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Harmonic Analysis on the Poincaré Group*

I. Generalized Matrix Elements

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Abstract. The generalized matrix elements of the unitary representations of the Poincaré group are computed as eigendistributions of a complete commuting set of infinitesimal operators on the group. The unitary representations of the Poincaré group can be reconstructed from these matrix elements in a Hilbert space of square integrable functions over the space of eigenvalues. Some properties of these distributions (which are measures) are given and some characters are computed from the explicit formulae for the matrix elements.

Introduction

It was suggested by LURÇAT [1] to construct the quantum fields on the Poincaré group instead of Minkowskian space (which is a homogeneous space of Poincaré group). Numerous attempts were made to enlarge the homogeneous space since then [2, 3]. In the attempt to construct this field theory, it was necessary to obtain the "exponentials" of the Poincaré group, that is to say to begin to study the Fourier transform on the group.

The harmonic analysis on the universal covering group of the proper orthochronous Poincaré group (in this paper this covering group is named improperly Poincaré group) is led trough with the simple idea of using the "exponentials" of the group. These "exponentials" must be the eigenfunctions of the infinitesimal operators and the method is therefore straightforward. In this paper, we first compute the eigenfunctions and in the following one, we shall study the Fourier transform. Chapter 0 is devoted to notations and to some useful definitions.

In Chapter I, we derive the infinitesimal operators (in more mathematical terms: the one-sided invariant vector fields on the group manifold) by using the Lie algebra of the group. We get ten left generators (which are ten independent right invariant vector fields) and ten right generators. In the enveloping algebra we choose a complete subset of commuting operators, which include the two Casimir operators (namely P^2 and W^2), four left operators, and four right operators.

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In Chapter II, we compute the eigendistributions of this Abelian subset. The eigendistributions, which depend on the set of eigenvalues, are in fact measures on the Poincaré group and when properly normalized can be handled as matrix elements of irreducible representations of the group, expressed on a continuous basis. By this way, we compute the matrix elements for all the unitary irreducible representations, and we get a fairly simple expression for them.

In Chapter III, we reconstruct the unitary irreducible representations from its matrix elements and we give some properties of the matrix elements. As an application, we compute the characters for non-zero mass representations.

Chapter 0. Notation and Symbols

A point in space time is denoted by a latin letter or by its components:

$$x = (x_0, \boldsymbol{x}) = (x_0, x_1, x_2, x_3) .$$

The metrics is positive on time like vectors, the Minkowskian product is

$$xy = x_0 y_0 - \boldsymbol{x}\boldsymbol{y} .$$

C and R are the complex and real number fields, N the set of integers.

The $SL(2, C)$ group is the group of two by two complex unimodular matrices, its elements are denoted by capitals: X, Y, Z, and the matrix elements of X are X_{ij} with $i, j = 1, 2$.

We need some subgroups of $SL(2, C)$:

$SU(2)$: unitary unimodular subgroup, its elements are U, V, W . . .

$SU(1, 1)$: an element of this group is called 0 (or P, Q . .) and satisfies the relations $0_{11} = 0^*_{22}$; $0_{12} = 0^*_{21}$.

$ST(2)$: triangular subgroup, its elements are called R (or S, T . . .) and satisfy the relations $R_{21} = 0$, $R_{11} = R^*_{22}$.

We set $\qquad \partial_{ij} = \dfrac{\partial}{\partial X_{ij}}$ and $\partial^*_{ij} = \dfrac{\partial}{\partial X^*_{ij}}$.

We define the differentiation matrices $\partial_X, \partial_{X^*}, \partial_{X^\dagger}$ by

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}; \quad \partial_X = \begin{pmatrix} \partial_{11} & \partial_{12} \\ \partial_{21} & \partial_{22} \end{pmatrix},$$

$$\partial_{X^*} = \begin{pmatrix} \partial^*_{11} & \partial^*_{12} \\ \partial^*_{21} & \partial^*_{22} \end{pmatrix}, \quad \partial_{X^\dagger} = \begin{pmatrix} \partial^*_{11} & \partial^*_{12} \\ \partial^*_{12} & \partial^*_{22} \end{pmatrix},$$

and we use the notation

$$A \cdot \partial_X = \sum_{i, j = 1, 2} A_{ij} \partial_{ij}$$

$$AB \ldots C \cdot \partial_X = (AB \ldots C) \cdot \partial_X .$$

In the same way we define

$$\partial_x = (\partial_{x_0}, \partial_{\boldsymbol{x}}) = \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) .$$

Some particular four vectors are useful:

$$\overset{0}{1} = (1,0,0,0)\,; \quad \overset{3}{1} = (0,0,0,1)\,; \quad \overset{03}{1} = (1,0,0,1)\,;$$

we shall use the same notation with an M replacing the $1's$ in these formulae:

$$\overset{0}{M} = (M,0,0,0)\,, \qquad \text{etc.}$$

Using the standard Pauli matrices $\sigma_0, \sigma_1, \sigma_2, \sigma_3$, we define

$$H(x) = x^0\sigma_0 + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3\,,$$

and any X belonging to $SL(2, C)$ realizes a Lorentz transformation when we give its result Xx on any x by the formula:

$$H(Xx) = XH(x)X^\dagger\,.$$

The stabilizator of a time-like (a light-like or a space-like) four vector is isomorphic to $SU(2)$ (to $ST(2)$ or to $SU(1, 1)$) which stabilizes $\overset{0}{1}$ $\left(\overset{03}{1} \text{ or } \overset{3}{1}\right)$. The homogeneous spaces i.e. the spaces of left cosets of these subgroups, can be identified to the corresponding orbits, that is:

1. the positive energy shell of the hyperboloid $k^2 = 1$;
2. the positive light cone;
3. the hyperboloid $k^2 = -1$.

For example, in the first case, the one-to-one correspondence is given by

$$k = X^{-1}\overset{0}{1}\,,$$

and the group acts on its homogeneous space by the canonical transformation

$$k \to Xk\,.$$

A canonical decomposition of $SL(2, C)$ is obtained once a section in the classes is given, i.e. when one has chosen an element in each class. The simplest choice is to take the unique positive Hermitian element in each class, which is named H_k. We then have the relations

$$\begin{aligned} H_k^{-1}\overset{0}{1} &= k = X^{-1}\overset{0}{1} \\ X &= UH_k \qquad (\text{with } U \in SU(2))\,. \end{aligned}$$

In the other cases, the results are the same and the formulae too are similar:

i)
$$\begin{aligned} &SU(2) : (U \in SU(2)) \\ &X = UH_k \\ &H_k = [1 + k_0 - \boldsymbol{k\sigma}]\,[2 + 2k_0]^{-1/2} \qquad (1) \\ &k = X^{-1}\overset{0}{1}\,. \end{aligned}$$

H_k is a C^∞ function of $\boldsymbol{k}$.

ii)
$$ST(2):(S \in ST(2))$$
$$X = S\,T_k \tag{1'}$$
$$T_k = \begin{pmatrix} \alpha/k_0 & -\gamma^*/k_0 \\ \gamma & \alpha \end{pmatrix} \quad \text{with} \quad \begin{matrix} \alpha = (k_0 + k_3)^{1/2}/\sqrt{2} \\ \gamma = (k_1 + i k_2)/2\alpha \end{matrix}$$
$$k = X^{-1} \overset{03}{1} \,,$$

T_k is a C^∞ function of $\boldsymbol{k}$ for any k on the light cone.

iii)
$$SU(1,1):(0 \in SU(1,1))$$
$$X = 0\,F_k$$
$$F_k = \{[k_0 + (1 - |\boldsymbol{k}|)\sigma_3]\,[2|\boldsymbol{k}| - 2]^{-1/2}\} \tag{1''}$$
$$\{[|\boldsymbol{k}| + k_3 - i k_2 \sigma_1 + i k_1 \sigma_2]\,[2\,|\boldsymbol{k}|\,(|\boldsymbol{k}| + k_3)]^{-1/2}\}$$
$$k = X^{-1} \overset{3}{1} \,.$$

F_k is a C^∞ function of k_0 and $\hat{k}$.
The invariant measure of $SL(2, C)$ splits into an invariant measure on the subgroup and an invariant measure on the homogeneous space. We take it to be
$$d^6X = d^3U\, d^3\boldsymbol{k}/k_0 = d^3S\, d^3\boldsymbol{k}/k_0 = d^3 0\, d^3\boldsymbol{k}/k_0 \tag{2}$$
where d^3U is the normalized Haar measure of $SU(2)$ and the Eq. (2) fix the normalization of the other measures:
$$d^3S = \frac{1}{32\pi^2}\, d\varphi\, dS_{12}\, dS_{12}^*$$
$$\text{if} \quad S = \begin{pmatrix} e^{i\varphi/2} & S_{12} \\ 0 & e^{-i\varphi/2} \end{pmatrix}$$
$$d^3 0 = \frac{1}{4\pi^2} \frac{d^3\boldsymbol{a}}{|a_0|}$$
$$\text{if} \quad 0 = a_0\sigma_0 + a_1\sigma_1 + a_2\sigma_2 + i a_3\sigma_3 \quad (a_0^2 + a_3^2 - a_1^2 - a_2^2 = 1)$$
$$d^3U = \frac{1}{2\pi^2} \frac{d^3\boldsymbol{a}}{a_0}$$
$$\text{if} \quad U = a_0\sigma_0 + i\boldsymbol{a}\boldsymbol{\sigma} \quad (a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1)\,.$$
We define the norm
$$\|X\|^2 = \frac{1}{2} \operatorname{Tr} XX^\dagger = \left(X \overset{0}{1}\right)_0 .$$

This function is left and right invariant by $SU(2)$ translations and its reciprocal hyperbolic cosine can be used to define a distance between the classes of $SU(2)$. We quote some relations:

$$\begin{aligned}
\|H_p H_k^{-1}\|^2 &= p \cdot k \\
\|H_k^{-1}\, U H_k\|^2 &= k \cdot U k \\
\|F_k^{-1}\, 0 F_k\|^2 &= \boldsymbol{k}^2(\|0\|^2 - 1) + 1 \\
\|T_k^{-1}\, S\, T_k\|^2 &= k_0^2(\|S\|^2 - 1) + 1 \\
\|H_k\|^2 = k_0; \quad \|F_k\|^2 &= |\boldsymbol{k}|; \quad \|T_k\|^2 = (k_0^2 + 1)/2k_0\,.
\end{aligned} \tag{3}$$

The Poincaré group.

We denote its elements by Greek letters $\alpha, \beta, \gamma \ldots$ or by a couple (x, X) or $(y, Y) \ldots$ where $x \in R^4$ and $X \in SL(2, C)$.

The product is denoted multiplicatively and is defined by

$$(x, X)\,(y, Y) = (x + Xx, XY)\,.$$

The two-sided invariant measure is taken to be

$$d^{10}\alpha = d^4 x\, d^6 X\,.$$

Chapter I. The Infinitesimal Operators

The Poincaré group $\mathscr{P}$ being endowed with its invariant measure, one can introduce the Hilbert space of square integrable functions on $\mathscr{P}$, which is called $\mathscr{L}^2(\mathscr{P})$. The group acts on this Hilbert space by the formulae:

$$\begin{aligned}
(U(\alpha)f)\,(\gamma) &= f(\alpha^{-1}\gamma) \\
(V(\alpha)f)\,(\gamma) &= f(\gamma\alpha)\,.
\end{aligned}$$

These are the left and right regular representations of the group. These representations are unitary and one can easily check the relations

$$\begin{aligned}
U(\alpha)\, U(\beta) &= U(\alpha\beta); \qquad U^\dagger(\alpha) = U(\alpha^{-1}) \\
V(\alpha)\, V(\beta) &= V(\alpha\beta); \qquad V^\dagger(\alpha) = V(\alpha^{-1})\,.
\end{aligned}$$

The involution I of $\mathscr{L}^2(\mathscr{P})$, defined by

$$(If)\,(\alpha) = f(\alpha^{-1})\,,$$

realizes a unitary equivalence between these two regular representations.

When a one-parameter subgroup is given by its generator Q, we can derive an infinitesimal operator on the group manifold. Giving the one-parameter subgroup in its exponential form, let us define

$$Q_g f = -\frac{i\partial}{\partial t}\, U(e^{itQ})f \qquad\qquad Q_d f = -\frac{i\partial}{\partial t}\, V(e^{itQ})f\,;$$

whenever f is a smooth function, for example C^∞, these expressions make sense and we get the infinitesimal operators associated to any Q. Each Q_g (resp. Q_d) is a right invariant (resp. left invariant) vector field on the group manifold. They have the same commutation relations as the generators of the group itself and any Q_g commutes with any Q_d.

In this way, one can exhibit a complete basis of right (resp. left) invariant vector fields by taking the Q_g's (resp. Q_d's) associated with the four components P_μ of the energy momentum and the six generators of the homogeneous Lorentz group. These last generators can be written as two vectors $\boldsymbol{M}$ and $\boldsymbol{N}$ (corresponding respectively to rotations and to boosts: see for example ref. [4]). We give here the explicit expression of the infinitesimal operators:

$$\begin{aligned} P_g &= i\partial_x \\ 2\boldsymbol{M}_g &= -2i\boldsymbol{x}\wedge\partial_{\boldsymbol{x}} - \boldsymbol{\sigma}X\cdot\partial_X + X^\dagger\boldsymbol{\sigma}\cdot\partial_{X^\dagger} \\ 2\boldsymbol{N}_g &= -2i\boldsymbol{x}\,\partial_{x_0} - 2ix_0\,\partial_{\boldsymbol{x}} - i\boldsymbol{\sigma}X\cdot\partial_X - iX^\dagger\boldsymbol{\sigma}\cdot\partial_{X^\dagger} \end{aligned} \tag{4}$$

and

$$\begin{aligned} P_d &= -X^{-1}P_g \\ 2\boldsymbol{M}_d &= X\boldsymbol{\sigma}\cdot\partial_X - \boldsymbol{\sigma}X^\dagger\cdot\partial_{X^\dagger} \\ 2\boldsymbol{N}_d &= iX\boldsymbol{\sigma}\cdot\partial_X + i\boldsymbol{\sigma}X^\dagger\cdot\partial_{X^\dagger}\,. \end{aligned} \tag{5}$$

The symbol $\wedge$ stands for the vector product in R^3.

We can then easily extend the domain of the infinitesimal operators to include the distributions on the Poincaré group.

First, let us define convolution on the group: if S and T are distributions let us set

$$\langle S * T, \varphi\rangle = \langle S(\alpha)\otimes T(\beta), \varphi(\alpha\beta)\rangle$$

whenever this expression makes sense, for any φ in $\mathscr{D}(\mathscr{P})$ (which is the classical Schwartz space of functions on $\mathscr{P}$) and with $S(\alpha)\otimes T(\beta)$ as the product distribution on the product space $\mathscr{P}\times\mathscr{P}$.

Let then ε_α be the Dirac measure at the point $\alpha\in\mathscr{P}$. We define

$$\begin{aligned} U(\alpha)T &= \varepsilon_\alpha * T \\ V(\alpha)T &= T * \varepsilon_{\alpha^{-1}}\,. \end{aligned}$$

This definition is consistent with the previous one when T is given by a function. Finally, we define

$$Q_g T = -i\frac{\partial}{\partial t}U(e^{itQ})T$$

$$Q_d T = -i\frac{\partial}{\partial t}V(e^{itQ})T$$

(this derivative is to be taken in distribution space [5]) then, one gets the following relations (which are expected!):

$$\begin{aligned} \langle Q_g T, \varphi\rangle &= -\langle T, Q_g\varphi\rangle \\ \langle Q_d T, \varphi\rangle &= -\langle T, Q_d\varphi\rangle \end{aligned}$$

and the Q's appear as first order differential operators.

Let us now consider the algebra generated by these infinitesimal operators. We can easily construct a set of commuting operators in a standard way. First, among those generated by left operators, as the last ones have the commutation rules of the Lie algebra of Poincaré group, we can choose the six operators:

$$P_g, W_{g3}, W_g^2$$

where, as usual, the W_μ's are the Pauli-Lubanski operators

$$W_\mu = \frac{1}{2} \varepsilon_{\mu\nu\varrho\sigma} P^\nu M^{\varrho\sigma}$$

and $M^{\varrho\sigma}$ is related to the M's and N's in the standard way. As the left operators commute with the right ones we can also take the six operators

$$P_d, W_{d3}, W_d^2$$

but in fact, we get the relations

$$P_d^2 = P_g^2 \quad \text{and} \quad W_d^2 = W_g^2$$

and we have only ten algebraically independent commuting infinitesimal operators.

We can compute explicitly the Pauli-Lubanski operators:

$$\begin{aligned}
2W_{g0} &= -(\boldsymbol{P}_g\boldsymbol{\sigma})X\cdot\partial_X + X^\dagger(\boldsymbol{P}_g\boldsymbol{\sigma})\cdot\partial_{X^\dagger} \\
2\boldsymbol{W}_g &= -P_{g0}\boldsymbol{\sigma}X\cdot\partial_X + P_{g0}X^\dagger\boldsymbol{\sigma}\cdot\partial_{X^\dagger} + i\boldsymbol{P}_g\wedge\boldsymbol{\sigma}X\cdot\partial_X + i\boldsymbol{P}_g\wedge(X^\dagger\boldsymbol{\sigma})\cdot\partial_{X^\dagger} \\
2W_{d0} &= X(\boldsymbol{P}_d\boldsymbol{\sigma})\cdot\partial_X - (\boldsymbol{P}_d\boldsymbol{\sigma})X^\dagger\cdot\partial_{X^\dagger} \qquad (6) \\
2\boldsymbol{W}_d &= P_{d0}X\boldsymbol{\sigma}\cdot\partial_X - P_{d0}\boldsymbol{\sigma}X^\dagger\cdot\partial_{X^\dagger} - i\boldsymbol{P}_d\wedge(X\boldsymbol{\sigma})\cdot\partial_X - i\boldsymbol{P}_d\wedge(\boldsymbol{\sigma}X^\dagger)\cdot\partial_{X^\dagger}\,.
\end{aligned}$$

Whenever $P_g^2 = M^2 > 0$, let us define (h stands for d or g)

$$\begin{aligned}
S_{h\mu} &= \varepsilon M^{-1} W_{h\mu} \qquad (\varepsilon \text{ is the sign of the eigenvalue of } P_{h0}) \\
S_h^+ &= S_{h1} + iS_{h2} \\
S_h^- &= S_{h1} - iS_{h2}\,.
\end{aligned}$$

On the eigenspace defined by $P_g = -M$, we get the simpler explicit form:

$$\begin{aligned}
S_{g0} &= 0 \\
2\boldsymbol{S}_g &= -\boldsymbol{\sigma}X\cdot\partial_X + X^\dagger\boldsymbol{\sigma}\cdot\partial_{X^\dagger} \\
S_g^+ &= X_{11}^*\partial_{21}^* + X_{12}^*\partial_{22}^* - X_{21}\partial_{11} - X_{22}\partial_{12} \\
S_g^- &= X_{21}^*\partial_{11}^* + X_{22}^*\partial_{12}^* - X_{11}\partial_{21} - X_{12}\partial_{22} \qquad (7) \\
2S_{g3} &= X_{11}^*\partial_{11}^* + X_{12}^*\partial_{12}^* - X_{21}^*\partial_{21}^* - X_{22}^*\partial_{22}^* \\
&\quad - X_{11}\partial_{11} - X_{12}\partial_{12} + X_{21}\partial_{21} + X_{22}\partial_{22}
\end{aligned}$$

and when $P_d = \overset{0}{M}$

$$\begin{aligned}
S_{d0} &= 0 \\
2\boldsymbol{S}_d &= X\boldsymbol{\sigma}\cdot\partial_X - \boldsymbol{\sigma}X^\dagger\cdot\partial_{X^\dagger} \\
S_d^+ &= -X_{12}^*\partial_{11}^* - X_{22}^*\partial_{21}^* + X_{11}\partial_{12} + X_{21}\partial_{22} \\
S_d^- &= -X_{11}^*\partial_{12}^* - X_{21}^*\partial_{22}^* + X_{12}\partial_{11} + X_{22}\partial_{21} \\
2S_{d3} &= -X_{11}^*\partial_{11}^* - X_{21}^*\partial_{21}^* + X_{12}^*\partial_{12}^* + X_{22}^*\partial_{22}^* \\
&\quad + X_{11}\partial_{11} + X_{21}\partial_{21} - X_{12}\partial_{12} - X_{22}\partial_{22}\,.
\end{aligned} \tag{8}$$

In fact these operators are first order differential operators on the stabilizator group, and in this case, on the $SU(2)$ subgroup.

Chapter II. The Matrix Elements as Eigendistributions

The solution of the equations

$$P_g\psi(x, X) = -k'\psi(x, X)$$

can be written in the form

$$\psi(x, X) = e^{ik'x}\psi'(X)\,.$$

The equations $P_d\psi(x, X) = k\psi(x, X)$ can then be written as

$$(X^{-1}k' - k)\ \psi'(X) = 0\,.$$

These equations have a non-zero solution only when k and k' belong to the same orbit and their solution is a distribution the support of which is the submanifold defined by the equation $X^{-1}k' - k = 0$.

1. The Non-Zero Physical Mass Case

The compatibility conditions are

$$k^2 = k'^2 = M^2 > 0$$

$$k_0k_0' > 0\,.$$

One can solve the particular case when $k = k' = \varepsilon\overset{0}{M}$ (ε is the sign of energy) by setting

$$\psi'(X) = \psi_0(X)\,\omega\,\delta^3(X)$$

where $\omega\,\delta^3(X)$ is the measure defined by

$$\int_{SL(2,C)} \omega\,\delta^3(X)\,\varphi(X)\,d^6X = \int_{SU(2)} \varphi(U)\,d^3U$$

(this is the invariant measure of the submanifold of $SU(2)$).

If one defines (with $p^2 = q^2$ and $p_0q_0 > 0$) the Dirac measure on the mass shell at the point q:

$$\omega\,\delta^3(p, q) = |p_1|\ \delta^3(\boldsymbol{p} - \boldsymbol{q})\ \Theta(p_0q_0)\,,$$

then one can write

$$\omega\,\delta^3(X) = \omega\,\delta^3\left(X\,\overset{0}{1},\overset{0}{1}\right)\cdot$$

In fact when one parametrizes X by setting $X = U H_p$, X becomes a function of $U \in SU(2)$ and of p which belongs to the positive mass shell of the hyperboloid $p^2 = 1$, and one gets

$$X^{-1}\,\overset{0}{1} = p$$

$$d^6 X = d^3 U\,\frac{d^3\boldsymbol{p}}{p_0}$$

$$\omega\,\delta^3\left(X\,\overset{0}{1},\overset{0}{1}\right) = p_0\,\delta^3(\boldsymbol{p})\,.$$

When the integration over $d^3\boldsymbol{p}$ is performed, one gets $p = \overset{0}{1}$ so that $X = U$ and

$$\int \omega\,\delta^3\left(X\,\overset{0}{1},\overset{0}{1}\right)\varphi(X)\,d^6X = \int_{SU(2)} \varphi(U)\,d^3U\,.$$

The remaining equations are

$$\begin{aligned}
(W^2 + M^2 s(s+1))\,\omega\,\delta^3(X)\,\psi_0(X) &= 0\\
(W_{g3} - M\lambda')\,\omega\,\delta^3(X)\,\psi_0(X) &= 0\\
(W_{d3} - M\lambda)\,\omega\,\delta^3(X)\,\psi_0(X) &= 0\,.
\end{aligned}$$

With the conditions $\overset{0}{P}_g = -M$ and $\overset{0}{P}_d = M$, the $W'_\mu s$, up to the constant factor M, are the infinitesimal operators of the $SU(2)$ subgroup, and these equations lead to the matrix elements of many representations of the $SU(2)$ group. We can exhibit them easily by finding a generating function, namely the function

$$\Phi(X) = (X_{11} + X_{12} + X_{21} + X_{22})^{2s}$$

which is an eigenfunction of W^2 with $-M^2 s(s+1)$ as eigenvalue.

The matrix elements are obtained by expanding Φ into eigenfunctions of W_{g3} and W_{d3}; this can be easily done by performing a Fourier integration over the two one-parameter subgroups generated by W_{d3} and W_{g3}. The normalization is then computed in such a way as to satisfy the correct multiplicative property for the matrix elements.

The results are

$$\begin{aligned}
f_{s\lambda'\lambda}(X) = \frac{C}{4\pi^2}\int_0^{2\pi} d\eta \int_0^{2\pi} d\varphi\,\{&(X_{11}e^{-i\eta} + X_{21}e^{i\eta})\,e^{i\varphi}\\
&+ (X_{12}e^{-i\eta} + X_{22}e^{i\eta})\,e^{-i\varphi}\}^{2s} e^{2i\lambda'\eta - 2i\lambda\varphi}
\end{aligned} \tag{9}$$

$$C = [(s-\lambda')!\,(s+\lambda')!\,(s-\lambda)!\,(s+\lambda)!]^{1/2}/(2s)!$$

(throughout this paper $a! = \Gamma(a+1)$ is the factorial function).

These functions are matrix elements of irreducible representations of $SU(2)$. When $2s \in N$, the representations are unitary and the matrix elements have the usual properties:

$$\sum_{\lambda=-s}^{+s} f_{s\lambda'\lambda}(X)\, f_{s\lambda'\lambda_1}(Y) = f_{s\lambda'\lambda_1}(XY)$$

$$f_{s\lambda'\lambda}(X^*) = f^*_{s\lambda'\lambda}(X)$$

$$f_{s\lambda'\lambda}(\tilde{X}) = f_{s\lambda\lambda'}(X) \quad (\tilde{X} \text{ is the transposed matrix of } X)$$

and when X belongs to $SU(2)$, the two last relations give

$$f_{s\lambda\lambda'}(X^\dagger) = f^*_{s\lambda'\lambda}(X)\,.$$

In fact, these functions are defined over the whole group $SL(2, C)$.

Now we can write the solution of our problem in the particular case:

$$\psi(x, X) = e^{i\varepsilon \overset{0}{M} x}\, \omega\, \delta^3(X)\, f_{s\lambda'\lambda}(X)\,,$$

then the general solution can be obtained by translating this particular solution:

$$\psi(s k' k \lambda' \lambda;\, (x, X)) = e^{ik'x}\, \omega\, \delta^3(H_{k'} X H_k^{-1})\, f_{s\lambda'\lambda}(H_{k'} X H_k^{-1}) \tag{10}$$

and we write it in a short-hand notation

$$= e^{ik'x}\, \omega\, \delta^3 f_{s\lambda'\lambda}(H_{k'} X H_k^{-1})\,.$$

The ranges of the eigenvalues are restricted by the conditions:

1. k and k' belong to the same sheet of the mass hyperboloid

$$k^2 = k'^2 = M^2 > 0$$

2. $2s$ belongs to N and $\lambda, \lambda' = -s, -s+1, \ldots, +s$, and H_k is defined by

$$H_k^{-1} \overset{0}{M} = \varepsilon k\,.$$

2. *The Zero-Mass Case*

The compatibility conditions are now

$$k'^2 = k^2 = 0 \qquad \text{and} \qquad k'_0 k_0 > 0\,.$$

In the particular case where we have $k_0 = k'_0 = \varepsilon \overset{03}{1}$ (ε is the sign of the energy) the solution is

$$\psi'(X) = \psi_1(X)\, \nu\, \delta^3(X)$$

where $\nu\, \delta^3(X)$ is an invariant measure carried by $ST(2)$ and defined by

$$\int_{SL(2,C)} \nu\, \delta^3(X)\, \varphi(X)\, d^6X = \int_{ST(2)} \varphi(S)\, d^3S\,.$$

In the same way as for non-zero mass, we define the Dirac measure on the zero-mass shell at the point q:

$$\nu\,\delta^3(p, q) = |p_0|\;\delta^3(\boldsymbol{p} - \boldsymbol{q})\;\Theta(p_0 q_0)$$

and we can write

$$\nu\,\delta^3(X) = \nu\,\delta^3\left(\overset{03}{X}\overset{03}{1, 1}\right).$$

Now, the W_μ's are operators of the Euclidian group of the plane and the remaining equations give rise to the matrix elements of this group, which can be found for example in VILENKIN's book [6]. For us, it is very easy to get them by the same method, with a generating function. We take

$$\psi(X) = \exp\frac{1}{2}\,i\,\{r X_{12} + r X_{12}^* + 2 s\,\varphi\}$$

when $X = \begin{pmatrix} e^{i\varphi/2} & X_{12} \\ 0 & e^{-i\varphi/2} \end{pmatrix}$.

This function is an eigenfunction of W_g^2 with $-r^2$ as eigenvalue and we expand it (in the same way as before) into eigenfunctions of W_{g3} and W_{d3}; after normalization, the result is

$$f_{r\varepsilon'\lambda'\lambda}(X) = \frac{1}{2\pi}\int_0^{2\pi} d\eta \exp\frac{1}{2} \cdot i\,\{r X_{12} e^{-i\eta} + r X_{12}^* e^{i\eta} + 2(\lambda' - \lambda)\,\eta + (\lambda' + \lambda)\,\varphi\} \tag{11}$$

and when $r = 0$, this matrix element is zero unless $\lambda = \lambda'$; r must be a real positive number; the eigenvalue of W^2 is $-r^2$, that of $S_{d3}(= W_{d3})$ is λ and that of $W_{g3}(= -S_{g3})$ is λ'; ε' indicates the parity of 2λ and $2\lambda'$ (if $\varepsilon' = 0$, λ and λ' are integers and if $\varepsilon' = 1$, λ and λ' are half integers). In the case where $r = 0$ (finite spin), one has $\lambda = \lambda'$.

With these expressions, one can compute the character of $ST(2)$ $(r \neq 0)$

$$\chi_{r\varepsilon}(X) = \sum_\lambda{}' f_{r\varepsilon\lambda\lambda}(X) = (\delta(\varphi) + (-1)^\varepsilon\,\delta(\varphi - 2\pi))\int_0^{2\pi} e^{ir|X_{12}|\cos\theta}\,d\theta$$

$\left(\sum_\lambda{}'\right.$ means a summation over all λ such that 2λ has the parity of $\left.\varepsilon\right)$ and also the orthogonality relations:

$$\int_{ST(2)} f_{r\varepsilon\lambda'\lambda}(S)\,f^*_{r_1\varepsilon_1\lambda_1'\lambda_1}(S)\,d^3S = \delta_{\varepsilon\varepsilon_1}\,\delta_{\lambda'\lambda_1'}\,\delta_{\lambda\lambda_1}\,\delta(r - r_0)/2r_0\,.$$

These functions are matrix elements and we have the relations

$$\sum_\lambda{}' f_{r\varepsilon\lambda'\lambda}(X)\,f_{r\varepsilon\lambda\lambda_1}(Y) = f_{r\varepsilon\lambda'\lambda_1}(XY) \quad (\text{for } r \neq 0)$$

$$f_{r\varepsilon\lambda'\lambda}(X^\dagger) = f^*_{r\varepsilon\lambda\lambda'}(X)\,.$$

We can then write the particular solution and obtain the general one by translating it:

$$\begin{aligned} \psi(r\varepsilon\, k' k \lambda' \lambda;\ (x, X)) &= e^{i k' x} \nu\, \delta^3 (T_{k'} X T_k^{-1})\, f_{r\varepsilon\lambda'\lambda}(T_{k'} X T_k^{-1}) \\ &= e^{i k' x} \nu\, \delta^3\, f_{r\varepsilon\lambda'\lambda}(T_{k'} X T_k^{-1}) \end{aligned} \tag{12}$$

in short-hand notation, and the result is similar for the case $r = 0$.

The ranges of the parameters are given by:

- k and k' belong to the same sheet of the light cone.
- if $r > 0$, $\varepsilon = 0$, λ and λ' are integers
  $\varepsilon = 1$, λ and λ' are half integers
- if $r = 0$, the conditions are the same, but moreover $\lambda = \lambda'$.

3. *The Unphysical Mass Case*

The compatibility conditions reduce to

$$k^2 = k'^2 = -M^2 \leqq 0\,.$$

In the particular case where one has $k = k' = \overset{3}{M}$, the solution is

$$\psi'(X) = \psi_1(X)\, \zeta\, \delta^3(X)$$

where $\zeta\, \delta^3(X)$ is an invariant measure of $SU(1, 1)$ defined by

$$\int_{SL(2,C)} \zeta\, \delta^3(X)\, \varphi(X)\, d^6 X = \int_{SU(1,1)} \varphi(0)\, d^3 0\,.$$

As for the other cases, one can define the Dirac measure on the imaginary mass shell (defined by q) at the point q:

$$\zeta\, \delta^3(p, q) = |p_0|\, \delta^3(\boldsymbol{p} - \boldsymbol{q})\, \Theta(p_0 q_0)$$

and then one can write

$$\zeta\, \delta^3(X) = \zeta\, \delta^3 \left(X \overset{3}{1}, \overset{3}{1} \right) \cdot$$

Now, the W_μ's are operators of the $SU(1, 1)$ subgroup and the remaining equations lead to its matrix elements. Matrix elements of the $SU(1,1)$ group are given by BARGMANN [7] or BARUT and FRONSDAL [8]. From our standpoint, they are easily obtained by using a generating function, the same as for the $SU(2)$ subgroup;

$$\psi(X) = (X_{11} + X_{12} + X_{21} + X_{22})^{2s}$$

and the corresponding eigenvalue for W^2 is now $M^2 s(s + 1)$. One term of its formal expansion (with $\alpha + \beta = 2s$) written as

$$(X_{11} + X_{12})^\alpha (X_{21} + X_{22})^\beta$$

is an eigenfunction of S_{g0} (S_{g0} is defined to be $-M^{-1} \cdot W_{g0}$) with $\frac{1}{2}(\beta - \alpha)$ as eigenvalues. Now, we expand it into eigenfunctions of

$S_{d0}(S_{d0} = M^{-1} \cdot W_{d0})$ by using an integration over the one-parameter rotation subgroup generated by S_{d0}, and we normalize it by a factor $N_s(\lambda', \lambda)$. The result is

$$\begin{aligned}\psi_{s\lambda'\lambda}(X) = \frac{N_s(\lambda', \lambda)}{2\pi} \int_0^{2\pi} d\varphi (X_{11} e^{i\varphi} + X_{12} e^{-i\varphi})^{s+\lambda'} \\ \cdot (X_{21} e^{i\varphi} + X_{22} e^{-i\varphi})^{s-\lambda'} e^{-2i\lambda\varphi} .\end{aligned} \tag{13}$$

When the unitarity relation and the composition relations

$$\sum_{\lambda} \psi_{s\lambda'\lambda}(X)\, \psi_{s\lambda\lambda_1}(Y) = \psi_{s\lambda'\lambda_1}(X\,Y)$$

$$\psi_{s\lambda'\lambda}(X^{-1}) = \psi_{s\lambda\lambda'}(X)^* \quad (\text{for } X \in SU(1,1))$$

are required, the ranges of the parameters are restricted and the normalization factor is settled up to a phase factor:

1. $s = -\frac{1}{2} + i\varrho$ $(\varrho > 0)$; 2λ and $2\lambda'$ are integers having same parity and any sign

$$N_s(\lambda', \lambda) = 1$$

2. $-1 \leqq 2s < 0$; λ, λ' are integers of any sign

$$N_s(\lambda', \lambda) = [(s+\lambda)!\,(\lambda' - s - 1)!]^{1/2}\, [(s+\lambda')!\,(\lambda - s - 1)!]^{-1/2}$$

3. $2s$ is an integer $\geqq -1$, 2λ and $2\lambda'$ are integers of the same parity as $2s$ and

a) λ and $\lambda > s$

$$N_s(\lambda', \lambda) = [(s+\lambda)!\,(\lambda' - s + 1)!]^{1/2}\, [(s+\lambda')!(\lambda - s - 1)!]^{-1/2}$$

b) λ and $\lambda' < s$

$$N_s(\lambda', \lambda) = [(s-\lambda)!\,(-\lambda' - s - 1)!]^{1/2}\, [(s-\lambda')!\,(-\lambda - s - 1)!]^{-1/2} .$$

These matrix elements have the same normalization as those of Bargmann in the discrete case (3a and 3b).

To distinguish between them, we introduce a parameter ε which can be

$\varepsilon = 0$: continuous series with λ and λ' integer (case 1 or 2)

$\varepsilon = 1$: continuous series with λ and λ' half integer (case 1)

$\varepsilon = +$: discrete series with positive λ, λ' (case 3a)

$\varepsilon = -$: discrete series with negative λ, λ' (case 3b).

Then the values of s and ε completely fix the representation and we can denote the matrix elements by the symbol

$$f_{\varepsilon s\lambda'\lambda}(X) .$$

From Bargmann's work (or directly from the expressions of the matrix elements given here in the case of the discrete series) we get the orthogonality relations:

1. Discrete series with $s \geqq 0$

$$\int\limits_{SU(1,1)} f^*_{\varepsilon s \lambda' \lambda}(0)\, f_{\varepsilon_1 s_1 \lambda'_1 \lambda_1}(0)\, d^3 0 = \frac{1}{2(2s+1)}\, \delta_{s s_1} \delta_{\varepsilon \varepsilon_1} \delta_{\lambda \lambda} \delta_{\lambda' \lambda'_1}\,. \tag{14}$$

2. Continuous series with $s = -\frac{1}{2} + i\varrho$ ($\varrho > 0$ and $\varepsilon = 0,1$)

$$\int f^*_{\varepsilon s \lambda' \lambda}(0)\, f_{\varepsilon_1 s_1 \lambda'_1 \lambda_1}(0)\, d^3 0 = \frac{(\tanh \pi \varrho)^{(2\varepsilon - 1)}}{4\varrho}\, \delta(\varrho - \varrho_0)\, \delta_{\lambda' \lambda'_1} \delta_{\lambda \lambda_1} \delta_{\varepsilon \varepsilon_1}\,. \tag{14'}$$

By summing up the diagonal matrix elements given in their explicit form, we can compute the characters $\chi_{\varepsilon s}$ for:

1. the continuous series

– If $\mathrm{Tr}\, X > 2$, we set $\mathrm{Tr}\, X = 2\tau\, \mathrm{ch}\, \eta/2$ with η real and $\tau = \pm 1$ and we get

$$\chi_{\varepsilon s}(X) = \tau^{\varepsilon}\, \mathrm{ch}\, (2s+1)\frac{\eta}{2} \cdot \left|\mathrm{sh}\, \frac{\eta}{2}\right|^{-1}. \tag{15}$$

– If $\mathrm{Tr}\, X < 2$, we get

$$\chi_{\varepsilon s}(X) = 0$$

2. The discrete series

– If $\mathrm{Tr}\, X > 2$, we set $\mathrm{Tr}\, X = 2\tau\, \mathrm{ch}\, \eta/2$ and we get

$$\chi_{\varepsilon s}(X) = \frac{1}{2}\, \tau^{2s} e^{-(2s+1)|\eta/2|}\, |\mathrm{sh}\, \eta/2|^{-1}\,. \tag{15'}$$

– If $\mathrm{Tr}\, X < 2$, we set $\mathrm{Tr}\, X = 2 \cos \varphi/2$ (φ is taken to be the angle of the rotation conjugate to X), and we get

$$\chi_{\varepsilon s}(X) = \frac{1}{2}\, \varepsilon i e^{i\varepsilon(2s+1)\varphi/2}\, (\sin \varphi/2)^{-1}\,.$$

Now, we define F_k for $k^2 = -M^2$ by setting

$$F_k^{-1} \overset{3}{M} = k\,.$$

Then by translating the particular solution by F_k and $F_{k'}$, we get the general one

$$\begin{aligned} \psi(\varepsilon s k' k \lambda' \lambda;\, (x, X)) &= e^{ik'x} \zeta\, \delta^3 (F_{k'} X F_k^{-1})\, f_{\varepsilon s \lambda' \lambda}(F_{k'} X F_k^{-1}) \\ &= e^{ik'x} \zeta\, \delta^3 f_{\varepsilon s \lambda' \lambda}(F_{k'} X F_k^{-1})\,. \end{aligned} \tag{16}$$

Chapter III. Properties of the Eigendistributions

Let us define a double index (ν, σ) which, as we shall see, characterizes the representations:

a) Non-zero physical masses $M > 0$

$$\nu = M;\ \sigma = (\varepsilon, s) \quad (\text{with } \varepsilon = \pm 1 \quad \text{and} \quad 2s \in N)$$

ε is the sign of the energy, s is the spin of the representation and then $\lambda, \lambda' = -s, -s+1, \ldots, +s$.

b) Zero mass

$\nu = 0$; $\sigma = (\varepsilon, \varepsilon', \mu)$ ε is the sign of energy; ε' distinguishes between the representations:

– $\varepsilon' = +$ or $-$: finite spin representation, $2\mu \in N$ and the helicity is $\varepsilon' \mu$

– $\varepsilon' = 0, 1$: infinite spin representation, λ and λ' are integers if $\varepsilon' = 0$, half integers if $\varepsilon' = 1$, the eigenvalue of W^2 is $-\mu^2$.

c) Imaginary masses iM

$$\nu = iM, \quad \sigma = (s, \varepsilon) .$$

In the same way, ε distinguishes between the representations:

– if s is an integer $\geqq 0$, $\varepsilon = \pm$ gives the sign of λ and λ' which are integers and $\varepsilon\lambda, \varepsilon\lambda' > s$

– if s is a half integer $\geqq -\frac{1}{2}$, $\varepsilon = \pm$ gives the sign of λ, λ' which are half integers and $\varepsilon\lambda, \varepsilon\lambda' > s$

– if s is continuous and $-\frac{1}{2} \leqq s < 0$, $\varepsilon = 0$, λ and λ' are integers

– if $s = -\frac{1}{2} + i\varrho$ with $\varrho > 0$; $\varepsilon = 0, 1$ and 2λ and $2\lambda'$ are of the same parity as ε.

And in any case, k and k' belong to the same mass shell defined by $k^2 = k'^2 = \nu^2$ (and in the cases a and b, they must have the same sign of energy).

Now we reconstruct the unitary representations from the matrix elements. First, we define the space $\mathscr{T}_{\nu\sigma}$ of the parameters (k, λ):

$$\begin{aligned}
&\text{a)} && \mathscr{T}_{M\varepsilon s} = \mathscr{H}^{\varepsilon}_{M} \times I_s \\
&\text{b)} && \mathscr{T}_{0\varepsilon\varepsilon' r} = \mathscr{V}^{\varepsilon}_{0} \times I_{\varepsilon' r} \qquad (17)\\
&\text{c)} && \mathscr{T}_{iMs\varepsilon} = \mathscr{H}_{iM} \times I_{s\varepsilon} .
\end{aligned}$$

In these formulae, $\mathscr{H}^{\varepsilon}_{M}$ is the mass shell of mass M and sign of energy ε, $\mathscr{V}^{\varepsilon}_{0}$ is the ε-energy light cone, $\mathscr{H}_{iM}$ is the imaginary mass shell ($k^2 = -M^2$) and discrete spaces for the parameter λ are given by

$$\begin{aligned}
I_s &= \{-s, -s+1, \ldots, s\} \\
I_{\varepsilon' r} &= \{\varepsilon' r\} \text{ if } \varepsilon = \pm \ (2r \text{ then belongs to } N) \\
&= \{Z + \varepsilon'/2\} \text{ if } \varepsilon' = 0, 1
\end{aligned}$$

$\left(Z \text{ is then the additive group of integers and } Z + \frac{1}{2} \text{ its translated by } \frac{1}{2}\right)$

$$\begin{aligned}
I_{s\varepsilon} &= \{\varepsilon(s+1), \varepsilon(s+2), \ldots\} \quad \text{if} \quad \varepsilon = \pm \\
&= \{Z + \varepsilon/2\} \quad \text{if} \quad \varepsilon = 0, 1 .
\end{aligned}$$

We define the measure on $\mathscr{T}_{\nu\sigma}$ by taking the product of the invariant measure over the mass shell (we normalize it to be $d^3\boldsymbol{k}/M^2|k_0|$ in non-zero mass case and $d^3\boldsymbol{k}/|k_0|$ in the zero-mass case) by the natural measure over the discrete space I_σ for the index λ, which is denoted $\sum_{\lambda \in I_\sigma}$:

$$\tau \in \mathcal{T}_{\nu\sigma} \Rightarrow \tau = (k, \lambda)$$

$$d\mu(\tau) = \frac{d^3 \boldsymbol{k}}{M^2 |k_0|} \sum_{\lambda \in I_\sigma} \quad (\text{if} \quad \nu^2 \neq 0)$$

$$= \frac{d^3 \boldsymbol{k}}{|k_0|} \sum_{\lambda \in I_\sigma} \quad \text{if} \quad \nu^2 = 0$$

$\mathcal{T}_{\nu\sigma}$ is a locally compact space and we can define the matrix elements as measures over $\mathcal{T}_{\nu\sigma} \times \mathcal{T}_{\nu\sigma}$. Let us denote them by the symbol (τ and τ' belong to $\mathcal{T}_{\nu\sigma}$):

$$\mathcal{U}^{\nu\sigma}_{\tau\tau'}(\alpha)$$

and we realize the representations in the space $L^2(\mathcal{T}_{\nu\sigma}, d\mu(\tau))$:

$$(\mathcal{U}^{\nu\sigma}(\alpha)\varphi)(\tau) = \int \mathcal{U}^{\nu\sigma}_{\tau\tau'}(\alpha)\varphi(\tau')\, d\mu(\tau') .$$

In fact, it is easily seen that this integral exists for almost all τ' and defines an element of $L^2(\mathcal{T}_{\nu\sigma}, d\mu)$ which can be written as

$$\{\mathcal{U}^{M\varepsilon s}(x, X)\varphi\}(k, \lambda) = e^{ikx} \sum_{\lambda' = -s}^{+s} f_{s\lambda\lambda'}(H_k X H^{-1}_{X^{-1}k})\varphi(X^{-1}k, \lambda')$$

$$\{\mathcal{U}^{0\varepsilon\varepsilon' r}(x, X)\varphi\}(k, \lambda) = e^{ikx} \sum_{\lambda' \in I_{\varepsilon' r}} f_{\varepsilon' r\lambda\lambda'}(T_k X T^{-1}_{X^{-1}k})\varphi(X^{-1}k, \lambda') \quad (18)$$

$$\{\mathcal{U}^{iMs\varepsilon}(x, X)\varphi\}(k, \lambda) = e^{ikx} \sum_{\lambda' \in I_{s\varepsilon}} f_{\varepsilon s\lambda\lambda'}(F_k X F^{-1}_{X^{-1}k})\varphi(X^{-1}k, \lambda') .$$

In these formulae, k' and k belong to the right mass shell and the discrete space I_σ has been defined just before. With all the properties we established for the matrix elements of the respective subgroup, we can check that these formulae define all the unitary representations of the Poincaré group (of non-null energy momentum). They are Wigner's well-known formulae and we established them as a subsidary result from our computation of the "exponentials" of the group. These exponentials are needed for establishing the explicit formulae for Fourier transforms on the Poincaré group.

We might also note that this gives a method of determination of irreducible representations of a Lie group; one computes the eigendistributions and adjust the normalization to an adequate measure on the index spaces. This works for the Poincaré group and might work for other Lie groups.

The matrix elements, computed as eigendistributions are, in fact, fairly simple measures on the group, whose carriers are algebraic submanifolds. We shall study their properties from a measure-theoretic point of view.

1. *Unitarity*

One extends to distributions the involution I defined in Chapter I by setting

$$\langle I T, \varphi\rangle = \langle T, I \varphi\rangle$$

where T is a distribution, φ a test function and

$$(I \varphi)(\alpha) = \varphi(\alpha^{-1}) .$$

When the right-hand side is computed, the following relations are obtained:

$$\begin{aligned} I \psi(s k' k \lambda' \lambda; (x, X)) &= \psi^*(s k k' \lambda \lambda'; (x, X)) \\ I \psi(r \varepsilon k' k \lambda' \lambda; (x, X)) &= \psi^*(r \varepsilon k k' \lambda \lambda'; (x, X)) \\ I \psi(\varepsilon s k' k \lambda' \lambda; (x, X)) &= \psi^*(\varepsilon s k k' \lambda \lambda'; (x, X)) . \end{aligned} \tag{19}$$

This is the unitarity of the generalized matrix elements expressed over a continuous basis indexed by $(k\lambda)$.

2. *Composition Relations*

As we have established that the eigendistributions are matrix elements of unitary irreducible representations of the Poincaré group, the relation

$$\int \mathcal{U}^{\nu\sigma}_{\tau\tau'}(\alpha)\, \mathcal{U}^{\nu\sigma}_{\tau'\tau''}(\beta)\, d\mu(\tau') = \mathcal{U}^{\nu\sigma}_{\tau\tau''}(\alpha\beta)$$

must be interpreted as a weak integral of a measure:

$$\int \mathcal{U}^{\nu\sigma}_{\tau\tau'}(\alpha)\, \{\int \mathcal{U}^{\nu\sigma}_{\tau'\tau''}(\beta)\, \varphi(\tau'')\, d\mu(\tau'')\}\, d\mu(\tau') = \int \mathcal{U}^{\nu\sigma}_{\tau\tau''}(\alpha\beta)\, \varphi(\tau'')\, d\mu(\tau'')$$

and has a meaning for any φ continuous with compact support. We can also have another interpretation as a weak integral of measure over the group itself, that is

$$\int \mathcal{U}^{\nu\sigma}_{\tau\tau'}(\alpha)\, \{\int \mathcal{U}^{\nu\sigma}_{\tau'\tau''}(\beta)\, \varphi(\beta)\, d^{10}\beta\}\, d\mu(\tau') = I(\varphi) .$$

The left-hand side is easily computed by applying the Fubini theorem on the product space $\mathcal{T}_{\nu\sigma} \times \mathcal{P}$ and when the integration over $d\mu(\tau')$ is performed first, the expected result is obtained:

$$\begin{aligned} &\int_{\mathcal{H}^{\varepsilon}_{M}} \sum_{\lambda'=-s}^{+s} \psi(s k k' \lambda \lambda'; \alpha)\, \psi(s k' k'' \lambda' \lambda''; \beta) \frac{d^3 \boldsymbol{k}'}{|M^2 k_0'|} = \psi(s k k'' \lambda \lambda''; \alpha\beta) \\ &\int_{\mathcal{V}^{\varepsilon}_{0}} \sum_{\lambda' \in I_{\varepsilon' r}} \psi(r \varepsilon' k k' \lambda \lambda'; \alpha)\, \psi(r \varepsilon' k' k'' \lambda' \lambda''; \beta) \frac{d^3 \boldsymbol{k}'}{|k_0'|} = \psi(r \varepsilon' k k'' \lambda \lambda''; \alpha\beta) \\ &\int_{\mathcal{H}_{iM}} \sum_{\lambda' \in I_{s\varepsilon}} \psi(\varepsilon s k k' \lambda \lambda'; \alpha)\, \psi(\varepsilon s k' k'' \lambda' \lambda''; \beta) \frac{d^3 \boldsymbol{k}'}{M^2 |k_0'|} = \psi(\varepsilon s k k'' \lambda \lambda''; \alpha\beta) . \end{aligned} \tag{20}$$

3. Orthogonality Relations

They are similar to the relation

$$\int_{-\infty}^{+\infty} e^{ikx} e^{-ik'x} dx = 2\pi\, \delta(k-k')$$

which must be understood as

$$\int_{-\infty}^{+\infty} e^{ikx} \langle e^{-ik'x}, \varphi(k')\rangle\, dx = 2\pi \langle \delta(k-k'), \varphi(k')\rangle = 2\pi\, \varphi(k)\,.$$

This relation is true when φ is smooth enough (for example when φ is integrable and differentiable). This can be seen as a weak integral of distributions on the index spaces, and this method gives a justification for the formal computations, at least in the physical non-zero mass case.

Another way to understand such a formula is to take it as a Bessel-Parseval formula for the Fourier transform, then the orthogonality relations hold for all the representations present in the Plancherel measure (see Part II).

We just list now the result which are easily computed formally:

$$\begin{aligned}
&\int_{\mathscr{P}} \psi(s k' k \lambda' \lambda; \alpha)\, \psi^*(s_1 k_1' k_1 \lambda_1 \lambda_1'; \alpha)\, d^{10}\alpha \\
&\qquad = (2\pi)^4\, \delta^4(k'-k_1')\, M^2 \omega\, \delta^3(k, k_1)\, \frac{\delta_{ss_1}}{2s+1}\, \delta_{\lambda\lambda_1} \delta_{\lambda'\lambda_1'} \\
&\int_{\mathscr{P}} \psi(r \varepsilon' k' k \lambda' \lambda; \alpha)\, \psi^*(r_1 \varepsilon_1' k_1' k_1 \lambda_1' \lambda_1; \alpha)\, d^{10}\alpha \qquad (21)\\
&\qquad = (2\pi)^4 \delta^4(k'-k_1')\, \nu\, \delta^3(k, k_1)\, \frac{\delta(r-r_1)}{2r_1}\, \delta_{\lambda'\lambda_1'} \delta_{\lambda\lambda_1} \delta_{\varepsilon'\varepsilon_1'} \\
&\int_{\mathscr{P}} \psi(\varepsilon s k' k \lambda' \lambda; \alpha)\, \psi^*(\varepsilon_1 s_1 k_1' k_1 \lambda_1' \lambda_1; \alpha)\, d^{10}\alpha \\
&\qquad = \frac{(2\pi)^4}{2(2s+1)}\, \delta(k'-k_1')\, M^2 \zeta\, \delta^3(k, k_1)\, \delta_{\varepsilon\varepsilon_1} \delta_{ss_1} \delta_{\lambda\lambda_1} \delta_{\lambda_1\lambda_1'}
\end{aligned}$$

when $2s \in N$ and $\varepsilon = \pm$, and

$$\begin{aligned}
&\int_{\mathscr{P}} \psi(\varepsilon s k' k \lambda' \lambda; \alpha)\, \psi^*(\varepsilon_1 s_1 k_1' k_1 \lambda_1' \lambda_1; \alpha)\, d^{10}\alpha \\
&\qquad = (2\pi)^4 \delta^4(k'-k_1')\, M^2 \zeta\, \delta^2(k, k_1)\, \delta_{\varepsilon\varepsilon_1} \delta_{\lambda\lambda_1'} \delta_{\lambda'\lambda_1'} C_\varepsilon^{-1}(\varrho)\, \delta(\varrho-\varrho_0)
\end{aligned}$$

when $s = -\frac{1}{2} + i\varrho$ and $\varepsilon = 0, 1$; $C_\varepsilon(\varrho) = 4\varrho(\tanh \pi\varrho)^{(1-2\varepsilon)}$.

4. The Character of Non-Zero Mass Representations

A matrix element is a measure on the Poincaré group, so we can define, for any test function φ the quantity

$$\langle \mathscr{U}_{\tau\tau}^{\nu\sigma}, \varphi\rangle\,.$$

When this function of τ is integrable, it defines a functional of φ which is in fact the trace of the operator

$$\mathscr{U}(\varphi) = \int \mathscr{U}(\alpha)\, \varphi(\alpha)\, d^{10}\alpha$$

defined in the representation space.

This computation has been already done by Joos and Schrader [9], but we were able to prove that, with an adequate limiting process to define the integral

$$\int_{\mathscr{T}_{\nu\sigma}} \langle \mathscr{U}^{\nu\sigma}_{\tau\tau}, \varphi\rangle\, d_\mu(\tau)$$

it makes sense for any ψ continuous with compact support in the physical mass case; the character is a measure on the group.

In the non physical mass case, we need someting more, namely that φ is continuously differentiable on the group, then the integral makes sense and defines a measure.

The characters can be written pointwisely, they are invariant by conjugation and we give here their expressions for the diagonal element of each conjugation class:

a) Physical mass:

$$\mathscr{C}^{M(\varepsilon s)}(x, X) = \frac{\delta(\eta)\,\chi_s(\varphi)}{4\sin^2\varphi/2} \int_{-\infty}^{+\infty} e^{i\varepsilon(k_0 x_0 - k_3 x_3)} \frac{d k_3}{k_0} \tag{22}$$

where $X = \begin{pmatrix} e^{(\eta + i\varphi)/2} & 0 \\ 0 & e^{-(\eta + i\varphi)/2}\end{pmatrix}$

$$\chi_s(\varphi) = \frac{\sin(2s+1)\varphi/2}{\sin\varphi/2}$$

$$k_0 = \sqrt{k_3^2 + M^2}$$

b) Non physical mass

$$\begin{aligned}\mathscr{C}^{iM(s\varepsilon)}(x, X) &= \chi_{\varepsilon s}(X) \frac{\delta(\eta)}{4\sin^2\varphi/2} \int_{-\infty}^{+\infty} \{e^{i(k_0 x_0 - k_3 y_3)} + e^{i(k_0 x_0 + k_3 x_3)}\} \frac{d k_0}{k_3} \\ &+ \chi_{\varepsilon s}(X) \frac{\delta(\varphi) + \delta(\varphi + 2\pi)}{4\,\mathrm{sh}^2\eta/2} \int e^{iM(x_1\cos\theta + x_2\sin\theta)}\, d\theta \end{aligned} \tag{22'}$$

where X has the same definition as before, $k_3 = \sqrt{k_0^2 + M^2}$ and $\chi_{\varepsilon s}(X)$ are the characters of the representations of $SU(1, 1)$ we computed in Chapter II. For more details one can see ref. [9, 10].

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