary term (21).

In case (ii) we have the boundary term:

$$\int dS(R)\left\{\frac{\partial\mathscr{L}}{\partial(\partial^\nu\phi^i)}\partial_r\phi^i + \frac{\partial\mathscr{L}}{\partial(\partial^\nu A^i)}\partial_r A^i\right\}x_\nu + \int dS(R)\frac{x_\nu}{R}\frac{\partial\mathscr{L}}{\partial(\partial^\nu A^i)}A^i$$
$$+ \int dS(R)\frac{x_\nu}{R}\frac{\partial\mathscr{L}}{\partial(\partial^\nu\phi_i)}\phi^i. \tag{41}$$

Only the latter term can contribute since the first and second terms have to vanish for $R \to \infty$ because they arise already in case (i). The last term yields:

$$\int dS(R) \frac{x_\nu}{R} \frac{\partial \mathscr{L}}{\partial(\partial^\nu \phi^i)} \phi^i = \int d\mathbf{o}^\nu (\mathscr{D}^\nu \phi^i \cdot \phi_i)$$
$$\to \int d^3\mathbf{r}\, \partial_\nu (\mathscr{D}^\nu \phi^i \cdot \phi_i) = \int \mathscr{H}(\mathbf{r}) d^3\mathbf{r} \quad (\text{for } R \to \infty) \tag{42}$$

with $\mathscr{H}(\mathbf{r})$ the energy of the P–S solution.

In other words, the P–S solution is a nontrivial example where the usual scaling argument does not apply. A solution of the E–L equations need not be stationary under global dilations. Furthermore this example shows that it depends crucially on the choice of the scale dimensions whether the scaling argument is conclusive.

Note: Prof. A. Jaffe kindly informed me that a related argument will appear in Chapter Two of his book with C. Taubes on vortices and monopoles.

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