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# Non-Translation Invariant Gibbs States with Coexisting Phases

## I. Existence of Sharp Interface for Widom-Rowlinson Type Lattice Models in Three Dimensions*

Jean Bricmont, Joel L. Lebowitz, Charles E. Pfister**, and Enzo Olivieri***

Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903, USA

**Abstract.** We investigate the spatially inhomogeneous states of two component, $A-B$, Widom-Rowlinson type lattice systems. When the fugacity of the two components are equal and large, these systems can exist in two different homogeneous (translation invariant) pure phases one $A$-rich and one $B$-rich. We consider now the system in a box with boundaries favoring the segregation of these two phases into "top and bottom" parts of the box. Utilizing methods due to Dobrushin we prove the existence, in three or more dimensions, of a "sharp" interface for the system which persists in the limit of the size of the box going to infinity. We also give some background on rigorous results for the interface problem in Ising spin systems.

## 1. Introduction

This is the first part of a study of the interface between two coexisting phases. The desired goal of our study is an understanding of the interface of a continuum system; e.g., that of a liquid-vapour or a segregating binary mixture in $\mathbb{R}^3$. There are good reasons for believing that in the absence of external forces acting to spatially segregate the phases, e.g., gravity, the location of the interface would undergo very large fluctuations in the equilibrium state of the system [1, 2]. That is, if we imagine the gravitational field gradually turned off, then the mean square fluctuation of the height of the liquid-vapour interface would tend to a quantity $r(A)$, where $A$ is the cross-sectional area of the system, with $r(A)\to\infty$ as $A\to\infty$ [presumably $r(A)\sim\ln A$[1]]. It is expected nevertheless that it is possible to define

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*** Permanent address: Istituto Matematico, Universita di Roma, GNFM, I-Roma, Italy

1 This is in analogy with the behavior of coupled harmonic oscillators (Gaussian spins) in two dimensions

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"locally" an interface structure which is independent of $A$ for large $A$. The "width" of this interface should be related to the surface tension, becoming infinite only at the critical temperature when the two phases cease to be distinct [1, 2].

Needless to say we are very far from achieving our ultimate goal. We do not even know, at the present time, how to define, in a precise way, the position of an interface in a continuum system. Nor can we prove rigorously the existence of a liquid-vapour phase transition for a system with realistic interactions; e.g., Lenard-Jones pair potentials. For these reasons the rigorous study of interfaces has been so far confined entirely to lattice systems where there are essentially no conceptual problems, only difficult technical ones [3, 4]. Even for lattice systems, however, the results are very far from complete. Many important questions, such as the existence of a "roughening" transition, remain unanswered even for the spin 1/2 Ising system with nearest neighbor ferromagnetic interactions, the system to which most studies so far have been confined [3, 4]. Perhaps even more important, as far as our goal is concerned, these Ising spin systems have no reasonable limit which would correspond to a continuum fluid. Continuous spins on a lattice and the Euclidean field theory limit [5] do have their own interest and difficulties. The problems involved in these studies appear to us, however, as different from those encountered in continuum fluid systems.

We decided for this reason, as well as for its own intrinsic interest to start our study with the Widom-Rowlinson model on a lattice [6, 7]. This is a two-component, $A-B$, mixture in which there is an extended hard-core exclusion between the $A$ and $B$ particles favoring segregation at high densities. This model does have a direct extension to a continuum binary mixture. Both the lattice and continuum models are isomorphic to a one component fluid with many body interactions [7, 8]. Indeed, this system, or some variations on it, is the only continuum system with short range interactions for which the existence of a phase transition can be proven rigorously [6, 9]. In this paper we prove the existence of a "sharp" interface at high fugacity for certain Widom-Rowlinson type lattice models in three or more dimensions: high fugacity here corresponds to low temperatures in the isomorphic one component fluid system. Interestingly enough, our result does not hold, and we actually believe it to be untrue for some type of Widom-Rowlinson models; e.g., those with only nearest-neighbor exclusion. This will be discussed in Sect. 7.

The dimensionality here is important. For continuum systems (and for lattice systems with a continuous symmetry), one may expect sharp interfaces to occur in four and more dimensions. Put otherwise, interfaces for continuum fluids in three dimensions are expected to behave like lattice system interfaces in two dimensions which are known to fluctuate widely [10]. To clarify this point, as well as to set our results in their proper context – we rely very heavily on the work of Dobrushin [3] for the Ising spin system – we first briefly sketch the history and current status of the interface problem for Ising systems. (We include some generalization of previous results.) This enables us also to pose the problem in the general context of non-translation invariant Gibbs states, the context in which this problem naturally belongs from an abstract point of view. We then give a brief survey of relevant results (not relating to interfaces) for the $W-R$ model.

*Background*

An infinite volume Gibbs state is said to be translation invariant if the probability of a given spin configuration on a set of lattice sites $A$, $A \subset \mathbb{Z}^\nu$, is the same as on the set $A$ translated by any lattice vector $x$.

In the two dimensional Ising model, with ferromagnetic nearest neighbor interactions and no external field, there are exactly two extremal translation invariant Gibbs states (with opposite spontaneous magnetization) at low temperatures. $T < T_c$ [11, 12]. These two states, denoted by $+$ and $-$, are obtained from finite volume Gibbs states with boundary conditions (b.c.) in which all spins on the boundary are equal to $+1$ or $-1$. This implies that if we take the thermodynamic limit with any kind of b.c. and then average the resulting state over lattice translations, we get some linear combination of the states $+$ and $-$. The question is whether we need to make this space average or do we always get a translation invariant Gibbs state when taking the thermodynamic limit? The latter is always true for $T \geqq T_c$ when there is a unique Gibbs state [17].

A natural candidate for a non-translation invariant Gibbs state is one in which the two phases, $+$ and $-$, coexist and are separated by a flat interface. Such a state would look like the $+$ state, if one goes to infinity in one direction and like the $-$ state in the opposite direction. It seems natural that such a state should be obtained with the following b.c. denoted $\pm$ : all boundary spins on "top" $= +1$ and all boundary spins on "bottom" $= -1$. This is the b.c. studied by Gallavotti [10]. He found, however, that, at low temperatures, this state is in fact translation invariant. Later, Messager and Miracle-Sole [13] have shown that this state is actually translation invariant at all temperatures (using an explicit computation of Abraham and Reed [14]). They also proved that a large class of boundary conditions lead to translation invariant Gibbs states. But this result has not yet been proven for all possible boundary conditions.

The description given above of the translation invariant states also holds for the three-dimensional Ising ferromagnet. Furthermore Dobrushin has shown [3] that the $\pm$b.c. lead to a non-translation invariant Gibbs state with a "sharp" interface between the $+$ and $-$ state. Dobrushin's result was obtained only at very low temperatures. Using, however, a completely different method, namely, correlation inequalities, van Beijeren [4] has shown the following: the three-dimensional state with $\pm$ boundary condition is not translation invariant for any temperature such that there is a spontaneous magnetization in the corresponding two-dimensional Ising model. It is natural to ask whether this non-translation invariant state persists all the way to the critical temperature (for spontaneous magnetization) of the three-dimensional system or is there a "roughening transition" at a lower temperature? Indeed, numerical evidence [15] suggests that above some temperature $T_R$ [$\sim 0.57 T_c$ (3-dim.)] only translation invariant Gibbs states exist. This corresponds to a roughening of the interface present in the non-invariant states. Since $T_c(\text{2-d}) \sim 0.5 T_c(\text{3-d})$, van Beijeren's results, $T_R \geqq T_c(\text{2-d})$, seems a very good lower bound. However, no rigorous upper bound on $T_R$ [less than $T_c$(3-d)] has been proven.

We give in Appendix B a slightly more general version of van Beijeren's argument, using a remark of Hegerfeldt [16].

*Widom-Rowlinson Model*

The Widom-Rowlinson model on a lattice is a two-component, $A-B$, lattice gas, in which there is a finite diameter hard-core exclusion between the $A$ and $B$ particles. This model was first studied by Lebowitz and Gallavotti [6] who proved the existence of a phase transition in two and more dimensions, similar to the one in the ferromagnetic Ising model. Large activities correspond to low temperatures and equal activities for the two components are analogous to zero external field. Although their estimates did not allow them to prove the same result for the continuum model (by letting the lattice spacing go to zero), Ruelle [9] proved it, by superimposing a fixed lattice on the continuum model. This made the situation in the continuum, at least as far as phase transitions are concerned, look similar to the one on the lattice. In fact, results of [17] on the Ising model about the uniqueness of the phase when the free energy is differentiable with respect to the magnetic field have been carried through to the Widom-Rowlinson model [18].

On the other hand, no results have been obtained until now on the existence or non-existence of non-translation invariant Gibbs states for this model. We prove here the existence of a sharp interface (and therefore of non-translation invariant Gibbs states) in the three-dimensional lattice Widom-Rowlinson model (with some types of hardcores). We expect, but do not prove, the absence of such a sharp interface in the three dimensional continuum model.

As one can see in Appendix B, van Beijeren's argument depends strongly on the ferromagnetic nature of the interaction and we do not think that this argument can be generalized to lattice Widom-Rowlinson models (see e.g. [8] for rewriting of the Widom-Rowlinson model as a one component Ising spin with many-body interactions). This is why we use the original Dobrushin's techniques here.

These techniques actually give much more information about the interface than can be obtained from inequalities. This will become clear in a second paper where we prove

i) The probability of a displacement of the interface is exponentially small.

ii) Exponential clustering of the correlation functions in this non-translation invariant state.

iii) A simple formula for the surface tension in this model as well as for the 3-d Ising system.

In the third paper of this series we prove analyticity of the correlation functions of the $\pm$ state and of the surface tension in the variable $e^{-\beta J}$ for the Ising model, at low temperatures (and in the variable $e^{-\mu}$ for the Widom-Rowlinson model, at large fugacity).

*Formulation of Problem*

We consider a three-dimensional simple cubic lattice in $\mathbb{R}^3$. Each cell of the lattice is labelled by its center $x \in \mathbb{Z}^3$. Sometimes we also use $x$ for the cell with center $x$. At each site $x$ there can be either an $A$-particle or a $B$-particle or nothing. Let us define $|x-y| = \max_{i=1,2,3} |x^i - y^i|$ as the distance between $x$ and $y \in \mathbb{Z}^3$, and $C(x,d) = \{y \in \mathbb{Z}^3 : |x-y| \leq d\} \subset \mathbb{Z}^3$, where $x \in \mathbb{Z}^3$ and $d$ is a positive integer. The interaction between particles is expressed by two hard-core conditions: there is at most one particle in each cell, and if there is a particle of a given kind at $x$, then there is no particle of the other kind at $y \in C(x,d)$. For simplicity we restrict ourselves to the case $d=1$, but all our results are true for $d \geqq 1$ and for more general hard-cores. We shall say more about this later (Sect. 7).

Let $A_{L,M} = \{x \in \mathbb{Z}^3 : |x^1| \leqq M,\ |x^2| \leqq L,\ |x^3| \leqq L\}$ be a parallelepiped of base $(2L+1)^2$ and height $2M+1$, $L$, and $M$ being positive integers. We consider now the system enclosed in $A_{L,M}$, choosing the same chemical potential $\mu$ for the two kinds of particles, and putting an $A$-particle at each $x \in \mathbb{Z}^3 \backslash A_{L,M}$ with $x^1 \geqq 1$ and a $B$-particle at each $x \in \mathbb{Z}^3 \backslash A_{L,M}$ with $x^1 \leqq -1$. There are no particles at sites $x \in \mathbb{Z}^3 \backslash A_{L,M}$ with $x^1 = 0$. The configuration outside $A_{L,M}$ is thus compatible with the hard-core conditions. This boundary condition for $A_{L,M}$ corresponds to the $\pm$ boundary condition in the Ising system.

The Hamiltonian of the system is

$$H_{L,M} = - \sum_{x \in A_{L,M}} \mu(S_x^2 - 1),$$

where $S_x = +1$, resp. $-1$, if there is an $A$-particle, resp. $B$-particle, at $x$ and $S_x = 0$ if there is no particle at $x$. We consider only configurations compatible with the hard-core conditions. To each configuration of the system inside $A_{L,M}$ we can associate in a unique way a configuration on all $\mathbb{Z}^3$ by taking the boundary condition for the configuration outside $A_{L,M}$. This will be convenient for the definition of the interface.

As already mentioned, our analysis follows closely the paper of Dobrushin [3]. In particular, Sects. 3 and 5 are a direct adaptation of Dobrushin's results to our system. We present them here for the convenience of the reader.

## 2. Definition of the Interface

Let $E$ be a set of $\mathbb{Z}^3$. We say that $E$ is *connected* if for each pair $x$ and $y \in E$ we can find a sequence $x_1 = x,\ x_2, \ldots, x_n = y$ such that $x_i \in E$ and $|x_i - x_{i+1}| = 1$, $i = 1, \ldots, n-1$. Let $A_{L,M}$ with the boundary condition above be given. Let $w$ be a configuration of the system. We consider $w$ defined on all $\mathbb{Z}^3$. Let $E_0 = \{x \in \mathbb{Z}^3 : S_x = 0\}$. We decompose this set into maximally connected components. Obviously there is exactly one infinite component: we call it $\Delta = \Delta(w)$. The union of the cells whose centers are in $\Delta$ is also denoted by $\Delta$. Let $\Gamma(\Delta)$ be the boundary of this set in $\mathbb{R}^3$. We decompose $\Gamma(\Delta)$ into maximally connected components $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, $n \geqq 2$. (Here connected has the usual meaning for sets in $\mathbb{R}^3$.) We say that two cells $x$ and $y$ are *adjacent* if $|x-y|=1$. Geometrically this means that the two cells have a face or an edge or a vertex in common. We say that a cell $y$ is *adjacent to* $\Gamma(\Delta)$ if $y$ has a face or an edge or a vertex in common with $\Gamma(\Delta)$. The set of all cells $y$ of $\mathbb{Z}^3 \backslash \Delta$ adjacent to some cell of $\Delta$ is denoted by $\partial\Delta$. By definition all cells of $\partial\Delta$ are occupied by some particle, and by the definition of a connected component of $\Gamma(\Delta)$, say $\Gamma_1$, all cells of $\partial\Delta$ adjacent to $\Gamma_1$ form a connected set and hence all these cells are occupied by the same kind of particle, say of kind $A$. In this case we say that $\Gamma_1$ is an *$A$-border* of $\Delta$.

*Definition.* The interface $\bar{\Delta} = \bar{\Delta}(w)$ of a configuration $w$ is the couple given by the set $\Delta(w)$ and all particles of $w$ in $\partial\Delta$.

*Remark.* We stress the fact that when we give the interface $\bar{\Delta}$ we specify the kinds of particles in $\partial\Delta$. We therefore know which $\Gamma_i(\Delta)$ are $A$-borders and which are $B$-borders. We can have two different interfaces with the same $\Delta$ and hence $\partial\Delta$ but with different $A$-borders and $B$-borders. We make the convention that $\Gamma_1$ is always the infinite $A$-border of $\Delta$ and $\Gamma_2$ the infinite $B$-border of $\Delta$. Finally we will concentrate only on the part of $\bar{\Delta}$ which lies in $\Lambda_{L,M}$. We use the same symbols $\bar{\Delta}$, $\Delta$, $\partial\Delta$, $\Gamma_1$, $\Gamma_2$, etc.

## 3. The Probability Distribution of the Interface

Let us consider a particular fixed interface $\bar{\Delta}$ in $\Lambda_{L,M}$. Let $S(\bar{\Delta})=\Lambda_{L,M}\setminus\{\Delta\cup\partial\Delta\}$. We decompose this set into maximally connected components, $S_1(\bar{\Delta}),\dots,S_m(\bar{\Delta})$. We observe that $\partial S_i(\bar{\Delta})$, which is the set of all sites $y\in\partial\Delta$ adjacent to some site of $S_i(\bar{\Delta})$, is a connected set of occupied sites and hence all particles in $\partial S_i(\bar{\Delta})$ are of the same kind. Furthermore, the union of all $\partial S_i(\bar{\Delta})$ is $\partial\Delta$. We note that inside $S_i(\bar{\Delta})$ we have a system with pure $A$ boundary conditions or pure $B$ boundary conditions. Using the symmetry of the system with respect to the interchange of $A$-particles and $B$-particles, we can always consider that the system enclosed in a connected component $S_i(\bar{\Delta})$ has pure $A$-boundary condition. If $Z(S(\bar{\Delta})|A)$ is the partition function of the system enclosed in $S(\bar{\Delta})$ with pure $A$-boundary condition then we can write the probability of having a particular interface $\bar{\Delta}$ as

$$P_{L,M}(\bar{\Delta})=\frac{\exp(-\mu|\Delta|)Z(S(\bar{\Delta})|A)}{\sum\limits_{\bar{\Delta}\in D_{L,M}}\exp(-\mu|\Delta|)Z(S(\bar{\Delta})|A)}, \tag{3.1}$$

where $|\Delta|$ is the number of sites in $\Delta$ and $D_{L,M}$ is the set of all possible interfaces in $\Lambda_{L,M}$.

To study in more detail the expression (3.1), we will write it in a more convenient form using properties of the pure phases of the system which we state in Lemma 3.1. The proof is in Appendix A. We introduce the following concept: two sets in $\mathbb{Z}^3$, $E_1$ and $E_2$, are *congruent*, $E_1\sim E_2$, if we can obtain $E_2$ by a translation of $E_1$.

**Lemma 3.1.** a) *Let $\Lambda$ be a finite set in $\mathbb{Z}^3$. Then there exists a $\bar{\mu}<\infty$ independent of $\Lambda$ such that for all $\mu>\bar{\mu}$ we can define a function $g_\mu(t,\Lambda)$ for all $t\in\Lambda$ with the following properties:*

$$\log Z(\Lambda|A)=\sum_{t\in\Lambda}g_\mu(t,\Lambda),$$

$$|g_\mu(t,\Lambda)|\leqq K<\infty,$$

*with $K$ a constant independent of $\mu$ and $\Lambda$.*

b) *Let $\Lambda_1$ and $\Lambda_2$ be two finite sets in $\mathbb{Z}^3$. For $t_1\in\Lambda_1$, $t_2\in\Lambda_2$, let*

$$d(t_1,\Lambda_1;t_2,\Lambda_2)=\sup\{d:(\Lambda_1\cap C(t_1,d))\sim(\Lambda_2\cap C(t_2,d))\}$$

*then for all $\mu>\bar{\mu}$*

$$|g_\mu(t_1,\Lambda_1)-g_\mu(t_2,\Lambda_2)|\leqq J\exp(-\alpha d(t_1,\Lambda_1;t_2,\Lambda_2)),$$

*where $J<\infty$ and $\alpha>0$ are two constants independent of $\Lambda$ and $\mu$.*

To use Lemma 3.1, we replace $Z(S(\bar{\Delta})|A)$ in both the numerator and denominator of formula (3.1) by $Z(S(\bar{\Delta})|A)/Z(A_{L,M}|A)$. We then write the logarithm of this ratio as $\sum_{x\in\Delta} f_\mu(x,\Delta,A_{L,M})$ where the function $f_\mu(x,\Delta,A_{L,M})$ has properties similar to those of $g_\mu(t,\Lambda)$ of Lemma 3.1. In order to do this we introduce the lexicographic ordering in $\mathbb{Z}^3$ and for each fixed interface $\bar{\Delta}$ we associate to each $t\in A_{L,M}$ an element $x(t|\Delta)$ of $\Delta$ in the following manner:

$x(t|\Delta)=t \quad \text{if} \quad t\in\Delta$

$x(t|\Delta)$ is the first element of $\Delta$ (in the ordering introduced above) such that $|t-x(t|\Delta)|$ is minimal if $t\notin\Delta$.

We define now the function $f_\mu(x,\Delta,A_{L,M})$ for all $\mu>\bar{\mu}$ and all $x\in\Delta$ by

$$f_\mu(x,\Delta,A_{L,M})=\sum_{t\in S(\bar{\Delta})}^{x}(g_\mu(t,S(\bar{\Delta}))-g_\mu(t,A_{L,M}))-\sum_{t\in(\Delta\cap\partial\Delta)}^{x} g_\mu(t,A_{L,M}). \tag{3.2}$$

Here $\sum^x$ means the sum over all $t$ in $A_{L,M}$ such that $x(t|\Delta)=x$. We see immediately that $\sum_{x\in\Delta} f_\mu(x,\Delta,A_{L,M})=\log Z(S(\bar{\Delta})|A)-\log Z(A_{L,M}|A)$. We introduce now the quantity $d(x_1,\Delta_1,A_{L_1M_1};x_2,\Delta_2,A_{L_2M_2})$ as the supremum over all $d$ such that

$$(A_{L_1M_1}\cap C(x_1;d))\sim(A_{L_2M_2}\cap C(x_2;d)),$$
$$(\Delta_1\cap C(x_1;d))\sim(\Delta_2\cap C(x_2;d)).$$

**Lemma 3.2.** *Let $\mu>\bar{\mu}$, where $\bar{\mu}$ is defined in Lemma 3.1. Then there exist two constants $\bar{K}<\infty$ and $\bar{\alpha}>0$, depending only on $\bar{\mu}$ such that*

$$|f_\mu(x_1,\Delta_1,A_{L_1M_1})|\leqq\bar{K},$$
$$|f_\mu(x_1,\Delta_1,A_{L_1M_1})-f_\mu(x_2,\Delta_2,A_{L_2M_2})|$$
$$\leqq\bar{K}\exp(-\bar{\alpha}d(x_1,\Delta_1,A_{L_1M_1};x_2,\Delta_2,A_{L_2M_2})).$$

*Proof.* Let $A_1=A_{L_1M_1}$. If $t\in S(\bar{\Delta}_1)$ and $x_1=x(t|\Delta_1)$ then $d(t,S(\bar{\Delta}_1);t,A_1)\geqq|t-x_1|-2$. If $t\in\Delta_1\cup\partial\Delta_1$, then $|t-x(t|\Delta_1)|\leqq 1$. Hence the number of different $t$ in $\Delta_1\cup\partial\Delta_1$ having the same $x(t|\Delta_1)=x_1$ does not exceed $27-1=26$. Using Lemma 3.1 we obtain

$$|f_\mu(x_1,\Delta_1,A_1)|\leqq 26K+\sum_{t\in\mathbb{Z}^3} J\exp(-\alpha(|t-x_1|-2))\equiv\bar{K}.$$

To prove the second part of Lemma 3.2, let $A_2=A_{L_2M_2}$ and

$\hat{d}=d(x_1,\Delta_1,A_1;x_2,\Delta_2,A_2)>6.$

Let $t_0=x_2-x_1\in\mathbb{Z}^3$. Suppose that $t_1\in S(\bar{\Delta}_1)$ and $|t_1-x_1|\leqq\frac{d}{2}$. Then by definition of $\hat{d}$ the two sets $C(x_1;\hat{d})\cap A_1$ and $C(x_2;\hat{d})\cap A_2$ are congruent and the same is true for $C(x_1;\hat{d})\cap\Delta_1$ and $C(x_2;\hat{d})\cap\Delta_2$. Hence the sets

$$\left\{t_1:t_1\in S(\bar{\Delta}_1),|t_1-x_1|\leqq\frac{\hat{d}}{2}\right\} \quad\text{and}\quad \left\{t_2:t_2\in S(\bar{\Delta}_2),|t_2-x_2|\leqq\frac{\hat{d}}{2}\right\}$$

are congruent. Furthermore, we have that $d(t_1, S(\bar{\Delta}_1); t_1+t_0, S(\bar{\Delta}_2)) \geqq \frac{\hat{d}}{2}$, since $C\left(t_1; \frac{\hat{d}}{2}\right) \subseteqq C(x_1; \hat{d})^2$. Therefore

$$
\begin{aligned}
&|f_\mu(x_1, \Delta_1, A_1) - f_\mu(x_2, \Delta_2, A_2)| \\
&\leqq \sum_{t_1 \in S(\bar{\Delta}_1),\, |x_1 - t_1| \leqq \frac{\hat{d}}{2}}^{x_1} \{|g_\mu(t_1, S(\bar{\Delta}_1)) - g_\mu(t_1+t_0, S(\bar{\Delta}_2))| + |g_\mu(t_1, A_1) - g_\mu(t_1+t_0 A_2)|\} \\
&+ \sum_{t_1 \in S(\bar{\Delta}_1),\, |x_1 - t_1| > \frac{\hat{d}}{2}}^{x_1} |g_\mu(t_1, S(\bar{\Delta}_1)) - g_\mu(t_1, A_1)| \\
&+ \sum_{t_2 \in S(\bar{\Delta}_2),\, |t_2 - x_2| > \frac{\hat{d}}{2}}^{x_2} |g_\mu(t_2, S(\bar{\Delta}_2)) - g_\mu(t_2, A_2)| \\
&+ \sum_{t_1 \in (\bar{\Delta}_1 \cup \partial \Delta_1)}^{x_1} |g_\mu(t_1, A_1) - g_\mu(t_1+t_0, A_2)|\,. \qquad (3.3)
\end{aligned}
$$

We estimate the first sum.

$$
\sum_{t_1 \in S(\bar{\Delta}_1),\, |t_1 - x_1| \leqq \frac{\hat{d}}{2}}^{x_1} |g_\mu(t_1, S(\bar{\Delta}_1)) - g_\mu(t_1+t_0, S(\bar{\Delta}_2))| \leqq J(\hat{d}+1)^3 \exp\left(-\frac{\alpha\hat{d}}{2}\right),
$$

where $(\hat{d}+1)^3$ is larger than the number of points $t$ of $\mathbb{Z}^3$ such that $|t-x_1| \leqq \frac{\hat{d}}{2}$. We estimate the third sum.

$$
\begin{aligned}
&\sum_{t_1 \in S(\bar{\Delta}_1),\, |t_1 - x_1| > \frac{\hat{d}}{2}}^{x_1} |g_\mu(t_1, S(\bar{\Delta}_1)) - g_\mu(t_1, A_1)| \\
&\leqq \sum_{t_1 \in S(\bar{\Delta}_1),\, |t_1 - x_1| > \frac{\hat{d}}{2}}^{x_1} J \exp(-\alpha(|t_1 - x_1| - 2)) \leqq C_2 \exp\left(-\frac{\alpha}{3}\hat{d}\right)
\end{aligned}
$$

because $d(t_1, S(\bar{\Delta}_1); t_1, A_1) \geqq |t_1 - x_1| - 2$.

The second sum and the fifth sum in (3.3) are estimated like the first sum. The fourth sum is estimated like the third sum. □

We now study the probability distribution of the interface in the infinite parallelepiped $B_L = \{x \in \mathbb{Z}^3; |x^2| \leqq L, |x^3| \leqq L\}$. We denote by $D_L = \bigcup_M D_{LM}$ the set of all finite interfaces ($|\Delta| < \infty$) in $B_L$. From Lemma 2 we get the following corollary

**Corollary 3.1.** *The following limit exists for all $L$, $x \in \Delta$ and $\Delta \in D_L$*

$$
f_\mu(x, \Delta, L) = \lim_{M \to \infty} f_\mu(x, \Delta, A_{L,M}).
$$

*This limit has the properties:*

$$
\begin{aligned}
&|f_\mu(x_1, \Delta_1, L_1)| \leqq \bar{K} \\
&|f_\mu(x_1, \Delta_1, L_1) - f_\mu(x_2, \Delta_2, L_2)| \leqq \bar{K} \exp(-\bar{\alpha} d(x_1, \Delta_1, L_1; x_2, \Delta_2, L_2)),
\end{aligned}
$$

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2 We also remark that $x(t_1|\Delta_1) = x_1$ if and only if $x(t_1+t_0|\Delta_2) = x_2$ since the lexicographic order is translation invariant

*where* $\bar{K}<\infty$ *and* $\bar{\alpha}>0$ *are two constants and* $d(x_1, \Delta_1, L_1; x_2, \Delta_2, L_2)$ *is the supremum over all d such that*

$$(B_{L_1} \cap C(x_1; d)) \sim (B_{L_2} \cap C(x_2; d)),$$
$$(\Delta_1 \cap C(x_1; d)) \sim (\Delta_2 \cap C(x_2; d)).$$

The proof of this corollary is immediate. We mention only that we use the fact that for large $M_1$ and $M_2$,

$$d(x_1, \Delta_1, A_{L_1 M_1}; x_2, \Delta_2, A_{L_2 M_2}) = d(x_1, \Delta_1, L_1; x_2, \Delta_2, L_2).$$

**Lemma 3.3.** a) *The number of connected sets with k elements in* $\mathbb{Z}^3$ *which have a common element* $x_0 \in \mathbb{Z}^3$ *does not exceed* $(26)^k$.

b) *The number of interfaces with a given finite* $\Delta$ *in* $B_L$ *does not exceed* $2^{|\Delta|}$.

*Proof.* The first part of Lemma 3.3 is proved in the demonstration of Lemma 4 of [19]. To prove the second part we remark that the number of internal components of $\Gamma(\Delta)$ i.e. the components which are not $\Gamma_1$ or $\Gamma_2$ cannot exceed $|\Delta|$. So we have at most $2^{|\Delta|}$ possibilities of choosing the $A$-borders and $B$-borders. □

We consider now the Gibbs state for the system enclosed in $B_L$ with $A-B$ boundary condition: we put an $A$-particle at each $x \in \mathbb{Z}^3 \backslash B_L$ with $x^1 \geqq 1$ and a $B$-particle at each $x \in \mathbb{Z}^3 \backslash B_L$ with $x^1 \leqq -1$. We define the interface as before. We can prove the following proposition for all $\mu$ large enough.

**Proposition 3.1.** a) *The interface of a configuration is in* $D_L = \bigcup_M D_{LM}$ *with probability* 1.

b) *If the interface* $\bar{\Delta}$ *is in* $D_L$ *then* $\operatorname{Prob}(\bar{\Delta}) = P_L(\bar{\Delta})$ *where*

$$P_L(\bar{\Delta}) = \frac{\exp\left(-\mu|\Delta| + \sum_{x \in \Delta} f_\mu(x, \Delta, L)\right)}{Z_L}$$

*with*

$$Z_L = \sum_{\bar{\Delta} \in D_L} \exp\left(-\mu|\Delta| + \sum_{x \in \Delta} f_\mu(x, \Delta, L)\right).$$

*Proof.* Assume b). Then $\sum_{\bar{\Delta} \in D_L} P_L(\bar{\Delta}) = 1$. Therefore let us prove b). From (3.1) and (3.2) we have that (for $M$ large enough)

$$P_{L,M}(\bar{\Delta}) = Z_{L,M}^{-1} \exp\left(-\mu|\Delta| + \sum_{x \in \Delta} f_\mu(x, \Delta, A_{L,M})\right)$$

with

$$Z_{L,M} = \sum_{\bar{\Delta} \in D_{L,M}} \exp\left(-\mu|\Delta| + \sum_{x \in \Delta} f_\mu(x, \Delta, A_{L,M})\right).$$

Using Lemma 3.3 and Corollary 3.1 we obtain that $\lim_{M \to \infty} Z_{L,M} = Z_L < \infty$ and $P_L(\bar{\Delta}) = \lim_{M \to \infty} P_{L,M}(\bar{\Delta})$, if $\mu$ is large enough. □

## 4. The Geometrical Description of the Interface $\bar{\Delta}$

Let $\bar{\Delta}$ be a finite interface in $B_L$, $\bar{\Delta}\in D_L$, and let $\Gamma(\Delta)=\Gamma_1+\Gamma_2+\ldots+\Gamma_m$ where $\Gamma_1$ is an $A$-border and $\Gamma_2$ is a $B$-border as in our convention of Sect. 2. The other $\Gamma_i$ are either $A$-borders or $B$-borders but this does not play a role in most of the results of this section. We concentrate on the part of $\bar{\Delta}$ inside $B_L$. Let us define the *regular plane* $\sigma$ as $\sigma=\{x\in\mathbb{Z}^3 : x^1=0\}$ and the projection $p(x)$ of a point $x=(x^1,x^2,x^3)\in\mathbb{Z}^3$ on the regular plane $\sigma$ by $p(x)=(0,x^2,x^3)$. We define two kinds of points in $\Delta$. We say that $x\in\Delta$ is a *c-point* if $p(x)=p(y)$ and $y\in\Delta$ implies that $y=x$. All other points are *w-points*. Now we decompose $\Delta$ into maximally connected components of $c$-points and maximally connected components of $w$-points.

*Definition.* A *ceiling* $\bar{C}$ is a couple given by a maximally connected component of $c$-points $C$ and all particles of $\bar{\Delta}$ in $\partial\Delta\cap\partial C$.

A *wall* $\bar{W}$ is a couple given by a maximally connected component of $w$-points $W$ and all particles of $\bar{\Delta}$ in $\partial\Delta\cap\partial W$.

**Lemma 4.1.** a) *The A-border $\Gamma_1$ and the B-border $\Gamma_2$ divide each $B_L$ into two disjoint regions.*

b) *$\Gamma_1$ is always above $\Gamma_2$.*

c) *All $\Gamma_i$, $i\geqq 3$ are closed polyhedra and lie between $\Gamma_1$ and $\Gamma_2$.*

*Proof.* By definition $\Gamma_i$ is a connected set in $\mathbb{R}^3$ and each face belonging to $\Gamma_i$ is common to an empty cell and to an occupied cell. Therefore each edge of a face of $\Gamma_i$ belongs to an even number of faces of $\Gamma_i$. The lemma then follows from this and the choice of the boundary condition. □

We now state the crucial lemma of this section. It is here that the choice of the hard-core matters.

**Lemma 4.2.** *Let $x$ be a c-point, $x=(x^1,x^2,x^3)$ of $\Delta$. Let*

$$Q_x=\{y\in\mathbb{Z}^3 : y=(x^1,y^2,y^3), |x-y|=1\}.$$

*Then $Q_x\subseteqq\Delta$.*

*Proof.* Since $x$ is a $c$-point, $u=(x^1+1,x^2,x^3)$ is occupied by an $A$-particle and $v=(x^1-1,x^2,x^3)$ is occupied by a $B$-particle. [Lemma 4.1a) and b)] So all adjacent points to $x$ in the plane $\{y\in\mathbb{Z}^3 : y^1=x^1\}$ are empty points and so belong to $\Delta$. □

Lemma 4.2 leads immediately to a series of results. Let $\bar{C}$ be a particular ceiling in $\Delta$. By definition $C$ is a connected component of $c$-points of $\Delta$. So we get from Lemma 4.2 that $C$ lies in a plane $\sigma_1$ parallel to $\sigma$. Furthermore all adjacent points to $C$ in $\sigma_1$ are $w$-points. Consider now the projection $p(C)$ of $C$ in $\sigma$. The boundary of this set $p(C)$ is made up of the projections of all vertical faces separating a $c$-point of $C$ and a $w$-point adjacent to $C$. The boundary of $p(C)$ can be decomposed into $n$ connected components $\gamma_1,\ldots,\gamma_n$, and the complement of $p(C)$ in $\sigma$ consists of $n$ connected components $E_1,\ldots,E_n$ with boundaries $\gamma_1,\ldots,\gamma_n$. Let $\Delta_i=\{x\in\Delta : p(x)\in E_i\} i=1,\ldots,n$. Then $\Delta_i$ and $\Delta_j$ are disconnected if $i\neq j$ because the $E_i$ and $E_j$ are disconnected. Since $\bar{C}$ is a ceiling $\Delta\backslash C=\Delta_1+\ldots+\Delta_n$. Since $\Delta$ is connected, the $\Delta_i$ are connected. Indeed if we have a path $\gamma$ in $\Delta$ joining $x$ and $y$ in $\Delta_i$ (such a path exists because $\Delta$ is connected) and if this path does not lie entirely in $\Delta_i$, then there is a first point $x_j$ of $\gamma$ such that $x_j \in \Delta_i$ and $x_{j+1} \notin \Delta_i$ and there is also a last point $x_k$ of $\gamma$ such that $x_p \in \Delta_i$ for all $p \geqq k$. But $x_{j+1}$ and $x_{k-1}$ belong to $C$. So by Lemma 4.2 we can find another path joining $x$ and $y \in \Delta_i$ and lying entirely in $\Delta_i$ because we can join $x_j$ and $x_k$ by $w$-points of $\Delta_i$ adjacent to $C$. As a corollary of these simple facts we obtain Lemma 4.3.

**Lemma 4.3.** *Let $\bar{W}$ be a wall in $\bar{\Delta}$. Let $p(W)$ be the projection of $W$ on $\sigma$. Then*

a) $\{x \in \Delta : p(x) \in p(W)\} = W$,

b) *the complement of $p(W)$ in $\sigma$ contains exactly one infinite component,*

c) *suppose that the complement of $p(W)$ in $\sigma$ is composed of $m$ disjoint connected components $G_1, \ldots, G_m$. Then all points in $\Delta$ adjacent to $W$ are $c$-points which belong to exactly $m$ ceilings $\bar{C}_1, \ldots, \bar{C}_m$ such that $p(C_i) \subseteqq G_i$.*

d) *Two different walls in $\bar{\Delta}$, $\bar{W}_1$, and $\bar{W}_2$, are such that $p(W_1)$ and $p(W_2)$ are disconnected.*

*Proof.* To prove a) we consider any ceiling $\bar{C}$ in $\bar{\Delta}$ and we construct the connected subsets $\Delta_1, \ldots, \Delta_m$ as above. $W$ must be in some of the $\Delta_i$, say $\Delta_1$. Then we take a ceiling in $\Delta_1$ (if there is one ceiling in $\Delta_1$) and we repeat the procedure. After a finite number of such procedures we have a connected subset which contains $W$ and only $w$-points. So this subset coincide with $W$. The proof of b) is trivial. The proof of c) follows directly from Lemma 4.2 and the fact that $W$ is a connected set. The proof of d) follows from a) and c). □

At this point we introduce some further definitions which will help us describe more precisely the interface $\bar{\Delta}$. We know from Lemma 4.3 that the complement of $p(W)$ contains exactly one infinite connected component, say $G_1$, and that there is a ceiling $\bar{C}$ adjacent to $\bar{W}$ such that $p(C) \subset G_1$. We call this ceiling the *base* of the wall. We know also that each ceiling $\bar{C}$ of $\bar{\Delta}$ is such that $C$ lies in a plane $\{y \in \mathbb{Z}^3 : y^1 = s\}$ parallel to $\sigma$. The *height* of $\bar{C}$ is defined as $s$. We define also the *interior of the projection of a wall $\bar{W}$*, $\operatorname{Int}(W)$, as the set of points of $\sigma$ which belong to $p(W)$ or to one of the finite connected components of the complement of $p(W)$ in $\sigma$. Among all interfaces $\bar{\Delta}$ in $D_L$ there are very simple ones which contain only one wall $\bar{W}$. Of course the base of this wall lies in the regular plane $\sigma$. All walls which are obtained in this way are called *standard walls*. The end of this section is devoted to the proof of the fact that any interface $\bar{\Delta}$ in $D_L$ can be described in a unique way by a set of standard walls. We introduce now a total ordering in the plane $\sigma$ and we define the *origin* of a standard wall $\bar{W}$ as the first point of the projection $p(W)$ of $W$. We call a collection of standard walls *admissible* if $p(W_i)$ and $p(W_j)$ are disconnected, $i \neq j$ for all pairs of standard walls $\bar{W}_i$, $\bar{W}_j$ of the collection.

**Lemma 4.4.** *There is a one-to-one correspondence between the set of interfaces $D_L$ and the set of admissible collections of standard walls in $B_L$.*

*Proof.* Let $\bar{\Delta}$ be an interface in $D_L$. Let $\bar{W}_i$ and $\bar{W}_j$ be to different walls of $\bar{\Delta}$. Then by Lemma 4.3 $p(W_i) \cap p(W_j) = \emptyset$. Let $s_i$ be the height of the base of $\bar{W}_i$. We translate $\bar{W}_i$ by $(-s_i, 0, 0)$ in order that the base lies now in the regular plane $\sigma$. To prove that the translation of $\bar{W}_i$ is a standard wall we must construct an interface which has only this wall. This can be done in a unique way [see Lemma 4.3c)] by constructing ceilings $\bar{C}_1, \ldots, \bar{C}_m$ such that $p(C_j) = G_j$ where $G_j$, $j = 1, \ldots, m$ are the different disjoint components of $\sigma \backslash p(W_i)$.

Let $(\bar{W}_1, \ldots, \bar{W}_p)$ be an admissible collection of standard walls. Let $(\bar{\Delta}_1, \ldots, \bar{\Delta}_p)$ be the collection of interfaces associated to the standard walls $(\bar{W}_1, \ldots, \bar{W}_p)$ by the construction above. We observe that for any pair $(\bar{W}_i, \bar{W}_j)$ of standard walls of the collection either $\operatorname{Int} W_i \subset \operatorname{Int} W_j$, or $\operatorname{Int} W_i \supset \operatorname{Int} W_j$ or $\operatorname{Int} W_i \cap \operatorname{Int} W_j = \emptyset$. So we can introduce a partial ordering, $\bar{W}_i > \bar{W}_j$ if $\operatorname{Int} W_i \supset \operatorname{Int} W_j$, such that if $\bar{W}_i > \bar{W}_j$ and $\bar{W}_k > \bar{W}_j$ then $\bar{W}_k > \bar{W}_i$ or $\bar{W}_i > \bar{W}_k$. Now we pick out the minimal elements of the family $(\bar{W}_1, \ldots, \bar{W}_p)$, say $(\bar{W}_1, \bar{W}_2, \bar{W}_3)$, with respect to this ordering. Consider $\bar{W}_1$. We know that there is a unique next wall $\bar{W}_k > \bar{W}_1$. By Lemma 4.3 $\operatorname{Int} W_1$ lies in one component, say $G_{k_1}$, of $\sigma \backslash p(W_k)$. We modify now the interface $\bar{\Delta}_k$ by taking out the part of the ceiling $C_{k_1}$ whose projection is $\operatorname{Int}(W_1)$ and replacing it by the part of $\bar{\Delta}_1$ whose projection is $\operatorname{Int}(W_1)$ and which is translated in such a way that the base of $\bar{W}_1$ lies now in the plane containing $C_{k_1}$. We do this construction for all minimal elements, $(\bar{W}_1, \bar{W}_2, \bar{W}_3)$ and we obtain a new family of interfaces by taking out $\bar{\Delta}_1, \bar{\Delta}_2, \bar{\Delta}_3$ from $(\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_p)$ and replacing the interface $\bar{\Delta}_k$ by its modified interface etc. We repeat the procedure with the minimal elements of $(\bar{W}_4, \ldots, \bar{W}_p)$. Finally we get a family of interfaces such that any pair of walls $(\bar{W}, \bar{W}')$ belonging to different interfaces is such that $\operatorname{Int} W \cap \operatorname{Int} W' = \emptyset$. So in a trivial way we can construct an interface which is associated with the family $(\bar{W}_1, \ldots, \bar{W}_p)$ of standard walls. □

## 5. The Groups of Walls

Let $\bar{\Delta}$ be an interface and $(\bar{W}_1, \ldots, \bar{W}_p)$ be its set of admissible standard walls. Then $(\bar{W}_2, \ldots, \bar{W}_p)$ is also a set of admissible standard walls. Let $\bar{\Delta}_0$ be the associated interface. The ratio

$$P_L(\bar{\Delta})/P_L(\bar{\Delta}_0)$$

can be well estimated when $\bar{W}_1$ is far from $\bar{W}_2, \bar{W}_3, \ldots, \bar{W}_p$ in a sense made precise below. This is the reason why we introduce groups of walls, a group of walls being a set of standard walls which are close so that two different groups of walls are by definition far apart of each other. We will then study the effect of removing a group of walls instead of a standard wall.

For any standard wall $\bar{W}$ we define $\Pi(\bar{W}) = |W| - |p(W)|$. By definition of $w$-points we have $p(W) \leqq \frac{1}{2}|W|$ so that $\Pi(\bar{W}) \geqq \frac{1}{2}|W|$ and $\Pi(\bar{W}) \geqq |p(W)|$. If $t_1$ and $t_2 \in \operatorname{Int}(W)$ then $|t_1 - t_2| \leqq \Pi(W) - 1$. It is also clear that $|\Delta| = (2L+1)^2 + \sum \Pi(\bar{W})$ where the sum is over all different walls of $\bar{\Delta}$. So we obtain

$$P_L(\bar{\Delta}) = \frac{\exp(-\mu \sum \Pi(\bar{W})) \exp\left(\sum\limits_{x \in \Delta} f_\mu(x, \Delta, L)\right)}{Z_L \exp(\mu(2L+1)^2)}. \tag{5.1}$$

Let $u \in \sigma$. We define $d(u)$ as the number of elements of $\Delta$ which have projection $u$. We say that two standard walls $\bar{W}_1, \bar{W}_2$ are *close* if there exist $u_1 \in p(W_1)$ and $u_2 \in p(W_2)$ such that $|u_1 - u_2| \leqq \sqrt{d(u_1)} + \sqrt{d(u_2)}$. A family of standard walls is a *group of walls* if the family is admissible and if for any two walls $\bar{W}_1$ and $\bar{W}_2$ of the family there is a sequence of walls of the family $(\bar{W}_1 = \bar{V}_1, \bar{V}_2, \ldots, \bar{W}_2 = \bar{V}_n)$ such that $\bar{V}_i$ and $\bar{V}_{i+1}$ are close, $i = 1, \ldots, n-1$. The origin of the group of walls is the first origin among the origins of the walls of the group. We define for a group of walls $F$, $\Pi(F) = \sum \Pi(\bar{W})$ where the sum is over all walls of the group. A set of groups of walls is *admissible* if all the walls of this set of groups are admissible and if any $\bar{W}_1 \in F_i$ and $\bar{W}_2 \in F_j$, $i \neq j$, are not close. By Lemma 4.4 there is a one-to-one correspondence between the admissible collections of groups of walls in $B_L$ and the interfaces in $D_L$. Let $(F_{t_1}, \dots, F_{t_n})$ be a family of groups of walls. Then we define

$$\operatorname{Prob}(F_{t_1}, \dots, F_{t_n}) = \begin{cases} P_L(\bar{\Delta}) & \text{if the family is admissible} \\ 0 & \text{if the family is not admissible}, \end{cases} \tag{5.2}$$

where $\bar{\Delta}$ is the interface associated to $(F_{t_1}, \dots, F_{t_n})$ by Lemma 4.4.

**Lemma 5.1.** *For all sufficiently large $\mu$ and all $L$ and all $(F_{t_1}, \dots, F_{t_n})$ the conditional probability*

$$\operatorname{Prob}\{F_{t_1} | F_{t_2}, \dots, F_{t_n}\} \leqq \exp\left(-\frac{\mu}{3} \Pi(F_{t_1})\right)$$

*if $\{F_{t_2}, \dots, F_{t_n}\}$ is admissible.*

*Proof.* If $(F_{t_1}, \dots, F_{t_n})$ is not admissible then $\operatorname{Prob}\{F_{t_1} | F_{t_2}, \dots, F_{t_n}\} = 0$. Suppose that $(F_{t_1}, \dots, F_{t_n})$ is admissible. Let $\bar{\Delta}$ be the interface associated to $(F_{t_1}, \dots, F_{t_n})$ and $\bar{\Delta}^*$ the interface associated to $(F_{t_2}, \dots, F_{t_n})$. Then

$$\begin{aligned} \operatorname{Prob}\{F_{t_1} | F_{t_2}, \dots, F_{t_n}\} &\leqq \frac{P_L(\bar{\Delta})}{P_L(\bar{\Delta}^*)} \\ &= \exp(-\mu \Pi(F_{t_1})) \exp\left\{\sum_{x \in \Delta} f_\mu(x, \Delta, L) - \sum_{x \in \Delta^*} f_\mu(x, \Delta^*, L)\right\}. \end{aligned}$$

Let $(\bar{W}_{u_1}, \dots, \bar{W}_{u_k})$ be the standard walls in $F_{t_1}$. We decompose $\sigma \setminus \bigcup_{i=1}^{k} p(W_{u_i})$ into disjoint connected components, $E_1, \dots, E_q$. Let $\Delta_i$, resp. $\Delta_i^*$, be the part of $\Delta$, resp. $\Delta^*$, which projects onto $E_i$, $i = 1, \dots, q$. Then by looking at the proof of Lemma 4.4 we see that there is a $1-1$ correspondence between $\Delta_i$ and $\Delta_i^*$. So the same is true for $\bigcup_{i=1}^{q} \Delta_i$ and $\bigcup_{i=1}^{q} \Delta_i^*$. This bijection is denoted by $\delta$. Let $x \in \Delta_i$. Then $d(x, \Delta, L; \delta(x), \Delta^*, L) \geqq r(x)$ where $r(x)$ is the distance from $p(x) = p(\delta(x))$ to $\bigcup_{i=1}^{k} p(W_{u_i})$. So

$$\begin{aligned} &\left|\sum_{x \in \Delta} f_\mu(x, \Delta, L) - \sum_{x \in \Delta^*} f_\mu(x, \Delta^*, L)\right| \\ &\leqq \sum_{x \in \bigcup_{i=1}^{q} \Delta_i} |f_\mu(x, \Delta, L) - f_\mu(\delta(x), \Delta^*, L)| \\ &\quad + \sum_{x \in \Delta \setminus \bigcup_{i=1}^{q} \Delta_i} |f_\mu(x, \Delta, L)| + \sum_{x \in \Delta^* \setminus \bigcup_{i=1}^{q} \Delta^*} |f_\mu(x, \Delta^*, L)| \\ &\leqq \bar{K} \sum_{x \in \bigcup_{i=1}^{q} \Delta_i} \exp(-\bar{\alpha} r(x)) + 4\bar{K} \Pi(F_{t_1}). \end{aligned}$$

Let $p(F_{t_1}) = \bigcup_{j=1}^{k} p(W_{u_j})$. For all $x \in \bigcup_{i=1}^{q} \Delta_i$ we have $d(p(x)) \leqq (r(x))^2 + 1$ where $d(p(x))$ is the number of elements in $\Delta$ which have projection $p(x)$ on $\sigma$. So we have

$$\begin{aligned} \sum_{x \in \bigcup_{i=1}^{q} \Delta_i} \exp(-\alpha r(x)) &\leqq \sum_{u \notin p(F_{t_1})} d(u) \exp(-\alpha r(u)) \\ &\leqq \sum_{u \notin p(F_{t_1})} \exp(-\alpha r(u))((r(u))^2 + 1) \leqq \tilde{K} \Pi(F_{t_1}). \quad \square \end{aligned}$$

**Lemma 5.2.** *The number of distinct groups of walls with $\Pi(F)=K$ and a common origin $t_0 \in \sigma$ does not exceed $S_3^k$.*

*Proof.* Let $F=(\bar{W}_1, \ldots, \bar{W}_p)$. For a given family $(W_1, \ldots, W_p)$ we have by Lemma 3.3 at most $S_2^{2\Pi(F)}$ different $F$. Let $u \in p(W_i)$. Then there are $d(u)$ points in $W_i$ having the same projection $u$. We define $E(u)=C(u, \sqrt{d(u)}) \cap \sigma$ and

$$\tilde{F}=\left(\bigcup_{i=1}^{p} W_i\right) \cup \left(\bigcup_{u \in p(F)} E(u) \backslash p(F)\right)$$

where $p(F)=\bigcup_{i=1}^{p} p(W_i)$. By the definition of a group of walls $\tilde{F}$ is connected and $\tilde{F}$ determines uniquely $F$. Thus we have

$$\begin{aligned} |\tilde{F}| &\leqq \sum_{i=1}^{p} |W_i| + \sum_{u \in p(F)} |E(u)| \\ &\leqq \sum_{i=1}^{p} 2\Pi(W_i) + \sum_{u \in p(F)} 4d(u) \leqq K\Pi(F). \end{aligned}$$

By applying again Lemma 3.3 we obtain Lemma 5.2. □

## 6. Proof of the Main Theorem

Let us begin with a simple remark. If $\bar{W}_1$ and $\bar{W}_2$ are two different walls of the interface $\bar{\Delta}$ they have disconnected projections in the regular plane $\sigma$. Therefore for any wall $\bar{W}$ of $\bar{\Delta}$ there is a path $\gamma=(x_1, \ldots, x_n)$ such that $x_i$ is a $c$-point of $\Delta$, $x_i$, and $x_{i+1}$ have a face in common and $p(\gamma)$ goes around $p(W)$. We say that such a path $\gamma$ is *regular* if it lies in $\sigma$. We say that a path $\gamma$ in $\sigma$ *goes to infinity* if it crosses the boundary of the squares $\{x \in \mathbb{Z}^3 : x^1=0, |x^2| \leqq L, |x^3| \leqq L\}$ for any finite $L$.

**Theorem 6.1.** 1) *The probability, for the distribution $P_L$ on the interfaces (Proposition 3.1), that there exists a regular path with starting point $x \in \sigma$ and going to infinity, is, for $\mu$ large, bigger than $(1-C\exp(-\mu/5))$ with $C<\infty$, uniformly in $x$ and $L$.*

2) *There exists a function $g(\mu)$, going to zero exponentially fast as $\mu \to \infty$, such that, for any Gibbs state $P$ obtained as a limit when $L \to \infty$ of the states $P_L$, $P(S_x=+1) \geqq 1-g(\mu)$ if $x^1 \geqq 1$, $P(S_x=-1) \geqq 1-g(\mu)$ if $x^1 \leqq -1$.*

*Proof.* To prove 1), let $x$ and $\bar{\Delta} \in B_L$ be given. If there is no regular $\gamma$ with starting point $x$ and going to infinity then there must exist a wall $\bar{W}$ in $\bar{\Delta}$ such that $x \in \operatorname{Int} W$; otherwise we could always find $\gamma$ as observed in the beginning of this section. Let $\mathscr{P}_x$ be the probability that there exists a wall $\bar{W}$ in $\bar{\Delta}$ with $x \in \operatorname{Int} W$. Each wall $\bar{W}$ is contained in a group of walls $F_t$ with origin $t$ and clearly $|t-x|+1 \leqq \Pi(F_t)$.

Therefore

$$\mathscr{P}_x \leqq \sum_{t \in \sigma} \operatorname{Prob}\{\Pi(F_t) \geqq |t-x|+1\}.$$

Now

$$\begin{aligned} \operatorname{Prob}\{\Pi(F_t) \geqq |t-x|+1\} &\leqq \sum_{k \geqq |t-x|+1} S_3^k \exp\left(-\frac{\mu}{3}k\right) \\ &\leqq C' \exp\left(-\frac{\mu}{4}(|t-x|+1)\right). \end{aligned}$$

Therefore

$$\mathscr{P}_x \leqq \sum_{t \in \mathbb{Z}^2} C' \exp\left(-\frac{\mu}{4}(|t-x|+1) \leqq C \exp\left(-\frac{\mu}{5}\right)\right)$$

with $C$ a constant independent of $L$.

To prove 2), observe that, if there is a regular path through $p(x) \in \sigma$, then $x$ is in a pure phase region and the estimates hold by [6] (or one can prove it directly with our definition of contours). On the other hand, the probability that there is no such path is exponentially small with $\mu$ by point 1). □

## 7. Discussion and Comments

We gave in detail the treatment of the hard-core with $d=1$. Suppose that we let $d>1$. We choose then

$$A_{L,M} = \{x \in \mathbb{Z}^3 : -M \leqq x^1 \leqq M+d-1,\ |x^2| \leqq L,\ |x^3| \leqq L\}$$

and we put an $A$-particle at each $x \in \mathbb{Z}^3 \backslash A_{L,M}$ with $x^1 \geqq d$ and a $B$-particle at each $x \in \mathbb{Z}^3 \backslash A_{L,M}$ with $x^1 \leqq -1$. There are no particles at sites $x \in \mathbb{Z}^3 \backslash A_{L,M}$ with $0 \leqq x^1 < d$. We then have the same definitions and results as before for Sects. 2 and 3 except that 26 in Lemma 3.3 is now $((2d+1)^3-1)$.

There are some changes in Sect. 4. We define the regular plane $\sigma$ as before but we say that $x \in \Delta$ is a *c-point* if there are exactly $d$ points $y \in \Delta$ including $x$ with $p(x) = p(y)$. The next change is Lemma 4.2 which we replace by

**Lemma 7.1.** *Let $x_1, \ldots, x_d$ be c-points in $\Delta$ with the same projection $x$ on $\sigma$. Let $\tilde{Q}_x = \bigcup_{i=1}^{d} Q_{x_i}$. Then $\tilde{Q}_x \subset \Delta$.*

Using this Lemma we prove that any ceiling $\bar{C}$ is such that $C$ lies between two parallel planes which are parallel to $\sigma$ and such that the distance between them is $d$. The rest of Sect. 4 is the same except for some minor changes in the definition of the base of a wall. The changes in Sect. 5 are the following: we define $\Pi(W) = |W| - d|p(w)|$ and we have therefore $|p(w)| \leqq \frac{1}{d+1}|W|$ so that $\Pi(W) \geqq \frac{1}{d+1}|W|$ and $\Pi(W) \geqq p(W)$. So we obtain Theorem 6.1 for cubical hard cores of side $d$.

Suppose now that the hard-core between unlike particles is not a cube but is specified by a symmetric set like in model 2 of [6]. Then we shall obtain Theorem 6.1 if we can prove Lemma 4.2. There are, however, models for which this Lemma is not true and this is the case for model 1 of [6]. In this model if we have an $A$-particle at $x$ then it is impossible to have a $B$-particle at $y$ if $x$ and $y$ have a *face* in common. We can define the interface for this model as before and by modifying slightly the definition of the $\Gamma_i$ we have that a $\Gamma_i$ is either an $A$-border or a $B$-border. (With this new definition two different $\Gamma_i$ can however intersect each other, but they cannot have a face in common.) The main difference between this model and the "cube" model with $d \geqq 1$ is that the ceilings do not lie in planes parallel to $\sigma$ because Lemma 4.2 is false. We do not have a well defined base for a wall and we cannot describe the interface even by specifying all walls contained in it. In fact we do not expect to find a sharp interface. It is easy to see that if we choose $x=(x^1,0,0)$ with $x^1$ arbitrary, then we can find an interface $\bar{\Delta}\in B_L$ with $\Delta$ containing only $c$-points and $x\in\Delta$, if $L$ is large enough. The ratio of the probability of $\bar{\Delta}$ and the probability of the interface $\bar{\Delta}_0$ with $\Delta_0=\sigma$ converges to 1 when $L\to\infty$.

A study of the interface in this model might help in understanding what happens in the Widom-Rowlinson model in the continuous case where a similar phenomenon can occur. As mentioned in the Introduction, we are unable at the present time to either prove or disprove the existence of a sharp interface for such a system.

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## Appendix A. Proof of Lemma 3.1

The proof of this lemma follows the ideas of Martin-Löf [20] using correlation inequalities valid for the lattice Widom-Rowlinson model [21].

The statements of the lemma follow from the formula:

$$\log Z(\Lambda|A)=\int_{\mu}^{\infty}\frac{d}{d\mu'}\log Z(\Lambda,\mu'|A)d\mu'$$
$$=\sum_{t\in\Lambda}\int_{\mu}^{\infty}\langle(1-S_t^2)\rangle_{\Lambda,\mu'}d\mu'$$

and the estimate: $\exists\bar{\mu}$, $J$, $\alpha$ such that $\forall\mu>\bar{\mu}$, and any $\Lambda_1,\Lambda_2\subseteqq Z^3$ finite, $t_1\in\Lambda_1$, $t_2\in\Lambda_2$,

$$|\langle S_{t_1}^2\rangle_{\Lambda_1\mu}-\langle S_{t_2}^2\rangle_{\Lambda_2,\mu}|\leqq J\alpha\exp(-\alpha\mu d(t_1,\Lambda_1,t_2,\Lambda_2))\,; \tag{A.1}$$

$\langle\ \rangle_{\Lambda,\mu}$ denotes the expectation value with respect to the Gibbs measure in $\Lambda$ with $A$ boundary conditions.

By translation invariance we have only to consider $t_1=t_2$, since

$$d(t_1,\Lambda_1,t_2,\Lambda_2)=d(t_1,\Lambda_1,t_2,\Lambda_2-t_2+t_1).$$

We write

$$S_t^2=n_A(t)+n_B(t),$$

where

$$\begin{aligned} n_A(t)&=+1 && \text{if there is an } A \text{ particle at } t\\ &=0 && \text{otherwise,}\\ n_B(t)&=+1 && \text{if there is a } B \text{ particle at } t\\ &=0 && \text{otherwise.}\end{aligned}$$

We have only to show (A.1) with $n_A(t)$ and $n_B(t)$ instead of $S_t^2$. We use the fact that $n_A(t)$ and $n_B(t)$ are monotone functions in the sense of F.K.G. (see [21]) which implies:

$$\langle n_A(t)\rangle_{\Lambda_1}\geqq\langle n_A(t)\rangle_{\Lambda_2}, \tag{A.2}$$

$$\langle n_B(t)\rangle_{\Lambda_1}\leqq\langle n_B(t)\rangle_{\Lambda_2}, \tag{A.3}$$

if $t\in\Lambda_1\subseteqq\Lambda_2$.

With this and the definition of $d(t_1, \Lambda_1, t_2, \Lambda_2)$ we take for $\Lambda_1, \Lambda_2$, cubes centered at $t$ and $\Lambda_1 \subsetneqq \Lambda_2$. We show now $(A_1)$ for $n_A(t)$, the proof for $n_B(t)$ being similar.

Given some event $E$ in the configuration space in $\Lambda_2$, we call $\langle\ \rangle_{\Lambda_2,E}$ the conditional expectation with respect to $E$. $P(E)$ = probability of $E$

$$\langle n_A(t)\rangle_{\Lambda_2} = \langle n_A(t)\rangle_{\Lambda_2,E} P(E) + (1-P(E))\langle n_A(t)\rangle_{\Lambda_2,\bar{E}}, \tag{A.4}$$

where $\bar{E}$ = complement of $E$.

*Definitions.* A contour is a connected set of empty sites.

The interior of a contour is the union of the points in that contour and in the finite connected components of the complement of that contour.

An outer contour is a contour that is not contained in the interior of any other contour.

Let $E$ be the event: there exists an outer contour of empty sites which contains in its interior the point $t$, as well as a point of the boundary of $\Lambda_1$ (i.e. the points in $\Lambda_1$ adjacent to the complement of $\Lambda_1$).

Then, (A.1) follows from (A.2), (A.4) and the two points below:

$$\text{i)}\quad \langle n_A(t)\rangle_{\Lambda_2,\bar{E}} \geqq \langle n_A(t)\rangle_{\Lambda_1}, \tag{A.5}$$

$$\text{ii)}\quad P(E) \leqq C\exp(-\alpha\mu d(t,\partial\Lambda_1)), \tag{A.6}$$

$d(t,\partial\Lambda_1)$ = distance between $t$ and $\partial\Lambda_1 = \frac{1}{2}d(t,\Lambda_1,t,\Lambda_2)$.

To prove i) we notice that:

$$\langle n_A(t)\rangle_{\Lambda_2,\bar{E}} = \sum_{\{\gamma\}\in\bar{E}} \langle n_A(t)\rangle_{\Lambda_2,\{\gamma\}} P(\{\gamma\}|\bar{E}), \tag{A.7}$$

where the sum runs over the sets of outer contours containing points in the boundary of $\Lambda_1$ in their interior but not the point $t$, and over the empty set. $\langle\ \rangle_{\Lambda_2,\{\gamma\}}$ is the conditional expectation with respect to the set of configurations that have exactly $\{\gamma\}$ as outer contours containing points of the boundary of $\Lambda_1$ in their interior.

Now define $\Lambda_{\{\gamma\}} = \Lambda_1 \Big\backslash \Big(\bigcup_{\{\gamma\}} \text{Interior } \gamma\Big)$. For all the configurations entering the expectation $\langle\ \rangle_{\Lambda_2,\{\gamma\}}$, there is an $A$ particle at all the points in the boundary of $\Lambda_{\{\gamma\}}$ (i.e. inside $\Lambda_{\{\gamma\}}$ but adjacent to the complement of $\Lambda_{\{\gamma\}}$).

Indeed, the points adjacent to an outer contour must have an $A$ particle because otherwise the contour would be modified, and the points in the boundary of $\Lambda$ (which are in $\Lambda_{\{\gamma\}}$ if there is no contour in $\{\gamma\}$ containing them) must also have an $A$ particle, otherwise they would be contained in some contour. This is due to the pure $A$ boundary conditions on $\Lambda_2$ that forces all $B$ particles to be in the interior of some contour.

Given this, we have

$$\langle n_A(t)\rangle_{\Lambda_2,\{\gamma\}} \geqq \langle n_A(t)\rangle_{\Lambda_1}$$

because the l.h.s. is in fact the expectation in a region, $\Lambda_{\{\gamma\}}\backslash\{$the boundary of $\Lambda_{\{\gamma\}}\}$, contained in $\Lambda_1$, with pure $A$ b.c. and we use (A.2). This inequality and (A.7) implies (A.5).

The proof of (A.6) follows as in [20] once we have the following estimate on the probability of a contour:

$$P(\gamma) \leqq 2^{|\gamma|} e^{-\mu|\gamma|}. \tag{A.8}$$

Indeed, one can fill all the points in $\gamma$ with $A$ particles provided one first changes all the $B$ particles adjacent to $\gamma$ into $A$ particles. These $B$ particles can only be in the finite connected component of the complement of $\gamma$ and therefore the indeterminacy in the change of $B$ in $A$ is certainly less than $2^{|\gamma|}$ (see Lemma 3.3). On the other hand, filling with $A$ particles gives a factor $e^{-\mu|\gamma|}$.

*Remarks.* 1) One can prove (A.1) with any monotone function $f$ (in F.K.G. sense) instead of $n_A(t)$.

2) Inequality (A.8) gives an alternative proof of the result of [6] because any $B$ particle must be in the interior of some contour. For the continuum Widom-Rowlinson model, using the technique of Ruelle [9] one can define contours of empty sites and obtain inequality (A.8). This also gives an alternative proof of Ruelle's result.

## Appendix B[3]

Let us consider an Ising model on $\mathbb{Z}^d$, i.e. a spin $\sigma_x = \pm 1$ at each point and the Hamiltonian in a finite region $\Lambda$ given by $-H = \sum_{x,y\in\Lambda} J_{xy}\sigma_x\sigma_y$.

We assume the following properties of $J_{xy}$, $\forall x, y \in \mathbb{Z}^d$:

i) $J_{x,y} \geqq 0$ (ferromagnetism).

For

$$x = (x_1, \ldots, x_d)$$

let

$$\bar{x} = (-x_1, \ldots, x_d)$$

then

ii) $J_{xy} = J_{\bar{x}\bar{y}}$ (reflection invariance).

iii) $J_{xy} \geqq J_{x\bar{y}}$ for $x_1, y_1 \geqq 0$ (growth condition).

iv) $\sum_{y\in\mathbb{Z}^d} J_{xy} < \infty$, $\forall x \in \mathbb{Z}^d$.

Let

$$\Lambda = \{x \in \mathbb{Z}^d \mid |x_i| \leqq M,\ i = 1 \ldots d\}$$

$$\Lambda_+ = \{x \in \Lambda \mid x_1 > 0\},$$

$$\Lambda_- = \{x \in \Lambda \mid x_1 < 0\},$$

$$\Lambda_0 = \{x \in \Lambda \mid x_1 = 0\}.$$

3 This generalization of [4] was also noticed by H. van Beijeren (private communication)

We take the $\pm$ b.c. where, for $x \in \mathbb{Z}^d \backslash \Lambda$

$\tilde{\sigma}_x = +1$ if $x_1 \geqq 0$.

$\tilde{\sigma}_x = -1$ if $x_1 < 0$.

This gives an additional term in the Hamiltonian. Let $H'$ be the Hamiltonian of a system confined to the plane $x_1 = 0$ with the same interactions $J_{xy}$ in the plane and $+$ b.c.:

$$-H' = \sum_{x, y \in \Lambda_0} J_{xy} \sigma'_x \sigma'_y + \sum_{\substack{x \in \Lambda_0 \\ y \notin \Lambda \\ y_1 = 0}} J_{xy} \sigma'_x .$$

**Theorem.** *For the expectation values in the Gibbs states corresponding to $H$ (resp. $H'$) with $\pm$ (resp. $+$)* b.c.*:*

$$\forall x \in \Lambda_0, \quad \langle \sigma_x \rangle_{\Lambda, \pm} \geqq \langle \sigma'_x \rangle'_{\Lambda_0, +} .$$

*Proof*. We follow van Beijeren's argument [4] who considered nearest neighbor interactions and notice, as in [16], that the proof actually goes through for the above Hamiltonian.

Let

$s_x = \sigma_x + \sigma_{\bar{x}}$,

$t_x = \sigma_x - \sigma_{\bar{x}}$ for $x \in \Lambda_+$,

$s_x = \sigma_x + \sigma'_x$,

$t_x = \sigma_x - \sigma'_x$ for $x \in \Lambda_0$.

We have to show:

$\langle t_x \rangle \geqq 0 \quad x \in \Lambda_0$

for the expectation with the Hamiltonian [that we write using ii)]:

$$\begin{aligned} -H(\sigma, \sigma') = & \sum_{x, y \in \Lambda_+} \frac{J_{xy}}{2} (s_x s_y + t_x t_y) \\ & + \frac{J_{x\bar{y}}}{2} (s_x s_y - t_x t_y) \\ & + \sum_{\substack{x \in \Lambda_+ \\ y \in \Lambda_0}} \frac{J_{xy}}{2} s_x (s_y + t_y) \\ & + \sum_{x, y \in \Lambda_0} \frac{J_{xy}}{2} (s_x s_y + t_x t_y) \\ & + \sum_{x \in \Lambda_+} \sum_{\substack{y \notin \Lambda \\ y_1 > 0}} (J_{xy} - J_{x\bar{y}}) t_x \\ & + \sum_{x \in \Lambda_+} \sum_{\substack{y \notin \Lambda \\ y_1 = 0}} J_{xy} s_x + \sum_{x \in \Lambda_0} \sum_{\substack{y \notin \Lambda \\ y_1 = 0}} J_{xy} s_x \\ & + \sum_{x \in \Lambda_0} \sum_{\substack{y \notin \Lambda \\ y_1 > 0}} \frac{(J_{xy} - J_{x\bar{y}})}{2} (s_x + t_x) . \end{aligned}$$

Using conditions i) and iii), we see that $H(\sigma, \sigma')$ is a polynomial with positive coefficients in $s_x$, $t_x$ and therefore the positivity of $\langle t_x \rangle$ follows as usual.

*Remarks.* 1) With this theorem one can deduce, as in [4], the existence of non-translation invariant Gibbs states for $d=3$ whenever the corresponding system with $d=2$ exhibits spontaneous magnetization.

2) The theorem still holds if $\sigma_x$, instead of being $= \pm 1$, is distributed with an even measure on $\mathbb{R}$.

3) For the Widom-Rowlinson model, the hard-core (or the many-body forces in the one-component version) destroys the ferromagnetic nature of the interaction in the $s_x, t_x$ variables used in the proof.

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