

Operator Product Expansions in Conformally Covariant Quantum Field Theory

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Abstract. Operator products in quantum field theory on two-dimensional Minkowski space are expanded into a series of local operators by means of the tensor product decomposition theorem for representations of the conformal group. The Thirring model is used as an explicit example. Two types of expansions result. If the operator product acts on the vacuum state, we obtain strictly covariant expansions. In general however, each term in the expansion is only semicovariant.

1. Introduction

Operator product expansions in conformal invariant quantum field theory have been studied [1–4] as a means to get insight into operator product expansions of more general and physically more relevant quantum field theories. In fact the requirement of conformal invariance restricts the structure of quantum field theory very strongly. This supports the hope that some problems of quantum field theory may become solvable by a nonperturbative construction in such models.

The general treatment of operator product expansions (“Wilson expansions” [5]) has been to multiply two local operators $A(x)$ and $B(y)$, guess an infinite set of other local operators $C_n(z)$, make an ansatz for the expansion of $A(x)B(y)$ in terms of the $C_n(z)$ in the sense of an asymptotic expansion for $x \rightarrow y$, and impose the requirement of conformal invariance on this expansion at the end.

Our approach is different. We intend to apply the reduction theorem of tensor products of representations for the conformal group into irreducible representations to this problem. Thus we want to find all terms in the expansion by construction. The idea is to guarantee “completeness” of the operators $C_n(z)$ this way. Of course this construction can only be matrix-element-wise and representation by representation (each local operator belongs to an infinite number of representations corresponding to different charge sectors). At the end there remains the problem to identify these matrix elements as elements of a known local field operator.

The main purpose of this work is to show that such conformally covariant or semi-covariant expansion exist and to study the difficulties connected with them. In order to give the operator product and its singular structure a well defined meaning, we consider a special solvable model in two-dimensional Minkowski space: the Thirring model. Two-dimensional Minkowski space has the advantage of possessing a small conformal group, namely the universal covering group of $SU(1, 1) \otimes SU(1, 1)$ whose representations and tensor product decomposition have been studied earlier [6]. This conformal group is sufficiently small to render algebraic calculations feasible.

In Section 2 we describe the Thirring model as a local field on the universal covering space of compactified Minkowski space that seems to us the most elegant presentation for our purposes. A special realization turns out to be particularly useful. In Sections 3 and 4 we solve the operator product expansion problem on the vacuum sector.

After deriving a completeness relation for the covariant kernels of the discrete series that was not given in Ref. [6] we apply it to the regularized product of two field operators $f_\gamma^\dagger(\psi_1)$ and $f_\gamma^\dagger(\psi_2)$. It results an expansion into a series of local operators with increasing dimension, each term of which is conformally covariant on the *full* conformal group ("strictly covariant Wilson expansion"). In Sections 5 and 6 we treat nonvacuum sectors. We are led to introduce semicovariant kernels that are infinitesimally but not globally covariant. A completeness relation for such kernels is derived from the general completeness relation of Ref. [6]. It is shown that this completeness relation if applied to regularized bilocal operators always leads to a Wilson expansion into local operators, each term of which is semicovariant under the conformal group. In Section 6 we study the regularized operator product of $f_\gamma(\phi)$ and $f_\gamma^\dagger(\psi)$. Because this bilocal operator is singular in a representation theoretic sense, our formulas must be adapted to this case first.

We use the notations of Ref. [6], in particular the abbreviations α_{lmn} , β_{mn} that are explained in the table of that article. The C -coefficients defined in Ref. [6] are normalized so that $N = N^d = (2\pi)^{-3}$.

2. Remarks on the Thirring Model

The Thirring model has been defined originally [7] by a field equation

$$-i\gamma^\mu \partial_\mu \phi(x) = g\gamma^\mu [J_\mu^{(-)}(x)\phi(x) + \phi(x)J_\mu^{(+)}(x)] \quad (1)$$

where $\phi(x)$ is a twocomponent field (whose components are labelled by $\gamma = 1, 2$) of arbitrary spin s and dimension d . $J_\mu(x)$ is the current which in two-dimensional Minkowski space is a free zeromass vector field and can thus be decomposed into a negative and a positive frequency part. J_μ and ϕ are operators in the Fock space of the free spin 1/2 zero-mass field. Solving the model means constructing these operators explicitly so that (1) is fulfilled.

This solution has been achieved by Klaiber in his classic paper [8]. A bigger class of solutions that are all isomorphic to Klaiber's have been constructed later by Dell'Antonio et al. [9] and by Kupsch et al. [10]. Dell'Antonio's solutions are parametrized by a pair of functions $\tilde{J}_\pm(p)$ of a certain class. This class of functions

describes an equivalence class of irreducible representations of the operator algebra of the zero-mass vector field $J_\mu(x)$. Each equivalence class belongs to a definite charge sector, i.e. an eigenspace of the charge Q and the axial charge \tilde{Q} . The Fock class belongs to the vacuum sector, the non-Fock classes to non-vanishing charge eigenvalues.

The class of solutions constructed by Kupsch et al. [10] is a subclass of the Dell'Antonio solutions that is parametrized by two complex numbers w_\pm , $\text{Im } w_\pm > 0$. In fact we find

$$\tilde{J}_\pm(p) = e^{ipw_\pm}. \quad (2)$$

In Ref. [10] the Thirring model has been investigated in order to establish its conformal covariance. The labels w_\pm are conformally invariant [thus the relation (2) may be not quite correct, since the conformal transformation property of the functions $\tilde{J}_\pm(p)$ is unknown].

The two charges Q, \tilde{Q} can be combined into the two renormalized chiral charge operators

$$Q_\pm = -\frac{1}{2}[(\alpha-1)Q \mp (\beta-1)\tilde{Q}] \quad (3)$$

where α, β are arbitrary real parameters introduced by Klaiber [8]. They allow us to express the three constants of the model g, d , and s . In fact, as auxiliary quantities we introduce C_\pm by

$$Q_\pm \phi(x) = \phi(x)(Q_\pm + C_\pm) \quad (4)$$

$$C_\pm = +\frac{1}{2}[(\alpha-1) \mp (\beta-1)\gamma^5] \quad (5)$$

$[\gamma^5]$ is a diagonal matrix with the elements $(-1)^v$ and

$$N_\pm = C_\pm^2. \quad (6)$$

Then g, d , and s are

$$g = \pi(\alpha - \beta) \quad (7)$$

$$d = \frac{1}{2}(N_+ + N_-) = \frac{1}{4}[(\alpha-1)^2 + (\beta-1)^2] \quad (8)$$

$$s = \frac{1}{2}|\alpha-1| |\beta-1|. \quad (9)$$

For technical reasons it is advantageous to replace the variables x_\pm by angles φ_\pm :

$$x_\pm = x^0 \pm x^3 = \text{tg}(\varphi_\pm/2), \quad -\pi < \varphi_\pm < +\pi \quad (10)$$

$$f(\varphi) = (\cos(\varphi_+/2))^{-N_+} (\cos(\varphi_-/2))^{-N_-} \phi(x(\varphi)). \quad (11)$$

Then we extend the field operators onto the infinitely sheeted universal covering space of compactified Minkowski space [11], that is: we let φ_\pm assume values from $-\infty$ to $+\infty$ by

$$f(\varphi_\pm + 2\pi, \varphi_\mp) = e^{i\pi Q_\pm^2} f(\varphi) e^{-i\pi Q_\mp^2}. \quad (12)$$

This definition allows us to present the conformal transformation as

$$U_g f(\varphi) U_g^{-1} = \prod_{\pm} \{ |\alpha_\pm e^{i\varphi_\pm} + \beta_\pm|^{-N_\pm} \} f(\varphi_g). \quad (13)$$

Namely in a "local" form. The Wightman functions for these field operators are boundary values of analytic functions of the variables $\varphi_{i\pm} - \varphi_{j\pm}$ in the lower half plane.

Restricting the field operator to a charge sector (n_+, n_-)

$$Q_{\pm} \Pi(n_+, n_-) = n_{\pm} \Pi(n_+, n_-) \quad (14)$$

yields

$$f(\varphi_{\pm} + 2\pi, \varphi_{\mp}) \Pi(n_+, n_-) = e^{i\pi(2C_{\pm} n_{\pm} + N_{\pm})} f(\varphi) \Pi(n_+, n_-). \quad (15)$$

Thus $f(\varphi) \Pi(n_+, n_-)$ is a covariant operator belonging to the representation

$$\chi_+ \otimes \chi_- = (j_+, \tau_+) \otimes (j_-, \tau_-)$$

of the conformal group [6] with

$$\begin{aligned} 2j_{\pm} - 1 &= -N_{\pm} \\ \tau_{\pm} &\doteq C_{\pm} n_{\pm} + \frac{1}{2} N_{\pm} \pmod{1}. \end{aligned} \quad (16)$$

This behaviour, namely that only projections of local operators on charge sectors are conformally covariant is typical for conformal covariant field theories [4, 11, 12].

The operator f depends on α and β . For $\alpha = \beta = 1$ we obtain the σ -field (a pathological case)

$$\sigma(\varphi) = f(\varphi)|_{\alpha=\beta=1}. \quad (17)$$

For this we need the explicit operator form of $f(\varphi)$ [10]. We apply a Klein transformation

$$\sigma(\varphi) = (2\pi)^{-\frac{1}{2}} \exp i(\pi/2)(Q + \tilde{Q}) \tilde{\sigma}(\varphi). \quad (18)$$

The operator field $\tilde{\sigma}_{\gamma}(\varphi)$ is then a constant unitary operator, e.g.

$$\tilde{\sigma}_{\gamma}(\varphi) \tilde{\sigma}_{\gamma}(\varphi)^{\dagger} = \mathbb{1} \quad (19)$$

$\tilde{\sigma}$ (the same as σ) is not invariant under conformal transformations [see (33)].

In a lightlike basis the vector field $J_{\mu}(\varphi)$ has the components $J_+(\varphi_+)$ and $J_-(\varphi_-)$. J_+ does not depend on φ_- etc. In the canonical basis [10] we can decompose the components

$$\begin{aligned} J_{\pm}(\varphi_{\pm}) &= \pi^{-1} \sum_{m=0}^{\infty} c_{\pm, m}^{\dagger} (m+1)^{\frac{1}{2}} e^{i(m+1)\varphi_{\pm}} \\ &+ \text{h.c.} + \pi^{-1} Q_{\pm} = J_{\pm}^{(-)}(\varphi_{\pm}) + J_{\pm}^{(+)}(\varphi_{\pm}) + \pi^{-1} Q_{\pm}. \end{aligned} \quad (20)$$

By definition the positive (negative) frequency part involves the negative (positive) powers of $e^{i\varphi_{\pm}}$. The simple form (20) is obtained from Equations (3.57), (3.72), Ref. [10], and (11) by putting

$$w_{\pm} = +i. \quad (21)$$

This realization of the Thirring field turns out to be particularly simple, namely σ commutes with both creation operators $c_{\pm, m}^{\dagger}$ and annihilation operators $c_{\pm, m}$

$$[c_{\pm, m}, \sigma_{\gamma}] = [c_{\pm, m}^{\dagger}, \sigma_{\gamma}] = 0. \quad (22)$$

Thus $J_{\pm}(\varphi_{\pm}) - \pi^{-1}Q_{\pm}$ commutes with the σ -field.

We introduce the source operators

$$\Gamma_{\pm}^{(-)}(\varphi_{\pm}) = \pi \int_{+i\infty}^{\varphi_{\pm}} d\chi J_{\pm}^{(-)}(\chi) = \Gamma_{\pm}^{(+)}(\varphi_{\pm})^{\dagger}. \quad (23)$$

Then the Thirring fields can be presented in the form

$$f_{\gamma}(\varphi) = 2^d \exp i \sum_{\pm} C_{\pm} [\Gamma_{\pm}^{(-)}(\varphi_{\pm}) + \frac{1}{2}Q_{\pm}\varphi_{\pm}] \\ \times \sigma_{\gamma} \times \exp i \sum_{\pm} C_{\pm} [\Gamma_{\pm}^{(+)}(\varphi_{\pm}) + \frac{1}{2}Q_{\pm}\varphi_{\pm}]. \quad (24)$$

A standard product of operators can be regularized by extraction of a singular factor

$$f_{\gamma_1}(\varphi_1) f_{\gamma_2}(\varphi_2) \dots f_{\gamma_n}(\varphi_n) f_{\gamma'_1}^{\dagger}(\psi_1) \dots f_{\gamma'_m}^{\dagger}(\psi_m) \\ = \prod_{\pm} \left\{ \prod_{i < j} [2i \sin \frac{1}{2}(\varphi_{i\pm} - \varphi_{j\pm} - i0)]^{C_{\pm}^i C_{\pm}^j} \prod_{i, j'} [2 \sin \frac{1}{2}(\varphi_{i\pm} - \psi_{j'\pm} - i0)]^{-C_{\pm}^i C_{\pm}^{j'}} \right. \\ \left. \cdot \prod_{i' < j'} [2i \sin \frac{1}{2}(\psi_{i'\pm} - \psi_{j'\pm} - i0)]^{C_{\pm}^{i'} C_{\pm}^{j'}} \right\} R_{\{\gamma\}_n; \{\gamma'\}_m} [\{\varphi\}_n; \{\psi\}_m] \quad (25)$$

with

$$R_{\{\gamma\}_n; \{\gamma'\}_m} [\{\varphi\}_n; \{\psi\}_m] \\ = 2^{(n+m)d} \exp i \left\{ \sum_{\pm, i} C_{\pm}^i [\Gamma_{\pm}^{(-)}(\varphi_{i\pm}) + \frac{1}{2}Q_{\pm}\varphi_{i\pm}] \right. \\ \left. - \sum_{\pm, j'} C_{\pm}^{j'} [\Gamma_{\pm}^{(-)}(\psi_{j'\pm}) + \frac{1}{2}Q_{\pm}\psi_{j'\pm}] \right\} \sigma_{\gamma_1} \sigma_{\gamma_2} \dots \sigma_{\gamma'_1}^{\dagger} \dots \sigma_{\gamma'_m}^{\dagger} \\ \times \exp i \{\text{positive frequencies}\}. \quad (26)$$

We start the investigation of conformal covariance with the currents. The transformation behaviour of $J_{\pm}(\varphi_{\pm})$ is

$$U_g J_{\pm}(\varphi_{\pm}) U_g^{-1} = |\alpha_{\pm} e^{i\varphi_{\pm}} + \beta_{\pm}|^{-2} J_{\pm}(\varphi_{g, \pm}), \quad J_{\pm}(\varphi_{\pm} + 2\pi) = J_{\pm}(\varphi_{\pm}). \quad (27)$$

Thus $J_{+}(\varphi_{+})(J_{-}(\varphi_{-}))$ transforms apparently like a vector of

$$\mathcal{D}_{\chi_{+}} \otimes \mathcal{D}_{\chi_{-}}, \chi_{\pm} = (j_{\pm}, \tau_{\pm})$$

with [compare (13), (15), (16)]

$$\frac{1}{2} - j_{+} - \tau_{+} = 1(0), \tau_{+} = 0 \\ \frac{1}{2} - j_{-} - \tau_{-} = 0(1), \tau_{-} = 0. \quad (28)$$

$\mathcal{D}_{\chi_{+}}$ and $\mathcal{D}_{\chi_{-}}$ possess invariant subspaces $\mathcal{F}_{\chi_{+}}^{(+)}$ and $\mathcal{F}_{\chi_{-}}^{(+)}$ [6] spanned by the elements $e^{iq_{\pm}\varphi_{\pm}}$ with

$$\left. \begin{aligned} \frac{1}{2} - j_{+} - q_{+} &= -m_{+} \\ \frac{1}{2} - j_{-} - q_{-} &= -m_{-} \end{aligned} \right\} m_{\pm} = 0, 1, 2, \dots \quad (29)$$

or

$$q_{+} = 1, 2, 3, \dots \quad (=0, 1, 2, \dots) \\ q_{-} = 0, 1, 2, \dots \quad (=1, 2, 3, \dots). \quad (30)$$

At the same time (because τ_+ and τ_- are zero) \mathcal{D}_{χ_+} and \mathcal{D}_{χ_-} possess invariant subspaces $\mathcal{F}_{\chi_+}^{(-)}$ and $\mathcal{F}_{\chi_-}^{(-)}$ spanned by the canonical basis vectors with the opposite sign of q_{\pm} as (30). Independence of $J_+(\varphi_+)$ on φ_- means in fact, that it lies in

$$\mathcal{F}_{\chi_-}^{(+)} \cap \mathcal{F}_{\chi_-}^{(-)}$$

($q_- = 0$) with respect to φ_- . From now on we shall neglect the superfluous variable in such cases.

Despite the fact that

$$J_{\pm}^{(-)}(\varphi_{\pm}) \in \mathcal{F}_{\chi_{\pm}}^{(+)}$$

as the expansion (20) of $J_{\pm}^{(-)}(\varphi_{\pm})$ shows, we obtain

$$\begin{aligned} U_g J_{\pm}^{(-)}(\varphi_{\pm}) U_g^{-1} &= |\alpha_{\pm} e^{i\varphi_{\pm}} + \beta_{\pm}|^{-2} J_{\pm}^{(-)}(\varphi_{g\pm}) \\ &\quad + \pi^{-1} [\bar{\alpha}_{\pm}(\bar{\alpha}_{\pm} + \bar{\beta}_{\pm} e^{i\varphi_{\pm}})^{-1} - 1] Q_{\pm}. \end{aligned} \quad (31)$$

Thus $J_{\pm}^{(-)}(\varphi_{\pm})$ is not properly covariant. This is due to the fact that U_g does not act on the basis vectors but on the operator coefficients. It is impossible to define another negative frequency part which would not have such second term in (31) involving the charge operator. Of course (31) is compatible with (27) and the conformal invariance of the charge operators.

The subtraction point $i\infty$ in (23) is not conformally invariant. It goes into a point ζ_g

$$e^{i\zeta_g} = (\alpha e^{i\varphi} + \beta)(\bar{\beta} e^{i\varphi} + \bar{\alpha})^{-1} |_{\varphi=i\infty} = \beta/\bar{\alpha}. \quad (32)$$

In order that the Thirring field (24) is covariant we must have compensating terms from the σ -field, namely

$$\begin{aligned} U_g \sigma_{\gamma} U_g^{-1} &= \prod_{\pm} \{ |\alpha_{\pm}|^{-N_{\pm}} \exp i C_{\pm} [\Gamma_{\pm}^{(-)}(\zeta_g) + Q_{\pm} \arg \alpha_{\pm}] \} \\ &\quad \times \sigma_{\gamma} \prod_{\pm} \{ \exp i C_{\pm} [\Gamma_{\pm}^{(+)}(\bar{\zeta}_g) + Q_{\pm} \arg \alpha_{\pm}] \}. \end{aligned} \quad (33)$$

Moreover (12) and (24) are compatible if σ is constant on the whole covering space (and not only on one sheet).

The multilocal operators (26) transform as

$$\begin{aligned} U_g R_{\{\gamma\}_n; \{\gamma\}_m} [\{\varphi\}_n; \{\psi\}_m] U_g^{-1} &= \prod_{\pm} \left\{ \prod_i |\alpha_{\pm} e^{i\varphi_{i\pm}} + \beta_{\pm}|^{-C_{\pm}^i \Sigma_{\pm}} \right. \\ &\quad \times \left. \prod_{j'} |\alpha_{\pm} e^{i\varphi_{j'\pm}} + \beta_{\pm}|^{C_{\pm}^{j'} \Sigma_{\pm}} \right\} R_{\{\gamma\}_n; \{\gamma\}_m} [\{\varphi_g\}_n; \{\psi_g\}_m] \end{aligned} \quad (34)$$

with

$$\sum_{\pm} = \sum_i C_{\pm}^i - \sum_{j'} C_{\pm}^{j'}. \quad (35)$$

3. A Completeness Relation for Discrete Series Representations

We consider the operator product

$$f_{\gamma}^{\dagger}(\varphi_1) f_{\gamma}^{\dagger}(\varphi_2).$$

There is no special motivation to study just this product. It is a simple but by no means trivial choice. This product can be regularized as described in the preceding section [Eq. (25)]. Under this regularization a conformally covariant Wilson expansion of the unregularized product goes into a conformally covariant expansion of the regularized product. Thus it suffices to treat the regularized product.

If it acts on the vacuum we obtain from (26)

$$\begin{aligned} & R_{;\gamma\gamma}[\varphi_1, \varphi_2]|0\rangle \\ &= 2^{2d} \exp \left\{ -i \sum_{\substack{i=1,2 \\ \pm}} C_{\pm} [\Gamma_{\pm}^{(-)}(\varphi_{i\pm}) - C_{\pm} \varphi_{i\pm}] \right\} \\ & \quad \times (\sigma_{\gamma}^{\dagger})^2 |0\rangle. \end{aligned} \quad (36)$$

Applying the conformal transformation (34) this state vector transforms as a representation

$$\prod_{\pm} \{\chi_{1\pm} \otimes \chi_{2\pm}\}$$

with the parameters

$$\begin{aligned} 2j_{1\pm} - 1 &= 2j_{2\pm} - 1 = -2N_{\pm} \\ \tau_{1\pm} &= \tau_{2\pm} \hat{=} +N_{\pm} \bmod 1. \end{aligned} \quad (37)$$

We assume that $2N_{\pm}$ are non-integral.

The two representations $\chi_{1\pm}$ and $\chi_{2\pm}$ are equivalent in this case. Both admit invariant subspaces of the type $\mathcal{F}_{\chi}^{(+)}$. These are spanned by the canonical basis elements with

$$\frac{1}{2} - j_{i\pm} - q_{i\pm} = -m_{i\pm}, m_{i\pm} = 0, 1, 2, \dots \quad (38)$$

Thus the state vector (36) lies in the representation space

$$\prod_{\pm} \{\mathcal{F}_{\chi_{1\pm}}^{(+)} \otimes \mathcal{F}_{\chi_{2\pm}}^{(+)}\}$$

and belongs to a tensor product of four discrete series representations. That is characteristic for acting with the operator product on the vacuum state.

We want to reduce this product representation (36) into irreducible representations. The reduction formula of Ref. [6] is not applicable. Therefore we use the formalism developed in Ref. [6] to derive in brief the correct formula. The starting point is the Burchnell-Chaundy expansion [6]

$$e_{q_1 q_1}^{-j_1}(z) e_{q_2 q_2}^{-j_2}(z) = \sum_{k=0}^{\infty} d_k e_{q_3 q_3}^{+j_3(k)}(z) \quad (39)$$

with

$$\begin{aligned} q_1 + q_2 &= q_3, \quad q'_1 + q'_2 = q'_3 \\ j_3(k) &= \frac{1}{2} - j_1 - j_2 + k. \end{aligned} \quad (40)$$

The d_k will be given below after some simplifications [that are not yet possible in (39)]. χ_1 and χ_2 are assumed to possess invariant subspaces of the type $\mathcal{F}_{\chi}^{(+)}$.

We choose the subscripts q_i, q'_i to belong to these subspaces. Then

$$e_{q_1 q'_1}^{-j_1}(z) = d_{q_1 q'_1}^{+j_1}(z). \quad (41)$$

The representation $\chi_3(k) = (j_3(k), \tau_3)$ admits an invariant subspace $\mathcal{F}_{\chi_3}^{(-)}$. We find

$$e_{q_3 q'_3}^{j_3(k)}(z) = d_{q_3 q'_3}^{-j_3(k)}(z) \quad (42)$$

if both q_3, q'_3 belong to $\mathcal{D}_{\chi_3}/\mathcal{F}_{\chi_3}^{(-)}$ and zero else. Thus the sum on the r.h.s. of (39) is restricted to the finite number of terms

$$j_3(k) \leq \min(q_3, q'_3) - \frac{1}{2}. \quad (43)$$

Inserting (41), (42) into (39) and going to $z=1$ we end up with

$$\delta_{q_1 q'_1} \delta_{q_2 q'_2} = \delta_{q_3 q'_3} \sum_{k=0}^{m_1+m_2} d_k. \quad (44)$$

Due to the Kronecker deltas in (44) we can set $q_3 = q'_3$ and in all common factors of the sum $q_1 = q'_1, q_2 = q'_2$. This gives a simplified expression for d_k

$$d_k = 2j_3(k)(-1)^{q_2 - \tau_2} \pi^2 [\sin \pi(\frac{1}{2} - j_2 + \tau_2) \sin \pi 2j_3(k)] F_p(5) F_n(5). \quad (45)$$

Our aim is to rewrite (44) in the form of a completeness relation for covariant trilinear kernels and their duals. First we need a kernel that maps $\mathcal{F}_{\chi_1}^{(+)} \otimes \mathcal{F}_{\chi_2}^{(+)}$ into \mathcal{D}_{χ_3} . We prefer χ_3^c over χ_3 because the dimension of the operator with covariance χ_3^c increases with k [see Eq. (58)] as is required for a Wilson expansion into local operators. The dimension of a local operator is namely bounded from below by positivity, whereas arbitrary high dimensions can be created by derivations.

In order to accomodate the form (45) we choose as C -coefficient (that is: the matrix element of a covariant trilinear kernel in the canonical basis)

$$\begin{aligned} C(\chi_3^c(k), q_1 + q_2 | \chi_1, q_1; \chi_2, q_2) \\ = \Gamma(\alpha_{023}) \Gamma(\alpha_{035}) \Gamma(\alpha_{245}) \Gamma(\alpha_{345}) F_p(1) / \Gamma(\alpha_{135}) \end{aligned} \quad (46)$$

with

$$\chi_3^c(k) = (-j_3(k), \tau_3), j_3(k) = \frac{1}{2} - j_1 - j_2 + k. \quad (47)$$

This is obtained from the coefficient C_1 (Ref. [6], Eq. (71a)) by a renormalization and is therefore properly covariant

$$\begin{aligned} C(\chi_3^c(k), q_1 + q_2 | \chi_1, q_1; \chi_2, q_2) \\ = \lim \pi^2 [\sin \pi(\frac{1}{2} - j_2 + \tau_2) \sin \pi(\frac{1}{2} + j_3 - \tau_1 - \tau_2) \Gamma(\alpha_{024}) \Gamma(\alpha_{135})]^{-1} \\ \times C_1(\chi_3^c, q_1 + q_2 | \chi_1, q_1; \chi_2, q_2) \\ = \lim -\pi^2 [\sin \pi(\frac{1}{2} - j_2 + \tau_2) \sin \pi(j_1 + j_2 + \tau_1 + \tau_2)]^{-1} \\ \times \operatorname{Res}_{j_3 = j_3(k)} C_1(\chi_3^c, q_1 + q_2 | \chi_1, q_1; \chi_2, q_2). \end{aligned} \quad (48)$$

The limit is understood as follows. Keep τ_1 and τ_2 off the discrete series values first (i.e. $\frac{1}{2} - j_i - \tau_i$ are nonintegral) and set $j_3 = j_3(k)$. Then go with τ_1 and τ_2 to the prescribed values.

The C -coefficient (46) has a remarkable symmetry property. If $\chi_1 = \chi_2$ it is symmetric (antisymmetric) in q_1 and q_2 for even (odd) k . Moreover it can be shown to map $\mathcal{F}_{\chi_1}^{(+)} \otimes \mathcal{F}_{\chi_2}^{(+)}$ into $\mathcal{F}_{\chi_3^c(k)}^{(+)}$. Thus for $\chi_1 = \chi_2$ the symmetry (antisymmetric) part of the tensor product is mapped into the direct sum (over k) of $\mathcal{F}_{\chi_3^c(k)}^{(+)}$ with even (odd) k .

Another property of the C -coefficient (46) is of utmost importance: It is a polynomial in q_1 and q_2 . Consequently the trilinear covariant kernel corresponding to it is a differential operator. In fact, we use the realization $F_p(1) = F_p(1; 0, 2)$ to rewrite (46) as

$$C = (\Gamma(\alpha_{023})\Gamma(\alpha_{245})/\Gamma(\alpha_{135})\Gamma(\alpha_{235}))Q_k(q_1, q_2) \quad (49)$$

where $Q_k(q_1, q_2)$ is the polynomial

$$Q_k(q_1, q_2) = \sum_{m=0}^k (-1)^m \binom{k}{m} (2j_1 - k)_m \left(\frac{1}{2} - j_2 - q_2\right)_m \\ \times (2j_2 - k)_{k-m} \left(\frac{1}{2} - j_1 - q_1\right)_{k-m}. \quad (50)$$

Both the symmetry of C and its property to map $F_{\chi_1}^{(+)} \otimes F_{\chi_2}^{(+)}$ into $F_{\chi_3(k)}^{(+)}$ can be inspected from this expression (50) for $Q_k(q_1, q_2)$.

If K is the kernel corresponding to C and if $(\varphi_1, \varphi_2) \in \mathcal{F}_{\chi_1}^{(+)} \otimes \mathcal{F}_{\chi_2}^{(+)}$

$$\int d\varphi_1 d\varphi_2 K(\chi_3^c(k), \varphi_3 | \chi_1, \varphi_1; \chi_2, \varphi_2) g(\varphi_1, \varphi_2) \\ = 2\pi (\Gamma(\alpha_{023})\Gamma(\alpha_{245})/\Gamma(\alpha_{135})\Gamma(\alpha_{235})) \\ \times \{Q_k(-i\partial/\partial\varphi_1, -i\partial/\partial\varphi_2)g(\varphi_1, \varphi_2)\}|_{\varphi_1=\varphi_2=\varphi_3}. \quad (51)$$

As dual kernel we use $K_3^d(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3^c(k), \varphi_3)$ (Ref. [6], Eq. (107)). It has a symmetry property related to that of K (46). If k is even (odd) and $\chi_1 = \chi_2$, the image of $\mathcal{F}_{\chi_3^c(k)}^{(+)}$ consists of symmetric (antisymmetric) functions of $\mathcal{F}_{\chi_1}^{(+)} \otimes \mathcal{F}_{\chi_2}^{(+)}$. With C , (46), and C_3^d we can write

$$d_k = 2j_3(k)(-1)^{\frac{1}{2}-j_2-\tau_2} C_3^d(\chi_1, q_1; \chi_2, q_2 | \chi_3^c(k), q_1 + q_2) \\ \times C(\chi_3^c(k), q_1 + q_2 | \chi_1, q_1; \chi_2, q_2). \quad (52)$$

Introducing the integral kernels themselves into (44) we have

$$S'_1(\varphi'_1 - \varphi_1)S'_2(\varphi'_2 - \varphi_2) = (2\pi)^3 \sum_{k=0}^{\infty} 2j_3(k) \int_0^{2\pi} d\varphi (-1)^{\frac{1}{2}-j_2-\tau_2} \\ \times K_3^d(\chi_1, \varphi'_1; \chi_2, \varphi'_2 | \chi_3^c(k), \varphi) \int_0^{2\pi} d\psi_1 \int_0^{2\pi} d\psi_2 K(\chi_3^c(k), \varphi | \chi_1, \psi_1; \chi_2, \psi_2) \\ \times S'_1(\psi_1 - \varphi_1)S'_2(\psi_2 - \varphi_2). \quad (53)$$

Here we made use of the intertwining operators $S'(\chi_i^c \rightarrow \chi_i) = S'_i(\varphi_i)$ defined by (Ref. [6], Eq. (37))

$$S'_i(\varphi_i) = \Gamma(1 - 2j_i)(2\pi)^{-1} [-2i \sin \frac{1}{2}(\varphi_i + i0)]^{+2j_i-1} \quad (54)$$

that serve as projection operators onto $\mathcal{F}_{\chi_i}^{(+)}$. It is the kernel K that restricts the space $\mathcal{D}_{\chi_3^c(k)}$ to $\mathcal{F}_{\chi_3^c(k)}^{(+)}$ in (53).

4. A Strictly Covariant Wilson Expansion

We apply the results of the preceding section to the operator $R;_{\gamma\gamma}[\varphi_1, \varphi_2]$. First we define

$$O_{k_+k_-}(\psi) = \prod_{\pm} \{Q_{k_{\pm}}(-i\partial/\partial\varphi_{1\pm}, -i\partial/\partial\varphi_{2\pm})\} R;_{\gamma\gamma}[\varphi_1, \varphi_2]|_{\varphi_{1\pm}=\varphi_{2\pm}=\psi_{\pm}}. \quad (55)$$

This operator is local as a finite derivative of a multilocal operator. Our construction guarantees that its projection on the vacuum charge sector

$$O_{k_+k_-}(\psi)\Pi(O, O) \quad (56)$$

is covariant according to the representation

$$\chi_{3+}^e(k_+) \otimes \chi_{3-}^e(k_-). \quad (57)$$

Projections on $\Pi(n_+, n_-)$ are also covariant with j_{\pm} the same as in (57) and τ_{\pm} the same as for the projection of $R;_{\gamma\gamma}[\psi, \psi]$ itself. The dimension of this operator is

$$d = d_1 + d_2 + k_+ + k_- \quad (58)$$

$$d_1 = d_2 = N_+ + N_-.$$

The explicit form of the operator $O_{k_+k_-}(\psi)$ is

$$\begin{aligned} O_{k_+k_-}(\psi) = & \prod_{\pm} \{Q_{k_{\pm}}(-\pi C_{\pm} J_{\pm}(\psi_{\pm}), -\pi C_{\pm} J_{\pm}(\psi_{\pm}))\} \\ & R;_{\gamma\gamma}[\psi, \psi]: \\ & + \text{derivative terms.} \end{aligned} \quad (59)$$

The normal product means: Place $J_{\pm}^{(-)}(\psi_{\pm}) + (2\pi)^{-1}Q_{\pm}$ to the left and $J_{\pm}^{(+)}(\psi_{\pm}) + (2\pi)^{-1}Q_{\pm}$ to the right of $R;_{\gamma\gamma}[\psi, \psi]$.

Inserting the operator $O_{k_+k_-}(\psi)$ into the completeness relation (53) yields

$$\begin{aligned} R;_{\gamma\gamma}[\varphi, \varphi](O, O) = & (2\pi)^6 \sum_{k_{\pm}=0}^{\infty} \prod_{\pm} \{2j_{3\pm}(k_{\pm}) \\ & \times \Gamma(4N_{\pm} - 1 + k_{\pm})\Gamma(1 - 2N_{\pm} - k_{\pm})(k_{\pm}!) \Gamma(2N_{\pm} + k_{\pm})\}^{-1} \\ & \times \int_0^{2\pi} d\psi_{\pm} (-1)^{[N_{\pm}]} K_3^d(\chi_{1\pm}, \varphi_{1\pm}; \chi_{2\pm}, \varphi_{2\pm} | \chi_{3\pm}^e(k_{\pm}, \psi_{\pm})) \\ & \times O_{k_+k_-}(\psi)\Pi(O, O). \end{aligned} \quad (60)$$

This is a Wilson expansion indeed.

In fact we have [see (54)]

$$\begin{aligned} & (\Gamma(\alpha_{245})/\Gamma(\alpha_{235})) K_3^d(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3^e(k), \varphi_3) \\ & = \int_0^{2\pi} d\psi K_3^d(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3(k), \psi) S'_3(\psi - \varphi_3). \end{aligned} \quad (61)$$

This intertwining operator maps $O_{k_+k_-}(\varphi_3)\Pi(O, O)$ into $\prod_{\pm} \mathcal{D}_{\chi_{3\pm}(k_{\pm})}/\mathcal{F}_{\chi_{3\pm}(k_{\pm})}^{(-)}$.

On this function (with operator expansion coefficients in the canonical basis) acts

$$\begin{aligned} K_3^d(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3(k), \psi) = & (2\pi)^{-3} (-1)^{[N]} \\ & \times [2i \sin \tfrac{1}{2}(\varphi_2 - \varphi_1)]^k [2i \sin \tfrac{1}{2}(\psi - \varphi_1 - i0)]^{-\alpha_{235}} \\ & \times [2i \sin \tfrac{1}{2}(\psi - \varphi_2 - i0)]^{-\alpha_{013}} \end{aligned} \quad (62)$$

with

$$\alpha_{013} = \alpha_{235} = 2N + k. \quad (63)$$

Because the second and third factors in (62) are boundary values both from below in the complex ψ -plane, their product is, as a distribution, C^∞ in φ_1 and φ_2 . Thus the first factor determines the asymptotic behaviour for $\varphi_1 \rightarrow \varphi_2$. Each term in the expansion (60) behaves like

$$\sim \prod_{\pm} [2i \sin \frac{1}{2}(\varphi_{2\pm} - \varphi_{1\pm})]^{k_{\pm}} \quad (64)$$

for $\varphi_{1\pm} \rightarrow \varphi_{2\pm}$.

Finally we consider the four-point function

$$\langle 0 | f_{\gamma}(\varphi_1) f_{\gamma}(\varphi_2) f_{\gamma}^{\dagger}(\psi_1) f_{\gamma}^{\dagger}(\psi_2) | 0 \rangle. \quad (65)$$

Its harmonic analysis “in the s -channel” reduces to the analysis of the function

$$\langle 0 | R_{;\gamma\gamma}[\varphi_1, \varphi_2]^{\dagger} R_{;\gamma\gamma}[\psi_1, \psi_2] | 0 \rangle \quad (66)$$

that is determined by (60). By covariance arguments we have

$$\begin{aligned} & \langle 0 | O_{k_+k_-}(\psi_1)^{\dagger} O_{k'_+k'_-}(\psi_2) | 0 \rangle \\ &= \delta_{k_+k'_+} \delta_{k_-k'_-} M_{k_+k_-} \prod_{\pm} [2i \sin \frac{1}{2}(\psi_{1\pm} - \psi_{2\pm} - i0)]^{-2j_{3\pm}(k_{\pm})-1} \end{aligned} \quad (67)$$

and consequently

$$\begin{aligned} & \langle 0 | R_{;\gamma\gamma}[\varphi'_1, \varphi'_2]^{\dagger} R_{;\gamma\gamma}[\varphi_1, \varphi_2] | 0 \rangle \\ &= (2\pi)^{12} \sum_{k_{\pm}=0}^{\infty} M_{k_+k_-} \prod_{\pm} \{ [2j_3(k_{\pm})]^2 \\ & \quad \times [\Gamma(4N_{\pm} - 1 + k_{\pm}) \Gamma(1 - 2N_{\pm} - k_{\pm}) / k_{\pm}! \Gamma(2N_{\pm} + k_{\pm})]^2 \\ & \quad \times \int d\psi'_{\pm} d\psi_{\pm} \overline{K_3^d(\chi_{1\pm}, \varphi'_{1\pm}; \chi_{2\pm}, \varphi'_{2\pm} | \chi_{3\pm}^c(k_{\pm}), \psi'_{\pm})} \\ & \quad \times [2i \sin \frac{1}{2}(\psi'_{\pm} - \psi_{\pm} - i0)]^{-2j_{3\pm}(k_{\pm})-1} K_3^d(\chi_{1\pm}, \varphi_{1\pm}; \chi_{2\pm}, \varphi_{2\pm} | \chi_{3\pm}^c(k_{\pm}), \psi_{\pm}) \}. \end{aligned} \quad (68)$$

Thus $M_{k_+k_-}$ enters the “reduced amplitude”.

If it were possible to find a simple expression for the state $|2N_+ + k_+, 2N_- + k_- \rangle$

$$O_{k_+k_-}(\psi) | 0 \rangle = \sum_{q_{\pm}=2N_{\pm}+k_{\pm}}^{\infty} e^{i \sum_{\pm} q_{\pm} \psi_{\pm}} | q_+, q_- \rangle \quad (69)$$

we could calculate $M_{k_+k_-}$ by projecting (67) onto these states. One form we obtain for $M_{k_+k_-}$ is

$$M_{k_+k_-} = (2\pi)^{-6} 2^{4d} \prod_{\pm} \{ \Gamma(2N_{\pm} + k_{\pm})^6 (k_{\pm}!)^2 \Gamma(1 - 2N_{\pm} - k_{\pm})^{-2} S_{k_{\pm}} \} \quad (70)$$

with

$$\begin{aligned} S_k &= \sum_{m=0}^k \sum_{r=0}^{k-m} \sum_{s=0}^{k-m} (-1)^{r-s} \binom{-N}{m} \binom{-N}{k-m-r} \binom{-N}{k-m-s} \\ & \quad \left(\binom{-N}{r+s+m-k} \right) \{ \Gamma(2N+r) \Gamma(2N+k-r) \Gamma(2N+s) \Gamma(2N+k-s) \}^{-1}. \end{aligned} \quad (71)$$

We have not been able to sum this series. From the symmetries of this series we can derive that

$$S_k = 0 \text{ for } k \text{ odd.}$$

5. Semicovariant Kernels

In the general case we use the completeness relation of Ref. [6], Equations (111), (115), (120), (122) instead of (52). One might think that a covariant expansion can be obtained by shifting the contour of the principal series integral such that the $\text{Re } j_3 \rightarrow \infty$. In fact, poles of the integrand at

$$j_3 = -\frac{1}{2} \pm j_1 \pm j_2 + k \quad (72)$$

(j -type poles) or at

$$j_3 = -\frac{1}{2} \pm \tau_3 \pm k \quad (73)$$

(τ -type poles) contribute several series of residues. In general a j -type pole does not belong to a discrete series representation. The residue of such pole contains a covariant kernel with the asymptotic behaviour for $\varphi_1 \rightarrow \varphi_2$ (as a distribution in φ_3)

$$|K^d| \sim \text{const} |\sin \frac{1}{2}(\varphi_1 - \varphi_2)|^{-\frac{1}{2} + j_1 + j_2 - |\text{Re } j_3|} \quad (74)$$

[see (80), (82), (86)] that becomes more and more singular with increasing k . Shifting the contour to the left, yields the same result. A reasonable asymptotic expansion (Wilson expansion) can therefore not be generated this way. In addition the residues contain matrix elements of operators that are nonlocal in general.

To overcome this difficulty we proceed as in the theory of Regge poles [14]. An analogous problem with the asymptotic behaviour is solved there by exploiting the symmetry under $\chi \rightarrow \chi^c$ and by splitting the ‘‘covariant’’ Legendre functions of the first kind into ‘‘semicovariant’’ Legendre functions of the second kind. Covariance (semicovariance) means in this context validity of the addition theorems on the whole (only on part) of the group $\text{SU}(1, 1)$ [15].

Since the asymptotic behaviour of the covariant kernels [6]

$$K^d(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3, \varphi_3)$$

for $\varphi_1 \rightarrow \varphi_2$ as distributions in φ_3 is relevant, we investigate the functions

$$\begin{aligned} L(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3, q_3) \\ = \int_0^{2\pi} d\varphi_3 e^{iq_3\varphi_3} K^d(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3, \varphi_3) \end{aligned} \quad (75)$$

for any element

$$g_{q_3}(\varphi_3) = e^{iq_3\varphi_3} \quad (76)$$

of the canonical basis of \mathcal{D}_{χ_3} [6]. The function L can be explicitly given in terms of hypergeometric functions ${}_2F_1$ of the argument

$$e^{-i(\varphi_2 - \varphi_1 - i0)}.$$

Using a well known identity [16] these functions are split into two hypergeometric functions of argument

$$1 - e^{-i(\varphi_2 - \varphi_1 - i0)}. \quad (77)$$

This way the function (75) is decomposed into two parts

$$\begin{aligned} L(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3, q_3) \\ = L_a(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3, q_3) + L_b(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3, q_3) \end{aligned} \quad (78)$$

of different asymptotic behaviour for $\varphi_1 \rightarrow \varphi_2$.

If we set (see Ref. [6], Eq. (83a))

$$\eta_3 = \frac{1}{2}\alpha_{013} + \tau_2 \quad (79)$$

we get $K^d = K_1^d$ and correspondingly $L_a = L_{1a}$

$$\begin{aligned} L_{1a} = (2\pi)^{-2} |2 \sin(\varphi_1 - \varphi_2)/2|^{-\alpha_{024}} e^{-2i(\frac{1}{2}\alpha_{013} + \tau_2)\sigma(\varphi_1 - \varphi_2)} \\ \times e^{-i/2\alpha_{013}(\varphi_2 - \varphi_1) + iq_3\varphi_1} (-1)^{q_3 - \tau_1 - \tau_2} \\ \times \Gamma(-2j_3)(\Gamma(\alpha_{124})\Gamma(\alpha_{045}))^{-1} {}_2F_1(\alpha_{013}, \alpha_{123}; \beta_{34}; 1 - e^{-i(\varphi_2 - \varphi_1 - i0)}). \end{aligned} \quad (80)$$

Similarly we have for

$$\eta_3 = +\frac{1}{2}\alpha_{235} - \tau_1 \quad (81)$$

$K^d = K_3^d$ and correspondingly $L_a = L_{3a}$

$$\begin{aligned} L_{3a} = (2\pi)^{-2} |2 \sin(\varphi_1 - \varphi_2)/2|^{-\alpha_{024}} e^{-2i(\frac{1}{2}\alpha_{235} - \tau_1)\sigma(\varphi_1 - \varphi_2)} \\ \times e^{-i/2\alpha_{235}(\varphi_2 - \varphi_1) + iq_3\varphi_2} (-1)^{q_3 - \tau_1 - \tau_2} \\ \times \Gamma(-2j_3)(\Gamma(\alpha_{124})\Gamma(\alpha_{045}))^{-1} {}_2F_1(\alpha_{035}, \alpha_{235}; \beta_{34}; 1 - e^{-i(\varphi_2 - \varphi_1 - i0)}). \end{aligned} \quad (82)$$

The C -coefficients may be subjected to an analogous decomposition

$$\begin{aligned} C^d(\chi_1, q_1; \chi_2, q_2 | \chi_3, q_3) \\ = A_a(\chi_1, q_1; \chi_2, q_2 | \chi_3, q_3) + A_b(\chi_1, q_1; \chi_2, q_2 | \chi_3, q_3). \end{aligned} \quad (83)$$

We find the symmetry relations

$$C_1(\chi_3^c, q_3 | \chi_1, q_1; \chi_2, q_2) = \frac{\Gamma(\alpha_{245})\Gamma(\alpha_{123})}{\Gamma(\alpha_{235})\Gamma(\alpha_{124})} C_1(\chi_3, q_3 | \chi_1, q_1; \chi_2, q_2) \quad (84a)$$

$$C_3(\chi_3^c, q_3 | \chi_1, q_1; \chi_2, q_2) = \frac{\Gamma(\alpha_{014})\Gamma(\alpha_{035})}{\Gamma(\alpha_{013})\Gamma(\alpha_{045})} C_3(\chi_3, q_3 | \chi_1, q_1; \chi_2, q_2) \quad (84b)$$

and

$$A_{1a}(\chi_1, q_1; \chi_2, q_2 | \chi_3^c, q_3) = \frac{\Gamma(\alpha_{013})\Gamma(\alpha_{045})}{\Gamma(\alpha_{014})\Gamma(\alpha_{035})} A_{1b}(\chi_1, q_1; \chi_2, q_2 | \chi_3, q_3) \quad (85a)$$

$$A_{3a}(\chi_1, q_1; \chi_2, q_2 | \chi_3^c, q_3) = \frac{\Gamma(\alpha_{124})\Gamma(\alpha_{235})}{\Gamma(\alpha_{123})\Gamma(\alpha_{245})} A_{3b}(\chi_1, q_1; \chi_2, q_2 | \chi_3, q_3). \quad (85b)$$

It follows that in the principal series contribution (Ref. [6], Eqs. (111), (113)) we can replace

$$C_3^d C_1' - C_1^d C_3' \quad \text{by} \quad 2[A_{3a} C_1' - A_{1a} C_3']$$

exploiting the symmetry of the integral contour and the principal series measure under the replacement of j_3 by $-j_3$.

Moreover it follows that asymptotically

$$|L_b| \sim |2 \sin(\varphi_1 - \varphi_2)/2|^{-\text{Re } \alpha_{023}} (1 + O(\varphi_1 - \varphi_2)). \quad (86)$$

Therefore $L_a(L_b)$ has a decreasing singular behaviour of $\varphi_1 - \varphi_2 \rightarrow 0$ with increasing (decreasing) $\text{Re } j_3$.

The corresponding semicovariant kernels are defined by

$$Q_a(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3, \varphi_3) = (2\pi)^{-1} \sum_{q_3} e^{-iq_3 \varphi_3} L_a(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3, q_3) \quad (87a)$$

$$Q_b(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3, \varphi_3) = (2\pi)^{-1} \sum_{q_3} e^{-iq_3 \varphi_3} L_b(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3, q_3). \quad (87b)$$

This summation can be performed. The semicovariant kernels turn out to be linear combinations of the covariant kernels with non-invariant coefficients

$$Q_{1a} = a K_1^d + b \cdot \exp \{ -i\pi(j_1 - j_2 + \tau_1 + \tau_2) \text{sign} \sin(\varphi_1 - \varphi_2) \} K_3^d \quad (88a)$$

$$Q_{3a} = a \exp \{ +i\pi(j_1 - j_2 + \tau_1 + \tau_2) \text{sign} \sin(\varphi_1 - \varphi_2) \} K_1^d + b K_3^d. \quad (88b)$$

The factors a, b are

$$a = +\sin \pi \alpha_{235} \sin \pi(\frac{1}{2} + j_3 - \tau_1 - \tau_2) / \sin \pi 2j_3 \sin \pi(j_1 - j_2 + \tau_1 + \tau_2) \quad (89a)$$

$$b = -\sin \pi \alpha_{013} \sin \pi(\frac{1}{2} + j_3 + \tau_1 + \tau_2) / \sin \pi 2j_3 \sin \pi(j_1 - j_2 + \tau_1 + \tau_2). \quad (89b)$$

It is obvious that Q_{1a} and Q_{3a} are related by

$$Q_{3a} = \exp \{ i\pi(j_1 - j_2 + \tau_1 + \tau_2) \text{sign} \sin(\varphi_1 - \varphi_2) \} Q_{1a}. \quad (90)$$

Due to this linear dependence between the relations (88a), (88b), K_1^d or K_3^d cannot be expressed by the semicovariant kernels Q_{1a}, Q_{3a} alone.

If the covariance property of the covariant kernels (Ref. [6], Eqs. (52), (74)) is expressed by differential equations we see from (88) that the semicovariant kernels satisfy the same differential equations as the covariant kernels except at a set of measure zero. For infinitesimal group transformations semicovariant and covariant kernels behave therefore in the same fashion except possibly for their boundary conditions.

We start from the completeness relation in the form given in Ref. [6], Equations (111), (115), (120), (122). We obtain as principal series part

$$\begin{aligned} (2\pi)^4 \sin^{-1} \pi(j_1 - j_2 + \tau_1 + \tau_2) \int d\mu(\chi_3)_{PS} \int_0^{2\pi} d\varphi_3 \{ Q_{3a}(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3, \varphi_3) \\ K_1(\chi_3, \varphi_3 | \chi_1, \varphi_1'; \chi_2, \varphi_2') - Q_{1a}(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3, \varphi_3) \\ \times K_3(\chi_3, \varphi_3 | \chi_1, \varphi_1'; \chi_2, \varphi_2') \}. \end{aligned} \quad (91)$$

With the shorthands

$$\alpha = j_1 - j_2 + \tau_1 + \tau_2, \text{sign} \sin \varphi = \varepsilon(\varphi) \quad (92)$$

and (90) we may write this

$$(2\pi)^4 \int d\mu(\chi_3)_{PS} \int_0^{2\pi} d\varphi_3 Q_{1a}(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3, \varphi_3) \\ \times \sin^{-1} \pi \alpha \{e^{+i\pi\alpha\varepsilon(\varphi_1 - \varphi_2)} K_1 - K_3\}. \quad (93)$$

This form is still equivalent with the original expression Ref. [6], Equation (111). We want to deform the contour into the right half plane and thus expand (93) into a series of residues of poles of the integrand.

There are poles of the analytically continued Plancherel measure (Ref. [6], Eq. (98)), poles of Q_{1a} , and poles of the curly bracket in (93). We skip the detailed calculation [17]. Most of the poles cancel against zeros and against the contributions of the discrete series. There remains only one series of poles whose residues survive, namely those at

$$\alpha_{024} = \frac{1}{2} - j_1 - j_2 - j_3 = -k, \quad k=0, 1, 2, \dots \quad (94)$$

or at $j_3(k)$ as in (47) and no discrete series terms. The resulting “asymptotic completeness relation” is

$$\sum_{k_1=-\infty}^{+\infty} e^{2\pi i \tau_1 k_1} \delta(\varphi_1 - \varphi'_1 - 2\pi k_1) \sum_{k_2=-\infty}^{+\infty} e^{2\pi i \tau_2 k_2} \delta(\varphi_2 - \varphi'_2 - 2\pi k_2) \\ \sim (2\pi)^4 \frac{e^{i\pi(\frac{1}{2} - j_2 + \tau_2)\varepsilon(\varphi_1 - \varphi_2)}}{\sin \pi(\frac{1}{2} - j_2 + \tau_2)} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!} \pi j_3 \\ \times \{\text{tg } \pi(j_3 + \tau_3) + \text{tg } \pi(j_3 - \tau_3)\} \\ \times \int_0^{2\pi} d\varphi_3 Q_{1a}(\chi_1, \varphi_1; \chi_2, \varphi_2 | \chi_3, \varphi_3) [\Gamma(\alpha_{024})^{-1} K_1(\chi_3, \varphi_3 | \chi_1, \varphi'_1; \chi_2, \varphi'_2)] \\ (j_3 = j_3(k)). \quad (95)$$

We have not investigated whether the right hand side of (95) converges in any sense and represents the left hand side distribution.

The kernel $\Gamma(\alpha_{024})^{-1} K_1$ contained in (95) consists of a differential operator that creates a vector in $\mathcal{D}_{\chi_3^{(k)}}$ and of an intertwining operator for the mapping $\mathcal{D}_{\chi_3^{(k)}} \rightarrow \mathcal{D}_{\chi_3(k)}$. Explicitly we find for the case that $\chi_3(k)$ does not belong to the discrete series

$$\Gamma(\alpha_{024})^{-1} K_1(\chi_3, \varphi_3 | \chi_1, \varphi_1; \chi_2, \varphi_2) |_{j_3=j_3(k)} \\ = (2\pi)^{-2} \frac{\Gamma(2j_3(k) - k)}{\Gamma(2j_3(k))} \sin \pi(\frac{1}{2} - j_2 + \tau_2) \cdot \pi^{-1} \\ \times |2 \sin(\varphi_3 - \varphi_1)/2|^{2j_3(k)-1} e^{2i(\tau_1 + \tau_2)\sigma(\varphi_3 - \varphi_1)} \\ \times [\delta(\varphi_1 - \varphi_2) + \text{periodic repetitions}] Q_k(-i\partial/\partial\varphi_1, -i\partial/\partial\varphi_2) \quad (96)$$

with the polynomial Q_k (50). Thus by applying the asymptotic completeness relation (95) to a (regularized) bilocal operator we obtain a semicovariant Wilson expansion into local operators.

6. A Semicovariant Wilson Expansion

We study now the regularized operator product

$$R[f_\gamma(\varphi)f_\gamma^\dagger(\psi)] \quad (97)$$

on a charge sector n_+, n_- . We find [labels 1(2) refer to the first (second) factor of (97)]

$$j_{1\pm} = j_{2\pm} = +\frac{1}{2}, \tau_{1\pm} \triangleq C_\pm n_\pm, \tau_{2\pm} \triangleq -C_\pm n_\pm \bmod 1. \quad (98)$$

We assume that we are dealing with a non-degenerate case in the sense that $\tau_{1\pm}$ are non-zero and therefore

$$\tau_{1\pm} + \tau_{2\pm} = 1, \tau_{3\pm} = 0. \quad (99)$$

We decompose the operator (97) into two commuting factors

$$R[f_\gamma(\varphi)f_\gamma^\dagger(\psi)] = M_+(\varphi_+, \psi_+)M_-(\varphi_-, \psi_-) \quad (100)$$

by

$$M_\pm(\varphi_\pm, \psi_\pm) = (2\pi)^{-\frac{1}{2}} 2^d \exp i\{C_\pm[\Gamma_\pm^{(-)}(\varphi_\pm) - \Gamma_\pm^{(-)}(\psi_\pm) + \frac{1}{2}Q_\pm(\varphi_\pm - \psi_\pm)]\} \exp i\{C_\pm[\Gamma_\pm^{(+)}(\varphi_\pm) - \Gamma_\pm^{(+)}(\psi_\pm) + \frac{1}{2}Q_\pm(\varphi_\pm - \psi_\pm)]\}. \quad (101)$$

We subject them independently to the asymptotic decomposition (95). Thus we may skip the labels \pm in the sequel wherever we deal with one of the factors (101) separately.

The asymptotic completeness relation is not directly applicable to the case $\tau_3 = 0$ (e.g. the measure factor is divergent), but we adapt it to this case by a certain limiting procedure. Since $\frac{1}{2} - j_3(k)$ is an integer, we have to deal with a singular case where $\mathcal{D}_{\chi_3(k)}$ possesses two invariant subspaces $\mathcal{F}_{\chi_3(k)}^{(+)}$ and $\mathcal{F}_{\chi_3(k)}^{(-)}$. Of course as intermediate representation we choose $\chi_3^c(k)$ instead of $\chi_3(k)$ because we want to get a local operator.

The trick of adjusting Equation (95) to the case (98) consists in splitting every term in (95) into factors each of which assumes a well defined limit. This limit is defined as follows:

1. Go with j_3 to $j_3(k)$, assume j_1, j_2 to be purely imaginary, $\tau_3 \neq 0$;
2. Go with j_1 to j_2 still on the imaginary axis, keep τ_3 fixed;
3. Go with j_2 to $+\frac{1}{2}$, keep τ_3 fixed;
4. Go with τ_3 to zero.

We denote these limits in this order by “Lim”. Then we define for $k = 1, 2, 3, \dots$

$$O_k(\varphi_3)\Pi(n_+, n_-) = \text{Lim } 2\pi^3 / \sin \pi\tau_3 \sin \pi\tau_2 \Gamma(\alpha_{024})\Gamma(\alpha_{245}) \int_0^{2\pi} d\varphi'_1 \int_0^{2\pi} d\varphi'_2 K_1(\chi_3^c, \varphi_3 | \chi_1, \varphi'_1; \chi_2, \varphi'_2) M(\varphi'_1, \varphi'_2) \Pi(n_+, n_-). \quad (102)$$

This definition can be seen to be valid on each charge sector (n_+, n_-) and leads to a charge sector independent operator $O_k(\varphi_3)$ at the end.

We investigate these operators first. With

$$M(\varphi_1, \varphi_2)\Pi(n_+, n_-) = \sum_{q_1 q_2} M_{q_1 q_2} e^{iq_1 \varphi_1 + iq_2 \varphi_2} \quad (103)$$

$$O_k(\varphi_3)\Pi(n_+, n_-) = \sum_{q_3} \omega_{q_3} e^{iq_3 \varphi_3} \quad (104)$$

we have

$$\omega_{q_3} = \sum_{q_1 q_2} P_k(q_1, q_2) M_{q_1 q_2} \delta_{q_3, q_1 + q_2}. \quad (105)$$

Here $P_k(q_1, q_2)$ is a polynomial of degree k in q_1 and q_2 . Explicitly it is

$$P_k(q_1, q_2) = (-1)^k \Gamma(1 - q_2) \Gamma(k - q_1 - q_2) / \Gamma(1 - q_1 - q_2) \Gamma(1 - k - q_2) \\ \times {}_3F_2(1 - k - q_1 - q_2, 1 - k, 1 - q_2; 1 - k - q_2, 1 - q_1 - q_2; 1). \quad (106)$$

Inserting (105) into (104) we obtain after summing over all charge sectors

$$O_k(\varphi_3) = P_k(-i\partial/\partial\varphi_1, -i\partial/\partial\varphi_2) M(\varphi_1, \varphi_2) |_{\varphi_1 = \varphi_2 = \varphi_3}. \quad (107)$$

Performing the differentiations leads to

$$O_k(\varphi_3) = (2\pi)^{-\frac{1}{2}} 2^d \{ :P_k(\pi C J(\varphi_3), -\pi C J(\varphi_3): \\ + \text{derivative terms} \}. \quad (108)$$

The derivative terms involve derivatives of J , they appear for $k \geq 4$ only.

In general we find easily

$$P_k(-q_2, +q_2) = (2k-2)! q_2^k / (k-1)! + O(q_2^{k-2}) \quad (109)$$

so that we may normalize

$$O_k(\varphi_3) = (2k-2)! O_k(\varphi_3)_{\text{nor}} / (k-1)! \quad (110)$$

$$(2\pi)^{\frac{1}{2}} 2^{-d} O_k(\varphi_3)_{\text{nor}} = :[-\pi C J(\varphi_3)]^k: \\ + \text{lower order and derivative terms}. \quad (111)$$

This definition can be used also for $k=0$

$$(2\pi)^{\frac{1}{2}} 2^{-d} O_k(\varphi_3)_{\text{nor}} |_{k=0} = \text{unit operator}. \quad (112)$$

Next we insert (102) into the asymptotic expansion (95). We apply the same limit as in (102) to the residual factors in the completeness sum (95). We introduce the kernel

$$\chi_k(\varphi_1, \varphi_2, \varphi_3) = (2\pi)^{-1} \sum_{q_3} e^{iq_3(\varphi_1 - \varphi_3)} {}_2F_1(k, k + q_3; 2k; 1 - e^{-i(\varphi_2 - \varphi_1 - i0)}) \quad (113)$$

and for $k=0$ (consider k as continuous variable in this context)

$$\chi_0(\varphi_1, \varphi_2, \varphi_3) = \lim_{k \rightarrow 0} \chi_k(\varphi_1, \varphi_2, \varphi_3) \\ = \frac{1}{2} [\delta(\varphi_1 - \varphi_3) + \delta(\varphi_2 - \varphi_3)] \\ + \text{periodic repetitions}. \quad (114)$$

In the interval $|\varphi_1 - \varphi_2| < \pi$ we end up with the following asymptotic expansion

$$M(\varphi_1, \varphi_2) \sim \sum_{k=0}^{\infty} \frac{1}{k!} [1 - e^{-i(\varphi_2 - \varphi_1)}]^k \int_0^{2\pi} \chi_k(\varphi_1, \varphi_2, \varphi_3) O_k(\varphi_3)_{\text{nor}} d\varphi_3. \quad (115)$$

For practical purposes the representation (113) of $\chi_k(\varphi_1, \varphi_2, \varphi_3)$ is very useful. We can expand the hypergeometric series into powers of $1 - e^{-i(\varphi_2 - \varphi_1)}$ and perform the summation over q_3 termwise. This way we can easily reorder (115) into an

asymptotic series with increasing powers of $(\varphi_2 - \varphi_1)$. The Taylor expansion of $M(\varphi_1, \varphi_2)$ results, as we expect.

The expansion (115) is valid for any charge sector. We sandwich the operator now between two states

$$\langle 0 | f_\gamma(\varphi_1) R[f_\gamma(\varphi_2) f_\gamma^\dagger(\psi_1)] f_\gamma^\dagger(\psi_2) | 0 \rangle \quad (116)$$

which can immediately be expressed by the Wightman four-point function. Inserting (115) yields an expansion of the matrix element (116) into the matrix elements

$$\begin{aligned} W(\varphi_1, \varphi_2, \varphi_3) \\ = \langle 0 | f_\gamma(\varphi_1) O_{k_+}(\varphi_{3+})_{\text{nor}} O_{k_-}(\varphi_{3-})_{\text{nor}} f_\gamma^\dagger(\psi_2) | 0 \rangle \end{aligned} \quad (117)$$

that represent covariant three-point functions. Conformal covariance of this function determines it up to a constant factor, namely

$$W(\varphi_1, \varphi_2, \varphi_3) = C_{k_+k_-} \prod K_2(\chi_{3\pm}^c(k_\pm), \varphi_{3\pm} | \chi_{1\pm}^d, \varphi_{1\pm}; \chi_{2\pm}^d, \varphi_{2\pm}) \quad (118)$$

with

$$\chi_{1,2} = (j_{1,2}, \tau_{1,2}), \chi_{1,2}^d = (-j_{1,2}, 1 - \tau_{1,2}) \quad (119)$$

$$2j_{1,2\pm} - 1 = -N_\pm, \tau_{1\pm} \cong -\frac{1}{2}N_\pm, \tau_{2\pm} \cong +\frac{1}{2}N_\pm \pmod{1}.$$

The normalization constant $C_{k_+k_-}$ can be determined by the following argument. $\chi_{1\pm}^d$ and $\chi_{2\pm}^d$ are discrete series representations that possess states of lowest respectively highest weight. We project both sides of (118) on these states. We find

$$\begin{aligned} C_{k_+k_-} &= (2\pi)^2 \int_0^{2\pi} \prod_{\pm} \{ d\varphi_{1\pm} d\varphi_{2\pm} e^{i(\varphi_{1\pm} - \varphi_{2\pm})\frac{1}{2}N_\pm} \} \\ &\quad \times W(\varphi_1, \varphi_2, \varphi_3). \end{aligned} \quad (120)$$

On the other hand this projection reduces the field operator in the four-point function (117) to the σ -field (17), (18)

$$C_{k_+k_-} = (2\pi)^5 2^{2d} \langle 0 | \tilde{\sigma}_\gamma O_{k_+}(\varphi_{3+})_{\text{nor}} O_{k_-}(\varphi_{3-})_{\text{nor}} \tilde{\sigma}_\gamma^\dagger | 0 \rangle. \quad (121)$$

In our realization of the Thirring field (21), (24) where the current operator commutes with the σ -field except its charge operator part, the matrix element (121) can be computed easily. We can neglect the derivative terms of (108), because they do not contain a charge operator part. This gives finally

$$C_{k_+k_-} = (2\pi)^{4+2d} \prod_{\pm} \{ (k_\pm - 1)! P_{k_\pm}(-N_\pm, +N_\pm) / (2k_\pm - 2)! \} \quad (122)$$

where the curly bracket is defined to be one for $k_\pm = 0$.

7. Conclusions

Though our concrete examples in the present work concern with the operator product expansion of two particular field operators in the Thirring model, the

principle should be obvious and further generalizations to other operator products are quite straightforward. As a byproduct of working in two-dimensional space-time with a soluble field theoretical model we see explicitly the fulfilment of the meromorphy hypothesis for conformal partial waves which was essential to get the operator product expansion in the general approach, for example, taken by Mack [18]. With all the group theoretical problems completely in our control (of course, in two dimensions) and the Thirring model it would be now very interesting to study in detail well-known difficulties related with the locality and crossing symmetry arising in the approach to the conformally covariant quantum field theory based on dynamical equations.

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Note Added in Proof. The authors proved recently that for the Thirring model and an operator product which does neither commute with the chiral charges nor is applied to the vacuum state the semi-covariant expansions do not recombine to a Wilson expansion in terms of local operators.

