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On Some Rajchman Measures and Equivalent Salem's Problem

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Abstract

We construct certain Rajchman measures by using integrability properties of the Fourier and Fourier-Stieltjes transforms. In particular, we state a problem and prove that it is equivalent to the known and still unsolved question posed by R. Salem (*Trans. Amer. Math. Soc.* **53** (3) (1943), p. 439) whether Fourier-Stieltjes coefficients of the Minkowski question mark function vanish at infinity.

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1 Introduction

Let φ be continuous function of bounded variation on an interval I. It is known (see, for instance, [6], Ch. 8) that φ supports a measure on I, i.e. there exists a regular Borel measure $\mu_\varphi(I)$ with bounded variation such that $\mu_\varphi([a,b)) = \varphi(b) - \varphi(a)$ for all $a, b \in I,\ a < b$. Moreover, since φ is continuous, then μ_φ is non-atomic and denoting it simply by $d\varphi$, the corresponding Fourier-Stieltjes integral will be written as

$$\widehat{\mu}_\varphi(t) = \int_I e^{ixt} d\varphi(x). \tag{1.1}$$

Further, if φ is absolutely continuous then $\widehat{\mu}_\varphi(t) \to 0$, as $|t| \to \infty$, because in this case the Fourier-Stieltjes transform $\widehat{\mu}_\varphi(t)$ is an ordinary Fourier transform of an integrable function.

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Thus φ supports a measure whose Fourier-Stieltjes transform vanishes at infinity. Such a measure is called a Rajchman measure (see details, for instance, in [5]). However, when φ is only continuous, the situation is quite different and the classical Riemann-Lebesgue lemma for the class L_1, in general, cannot be applied. The question is quite delicate when it concerns singular functions (see [11], Ch. IV). A singular function is defined as a continuous, bounded monotone function with a null derivative almost everywhere. Hence it supports a positive bounded Borel measure, which is singular with respect to Lebesgue measure. For such singular measures there are various examples whose Fourier transforms do not tend to zero, although some do (see, for instance, in [9], [10], [7]). In [13] (see also [4]) it was proved that for every $\varepsilon > 0$ there exists a singular monotone function, which supports a measure whose Fourier-Stieltjes transform behaves as $O(t^{-\frac{1}{2}+\varepsilon})$, $|t| \to \infty$.

Our goal here is to construct some Rajchman's measures, which are associated with continuous functions of bounded variation. In particular, it will be proved that the famous Minkowski question mark function $?(x)$ [2], which in the sequel will be denoted by $\psi(x)$, supports a Rajchman measure if and only if its Fourier-Stieltjes transform has a limit at infinity, and then, of course, the limit should be zero. This is still unsolved question posed by Salem in 1943 [9]. In fact, it is worth to mention that it is an old and quite attractive problem in the number theory and Fourier analysis. Moreover, recently [15], the author tried to give a solution to Salem's problem basing on Naylor's asymptotic formula for the Kontorovich-Lebedev transform of continuous functions at infinity (see the corresponding reference in [14], [15]). Unfortunately, it was found (see [16], [1]) that this asymptotic expansion does not work for extreme values of a parameter, and author's attempt was fallacious. We also mention about another approach towards the affirmative solution, which is exhibited in [1] and where some integral and discrete functional equations for the Fourier-Stieltjes transform of ψ are proved. Anyway we hope that the results, which will be presented in the sequel, have their own interest in Fourier analysis and can help to solve finally the Salem question.

2 Some Rajchman measures

In this section we prove three theorems, characterizing Rajchman measures, which are associated with Fourier-Stieltjes integrals (1.1) over $[0,1]$ and $[0,\infty[$.

We begin with the following general result.

Theorem 2.1. *Let $I = [0,\infty[$ and φ be a real- valued continuous integrable function of bounded variation on I such that $\varphi(x) = o(1)$, $x \to \infty$. Let $\Phi(t) = \widehat{\mu}_\varphi(t)$. Then φ supports a Rajchman measure (i.e. $\Phi(t) = o(1)$, $|t| \to \infty$), if and only if $\Phi(t)$ has a limit when $|t| \to \infty$.*

Proof. Without loss of generality we prove the theorem for positive t. Evidently, the necessity is trivial and we will prove the sufficiency. Suppose that the limit of $\Phi(t)$ when $t \to +\infty$ exists. Since $\Phi(t) = \Phi_c(t) + i\Phi_s(t)$, where $\Phi_c(t), \Phi_s(t)$ are real-valued and

$$\Phi_c(t) = \int_0^\infty \cos xt \, d\varphi(x), \tag{2.1}$$

$$\Phi_s(t) = \int_0^\infty \sin xt \, d\varphi(x), \tag{2.2}$$

we will treat these Fourier-Stieltjes transforms separately. Indeed, taking (2.1) and appealing to the formula for integration by parts in the Stieltjes integral [3], we get

$$\Phi_c(t) = -\varphi(0) + t\int_0^\infty \varphi(x)\sin xt\, dx. \tag{2.3}$$

However, since $\varphi \in L_1(\mathbb{R}_+; dx)$, we appeal to the integrated form of the Fourier formula (cf. [12], Th. 22) to write for all $x \geq 0$

$$\int_0^x \varphi(y)\, dy = \frac{2}{\pi}\int_0^{\to\infty} \frac{1-\cos yx}{y}\int_0^\infty \varphi(u)\sin uy\, du dy.$$

We note, that the integral with respect to u in the latter formula is understood in the Lebesgue sense, whereas the integral by y is in the Cauchy sense. But taking into account the previous equality after simple change of variable we come out with the relation

$$\frac{1}{x}\int_0^x \varphi(y)\, dy = \frac{2}{\pi}\int_0^\infty \frac{1-\cos y}{y^2}\left[\varphi(0) + \Phi_c\left(\frac{y}{x}\right)\right] dy,\ x > 0,$$

where the integral in the right-hand side is absolutely convergent due to the boundedness of Φ_c. Minding the value of elementary Fejer type integral

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos y}{y^2} dy = 1,$$

we establish an important equality

$$\frac{1}{x}\int_0^x [\varphi(y) - \varphi(0)]\, dy = \frac{2}{\pi}\int_0^\infty \Phi_c\left(\frac{y}{x}\right)\frac{1-\cos y}{y^2} dy,\ x > 0. \tag{2.4}$$

Meanwhile, the left-hand side of (2.4) evidently goes to zero when $x \to 0+$ via the continuity of φ on $[0,\infty)$. Further, since φ is of bounded variation on $(0,\infty)$ we obtain the uniform estimate

$$|\Phi_c(t)| \leq \int_0^\infty dV_\varphi(x) = \Phi_0,$$

where $V_\varphi(x)$ is a variation of φ on $[0,x]$ and $\Phi_0 > 0$ is a total variation of φ. This means that $\Phi_c(t)$ is continuous and bounded on $\mathbb{R}_+$. Furthermore, the integral with respect to y in the right-hand side of (2.4) converges absolutely and uniformly by virtue of the Weierstrass test. Consequently, since $\Phi_c(t)$ has a limit at infinity, which is finite, say a, one can pass to the limit through equality (2.4) when $x \to 0+$. Hence we find

$$0 = \lim_{x\to 0+}\frac{1}{x}\int_0^x [\varphi(y) - \varphi(0)]\, dy = \frac{2a}{\pi}\int_0^\infty \frac{1-\cos y}{y^2} dy = a.$$

So, $a = 0$ and in order to complete the proof, we need to verify whether the Fourier-Stieltjes transform (2.2) tends to zero as well. To do this, we appeal to the corresponding integrated form of the Fourier formula for the Fourier cosine transform

$$-\int_0^x \varphi(y)\, dy = \frac{2}{\pi}\int_0^\infty \frac{\sin yx}{y^2}\Phi_s(y)\, dy,\ x > 0, \tag{2.5}$$

where after integration by parts $\Phi_s(t)$ turns to be represented as follows

$$\Phi_s(t) = -t\int_0^\infty \varphi(u)\cos ut\, du,\ t > 0. \tag{2.6}$$

Hence it is easily seen that $\Phi_s(t) = O(t),\ t \to 0+$ and since $|\Phi_s(t)| \le \Phi_0$ we have that $\frac{\Phi_s(t)}{t} \in L_2(\mathbb{R}_+; dx)$. This means that the integral in the right-hand side of (2.5) converges absolutely and uniformly by $x \ge 0$. After a simple change of variable we split the integral on the right-hand side of (2.5) into two integrals and obtain

$$-\frac{1}{x}\int_0^x \varphi(y)\, dy = \frac{2}{\pi}\int_0^1 \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right) dy + \frac{2}{\pi}\int_1^\infty \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right) dy.$$

Considering again $x > 0$ sufficiently small and splitting the integral over $(0,1)$ on two more integrals over $(0, x\log^\gamma(1/x))$ and $(x\log^\gamma(1/x), 1)$, where $0 < \gamma < 1$, we derive the equality

$$\frac{2}{\pi}\int_{x\log^\gamma(1/x)}^1 \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right) dy = -\frac{1}{x}\int_0^x \varphi(y)\, dy - \frac{2}{\pi}\int_1^\infty \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right) dy$$
$$-\frac{2}{\pi}\int_0^{x\log^\gamma(1/x)} \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right) dy.$$

Minding the inequality (see (2.6)) $|\Phi_s(t)| \le t\|\varphi\|_{L_1(\mathbb{R}_+;dx)},\ t \ge 0$, the right-hand side of the latter equality has the straightforward estimate

$$\begin{aligned}&\left|\frac{1}{x}\int_0^x \varphi(y)\, dy + \frac{2}{\pi}\int_1^\infty \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right) dy\right.\\ &\left.+\frac{2}{\pi}\int_0^{x\log^\gamma(1/x)} \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right) dy\right| \le \sup_{y\ge 0}|\varphi(y)| + \frac{2}{\pi}\left[\Phi_0 + \|\varphi\|_{L_1(\mathbb{R}_+;dx)}\log^\gamma(1/x)\right].\end{aligned} \tag{2.7}$$

On the other hand, via the first mean value theorem

$$\frac{2}{\pi}\left|\int_{x\log^\gamma(1/x)}^1 \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right) dy\right| = \frac{2}{\pi}|\Phi_s(\xi(x))|\int_{x\log^\gamma(1/x)}^1 \frac{\sin y}{y^2}\, dy,$$

where

$$\log^\gamma\left(\frac{1}{x}\right) \le \xi(x) \le \frac{1}{x}.$$

Meanwhile, we have

$$\frac{2}{\pi}\int_{x\log^\gamma(1/x)}^1 \frac{\sin y}{y^2}\, dy > \frac{2\sin 1}{\pi}\int_{x\log^\gamma(1/x)}^1 \frac{dy}{y} = \frac{2\sin 1}{\pi}\log\left(\frac{1}{x\log^\gamma(1/x)}\right).$$

Consequently, combining with (2.7) we find

$$\begin{aligned}|\Phi_s(\xi(x))| &< \frac{1}{\sin 1}\left[\frac{\pi}{2}\sup_{y\ge 0}|\varphi(y)| + \Phi_0 + \|\varphi\|_{L_1(\mathbb{R}_+;dx)}\log^\gamma(1/x)\right]\log^{-1}\left(\frac{1}{x\log^\gamma(1/x)}\right)\\ &= o(1),\ x \to 0+.\end{aligned} \tag{2.8}$$

Thus making $x \to 0+$ we get $\xi(x) \to +\infty$ and therefore there is a subsequence $t_n = \xi(x_n) \to \infty$ such that $\lim_{n\to+\infty}|\Phi_s(t_n)| = 0$. But since the limit of $\Phi_s(t)$ exists, when $t \to +\infty$ it will be zero. So φ supports a Rajcman measure and the theorem is proved. □

Corollary 2.2. *Under the conditions of Theorem 2.1 φ supports a Rajchman measure if and only if the limit*

$$\lim_{t\to+\infty} t\int_0^\infty e^{itx}\varphi(x)\,dx$$

exists and equals $i\varphi(0)$.

A more general result deals with the smoothness of the Fourier-Stieltjes transform (1.1) and the behavior at infinity of its derivatives.

We have

Theorem 2.3. *Let $n\in\mathbb{N}_0$, $\varphi(x),\ x\geq 0$ be a real-valued continuous function such that $x^m\varphi(x)$ is of bounded variation on $[0,\infty)$ for each $m=0,1,\ldots,n$. If $\varphi(x)=o(x^{-n}), x\to\infty$ and $x^n\varphi(x)\in L_1(\mathbb{R}_+;dx)$, then the corresponding Fourier-Stieltjes transform (1.1) $\Phi(t)$ is n times differentiable on $\mathbb{R}_+$, its m-th order derivative is equal to*

$$\Phi^{(m)}(t)=\int_0^\infty (ix)^m e^{itx}\,d\varphi(x),\ m=0,1,\ldots,n \tag{2.9}$$

and vanishes at infinity if and only if there exists a limit of the integral

$$\widehat{\mu}_{x^m\varphi}(t)=\int_0^\infty e^{itx}\,d\left(x^m\varphi(x)\right)$$

when $|t|\to\infty$.

Proof. In fact, the case $n=0$ corresponds the previous Theorem 2.1. Further, for $m,n\in\mathbb{N}, m\leq n$ integrals

$$\int_0^\infty e^{itx}\,d\left(x^m\varphi(x)\right),\qquad \int_0^\infty e^{itx}x^{m-1}\varphi(x)dx$$

converge since $Var_{[0,\infty)}\left(x^m\varphi(x)\right)<\infty$ and $x^{m-1}\varphi(x)\in L_1(\mathbb{R}_+;dx)$. Hence

$$\int_0^\infty (ix)^m e^{itx}\,d\varphi(x)=\int_0^\infty e^{itx}\,d\left((ix)^m\varphi(x)\right)-mi^m\int_0^\infty x^{m-1}e^{itx}\varphi(x)dx$$

and

$$\left|\int_0^\infty (ix)^m e^{itx}\,d\varphi(x)\right|\leq Var_{[0,\infty)}\left(x^m\varphi(x)\right)$$

$$+m\int_0^\infty x^{m-1}|\varphi(x)|\,dx<\infty,\ m=1,\ldots,n.$$

Now, since $x^m d\varphi(x)=\mu_{x^m\varphi}-mx^{m-1}\varphi(x)dx$, it is indeed a measure with bounded variation. Therefore by induction and Lebesgue dominated convergence theorem we establish (2.9). Moreover, the Fourier integral of the function $x^{m-1}\varphi(x)$ vanishes at infinity via the Riemann-Lebesgue lemma. Consequently, $\Phi^{(m)}(t)\to 0,\ |t|\to\infty$ if and only if $\widehat{\mu}_{x^m\varphi}(t)\to 0,\ |t|\to\infty$, and this holds by Theorem 2.1 if and only if the corresponding Fourier-Stieltjes integral has a limit at infinity. □

Let us consider the case of the Fourier-Stieltjes integral (1.1) over the interval $I = [0,1]$. We will give sufficient conditions in order to guarantee that μ_φ is a Rajchman measure, involving the Kontorovich-Lebedev integrals with the modified Bessel function $K_{i\tau}(x)$ of the pure imaginary index [14]. It is known that the modified Bessel function $K_\mu(z)$ satisfies the differential equation

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2+\mu^2)u = 0$$

and has the following asymptotic behavior

$$K_\mu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1+O(1/z)], \qquad z\to\infty, \tag{2.10}$$

$$K_\mu(z) = O(z^{-|\mathrm{Re}\mu|}),\ z\to 0,\ \mu\neq 0, \tag{2.11}$$

$$K_0(z) = -\log z + O(1),\ z\to 0. \tag{2.12}$$

When $|\tau|\to\infty$ and $x>0$ is fixed it behaves as

$$K_{i\tau}(x) = O\left(\frac{e^{-\pi|\tau|/2}}{\sqrt{|\tau|}}\right). \tag{2.13}$$

We will appeal in the sequel to the uniform inequality for the modified Bessel function

$$|K_{i\tau}(x)| \le \frac{x^{-1/4}}{\sqrt{\sinh \pi\tau}},\ x,\tau>0 \tag{2.14}$$

and its representation via the following Fourier cosine integral

$$\cosh\left(\frac{\pi\tau}{2}\right)K_{i\tau}(x) = \int_0^\infty \cos(\tau u)\cos(x\sinh u)du,\ x>0. \tag{2.15}$$

Furthermore, employing relation (2.16.48.20) in [8] and making differentiation by a parameter, we derive useful integral with respect to an index of the modified Bessel function

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty} &\tau e^{\lambda\tau}\left(t+(1+t^2)^{1/2}\right)^{i\tau} K_{i\tau}(x)d\tau \\ &= x\exp\left(-x\left[(1+t^2)^{1/2}\cos\lambda - it\sin\lambda\right]\right)\left[(1+t^2)^{1/2}\sin\lambda + it\cos\lambda\right], \quad x,t>0,\end{aligned} \tag{2.16}$$

where $0\le\lambda<\frac{\pi}{2}$.

We have

Theorem 2.4. *Let $\varphi(x)$ be a real-valued continuous function of bounded variation on* $[0,1]$ *such that $\varphi(0)=\varphi(1)=0$ and $\varphi(x)/x\in L_2([0,1];dx)$. If*

$$i)\quad t^m\widehat{\varphi}^{(m)}(t)\in L_1([1,\infty);dx),\ m=0,1,2;\ ii)\quad t^{m+1}\widehat{\varphi}^{(m)}(t)=o(1),\ t\to\infty,\ m=0,1, \tag{2.17}$$

where $\widehat{\varphi}^{(m)}(t)$ is the m-th derivative of the Fourier transform

$$\widehat{\varphi}(t) = \int_0^1 e^{itx}\varphi(x)dx,$$

then φ supports a Rajchman measure, i.e. $\widehat{\mu}_\varphi(t)=o(1)$, $|t|\to\infty$.

Proof. Without loss of generality we prove the theorem for positive t. Taking the integral (1.1) over $I = [0,1]$, we integrate by parts and eliminating integrated terms come out with the equality

$$\widehat{\varphi}(t) = -it \int_0^1 e^{itx}\varphi(x)dx.$$

Meanwhile, passing to the limit in equality (2.16) when $\lambda \to \frac{\pi}{2}-$, we find

$$\frac{1}{\pi} \lim_{\lambda\to\frac{\pi}{2}-} \int_{-\infty}^{\infty} \tau e^{\lambda\tau} \left(t + (1+t^2)^{1/2}\right)^{i\tau} K_{i\tau}(x)d\tau = x(1+t^2)^{1/2}\, e^{ixt}, \quad x,t>0. \tag{2.18}$$

Hence

$$\widehat{\varphi}(t) = \frac{t}{\pi i(1+t^2)^{1/2}} \int_0^1 \varphi(x) \lim_{\lambda\to\frac{\pi}{2}-} \int_{-\infty}^{\infty} \tau e^{\lambda\tau} \left(t + (1+t^2)^{1/2}\right)^{i\tau} K_{i\tau}(x)d\tau\, \frac{dx}{x}. \tag{2.19}$$

But since for each $x,t>0$ and $0 \le \lambda < \frac{\pi}{2}$ (see (2.16))

$$\left|\int_{-\infty}^{\infty} \tau e^{\lambda\tau} \left(t + (1+t^2)^{1/2}\right)^{i\tau} K_{i\tau}(x)d\tau\right| \le x\left[t + (1+t^2)^{1/2}\right]$$

and φ is integrable, we can take out the limit in (2.19) and get the representation

$$\widehat{\varphi}(t) = \frac{t}{\pi i(1+t^2)^{1/2}} \lim_{\lambda\to\frac{\pi}{2}-} \int_0^1 \varphi(x) \int_{-\infty}^{\infty} \tau e^{\lambda\tau} \left(t + (1+t^2)^{1/2}\right)^{i\tau} K_{i\tau}(x)d\tau\, \frac{dx}{x}. \tag{2.20}$$

A change of the order of integration in (2.20) is allowed since $x^{-1/4} \in L_2((0,1);dx)$, and therefore the condition $\varphi(x)/x \in L_2((0,1);dx)$ implies $\varphi(x)x^{-5/4} \in L_1((0,1);dx)$. Hence by (2.14) one may apply Fubini's theorem to the integral in (2.20) for a fixed value of λ. Consequently,

$$\widehat{\varphi}(t) = \frac{t}{\pi i(1+t^2)^{1/2}} \lim_{\lambda\to\frac{\pi}{2}-} \int_{-\infty}^{\infty} \tau e^{\lambda\tau} \left(t + (1+t^2)^{1/2}\right)^{i\tau} \left(\int_0^1 K_{i\tau}(x)\varphi(x)\, \frac{dx}{x}\right) d\tau. \tag{2.21}$$

However, the inner integral with respect to x can be treated invoking Parseval equality for the Fourier cosine transform in L_2 [12]. In fact, since for each $\tau \in \mathbb{R}$ $K_{i\tau}(x) \in L_2(\mathbb{R}_+;dx)$ (see the asymptotic behavior of the modified Bessel function (2.10), (2.11)) and $\varphi(x)/x \in L_2((0,1);dx) \subset L_1((0,1);dx)$, we recall (2.15) and making simple substitutions, we deduce

$$\int_0^1 K_{i\tau}(x)\varphi(x)\, \frac{dx}{x} = \frac{1}{\cosh\left(\frac{\pi\tau}{2}\right)} \int_0^{\infty} \frac{\cos\left(\tau \log(u+\sqrt{u^2+1})\right)}{\sqrt{u^2+1}} \int_0^1 \cos(xu)\varphi(x)\, \frac{dxdu}{x}$$

$$= \frac{1}{\cosh\left(\frac{\pi\tau}{2}\right)} \int_0^{\infty} \cos(\tau u) \int_0^1 \cos(x\sinh u)\varphi(x)\, \frac{dxdu}{x}. \tag{2.22}$$

Our goal now is to show that

$$\int_0^1 K_{i\tau}(x)\varphi(x)\,\frac{dx}{x} = O\left(\frac{e^{-\frac{\pi}{2}|\tau|}}{\tau^3}\right),\ |\tau| \to \infty. \tag{2.23}$$

Indeed, integrating by parts in the outer integral by u from the right-hand side of the latter equality in (2.22) when $|\tau|$ is big, and taking into account that the integral by x vanishes when $u \to \infty$ due to the Riemann-Lebesgue lemma, we obtain

$$\begin{aligned}&\int_0^\infty \cos\tau u \int_0^1 \cos(x\sinh u)\varphi(x)\,\frac{dxdu}{x}\\&= \frac{1}{\tau}\int_0^\infty \cosh u\ \sin\tau u \int_0^1 \sin(x\sinh u)\varphi(x)\,dxdu.\end{aligned} \tag{2.24}$$

Moreover, the right-hand side of (2.24) converges absolutely and uniformly by $|\tau| > A > 0$ (see (2.17))

$$\int_0^\infty \cosh u\left|\sin\tau u\int_0^1 \sin(x\sinh u)\varphi(x)\,dx\right|du \le \int_0^\infty\left|\int_0^1 \sin(xy)\varphi(x)\,dx\right|dy < \infty.$$

Therefore, integrating by parts two more times in the right-hand side of (2.24), we appeal again to (2.17), (2.22) and derive (2.23). Returning to (2.21), we pass to the limit with respect to λ in (2.21). Then observing, that $\widehat{\varphi}(t)$ plainly goes to zero when $t \to +\infty$ via the Riemann-Lebesgue lemma, we complete the proof of the theorem. □

3 An equivalent Salem's problem

The results of the previous section can be applied to formulate an equivalent Salem problem, concerning the Fourier-Stieltjes transform of the Minkowski question mark function $?(x) \equiv \psi(x)$. This function is defined by [2] $\psi : [0,1] \mapsto [0,1]$

$$\psi([0,a_1,a_2,a_3,\ldots]) = 2\sum_{i=1}^{\infty}(-1)^{i+1}2^{-\sum_{j=1}^{i} a_j},$$

where $x = [0,a_1,a_2,a_3,\ldots]$ stands for the representation of x by a regular continued fraction. It is well known that $\psi(x)$ is continuous, strictly increasing and supports a singular measure. It satisfies the following relations

$$\psi(x) = 1-\psi(1-x),\ x \in [0,1], \tag{3.1}$$

$$\psi(x) = 2\psi\left(\frac{x}{x+1}\right),\ x \in [0,1]. \tag{3.2}$$

Equation (3.2) can be extended on $[0,\infty]$ by using the relation

$$\psi(x)+\psi\left(\frac{1}{x}\right) = 2,\ x > 0. \tag{3.3}$$

When $x \to 0$, it decreases exponentially $\psi(x) = O\left(2^{-1/x}\right)$. Key values are $\psi(0) = 0$, $\psi(1) = 1$, $\psi(\infty) = 2$. For instance, from (3.1) and asymptotic behavior of the Minkowski function near zero one can easily get the finiteness of the following integrals

$$\int_0^1 x^\lambda \, d\psi(x) < \infty, \ \lambda \in \mathbb{R},$$

$$\int_0^1 (1-x)^\lambda \, d\psi(x) < \infty, \ \lambda \in \mathbb{R}.$$

Further, as was proved by Salem [9], the Minkowski question mark function satisfies the Hölder condition

$$|\psi(x) - \psi(y)| < C|x-y|^\alpha, \ \alpha < 1,$$

where

$$\alpha = \frac{\log 2}{2\log \frac{\sqrt{5}+1}{2}},$$

where $C > 0$ is an absolute constant.

Let us consider the Fourier -Stieltjes transforms (1.1) of the Minkowski question mark function over $[0,1]$ and $[0,\infty[$

$$\widehat{\mu_\psi}(t) = \int_0^1 e^{ixt} \, d\psi(x), \quad \widehat{\nu_\psi}(t) = \int_0^\infty e^{ixt} \, d\psi(x), \quad t \in \mathbb{R}, \tag{3.4}$$

respectively. It is easily seen the estimates

$$|\widehat{\mu_\psi}(t)| \le \int_0^1 d\psi(x) = 1, \quad |\widehat{\nu_\psi}(t)| \le \int_0^\infty d\psi(x) = 2.$$

Further we observe that functional equation (3.1) implies $\widehat{\mu_\psi}(t) = e^{it}\widehat{\mu_\psi}(-t)$ and therefore $e^{-it/2}\widehat{\mu_\psi}(t)$ is real-valued. So, taking its imaginary part we obtain the equality

$$\cos\left(\frac{t}{2}\right)\mathrm{Im}\widehat{\mu_\psi}(t) = \sin\left(\frac{t}{2}\right)\mathrm{Re}\widehat{\mu_\psi}(t). \tag{3.5}$$

Hence, letting, for instance, $t = 2\pi n, \ n \in \mathbb{N}_0$ it gives $\mathrm{Im}\widehat{\mu_\psi}(2\pi n) = 0$ and $\mathrm{Re}\widehat{\mu_\psi}(2\pi n) = d_n$. In 1943 Salem asked [9] whether $d_n \to 0$, as $n \to \infty$.

Proposition 3.1. *The following functional equation takes place*

$$\widehat{\nu_\psi}(t) = \frac{2\widehat{\mu_\psi}(t)}{2 - e^{it}}. \tag{3.6}$$

Proof. The proof is based on functional equations for the Minkowski function and simple properties of the Stieltjes integral, including the formula for integration by substitution [3]

$$\int_J (f \circ g) \, d\nu = \int_I f d\varphi, \quad f \in L_1(I; d\varphi),$$

where $\varphi : I \to \mathbb{R}$ is a continuous function, which is defined on an interval I with bounded variation, $g : J \to I$ is a continuous bijective map on the interval J and $\varphi = \nu \circ g^{-1}$ is a pullback measure. So, employing (3.2), (3.3), we derive the chain of equalities

$$\begin{aligned}
\int_0^1 e^{ixt}\, d\psi(x) &= \int_0^\infty e^{ixt}\, d\psi(x) - \int_1^\infty e^{ixt}\, d\psi(x) \\
&= \int_0^\infty e^{ixt}\, d\psi(x) - e^{it}\int_0^\infty e^{ixt}\, d\psi(x+1) \\
&= \int_0^\infty e^{ixt}\, d\psi(x) - e^{it}\int_0^\infty e^{ixt}\, d\left(2-\psi\left(\frac{1}{x+1}\right)\right) \\
&= \int_0^\infty e^{ixt}\, d\psi(x) + e^{it}\int_0^\infty e^{ixt}\, d\psi\left(\frac{1/x}{1+1/x}\right) \\
&= \int_0^\infty e^{ixt}\, d\psi(x) + \frac{e^{it}}{2}\int_0^\infty e^{ixt}\, d\psi\left(\frac{1}{x}\right) \\
&= \int_0^\infty e^{ixt}\, d\psi(x) + \frac{e^{it}}{2}\int_0^\infty e^{ixt}\, d(2-\psi(x)) \\
&= \left(1-\frac{e^{it}}{2}\right)\int_0^\infty e^{ixt}\, d\psi(x),
\end{aligned}$$

which imply (3.6). □

Corollary 3.2. *The following equalities*

$$\mathrm{Re}\widehat{\nu_\psi}(t) = \frac{2}{5-4\cos t}\,\mathrm{Re}\widehat{\mu_\psi}(t), \tag{3.7}$$

$$\mathrm{Im}\widehat{\nu_\psi}(t) = \frac{6}{5-4\cos t}\,\mathrm{Im}\widehat{\mu_\psi}(t) \tag{3.8}$$

are valid.

Proof. In fact, taking real and imaginary parts in (3.6) and employing functional equation

(3.1) we get, for instance,

$$\int_0^\infty \cos xt \, d\psi(x) = \frac{2}{5-4\cos t}\left[(2-\cos t)\int_0^1 \cos xt \, d\psi(x) - \sin t \int_0^1 \sin xt \, d\psi(x)\right]$$

$$= \frac{2}{5-4\cos t}\left[2\int_0^1 \cos xt \, d\psi(x) - \int_0^1 \cos t(1-x) \, d\psi(x)\right]$$

$$= \frac{2}{5-4\cos t}\int_0^1 \cos xt \, d\psi(x)$$

and this yields relation (3.7). Analogously we prove (3.8). □

In particular, letting $t = 2\pi n, n \in \mathbb{N}_0$ in (3.7), (3.8), we find accordingly

$$\int_1^\infty \cos(2\pi nx) \, d\psi(x) = \int_0^1 \cos(2\pi nx) \, d\psi(x),$$

$$\int_1^\infty \sin(2\pi nx) \, d\psi(x) = 5\int_0^1 \sin(2\pi nx) \, d\psi(x) = 0$$

via (3.5). Generally, equalities (3.7), (3.8) yield

$$\int_1^\infty \cos xt \, d\psi(x) = \frac{1-8\sin^2(t/2)}{1+8\sin^2(t/2)}\int_0^1 \cos xt \, d\psi(x),$$

$$\int_1^\infty \sin xt \, d\psi(x) = \frac{5-8\sin^2(t/2)}{1+8\sin^2(t/2)}\int_0^1 \sin xt \, d\psi(x),$$

respectively. For instance,

$$\int_1^\infty \cos(xt_m) \, d\psi(x) = 0,$$

$$\int_1^\infty \sin(xt_k) \, d\psi(x) = 0$$

for any $t_m,\ t_k$, which are roots of the corresponding equations

$$\sin(t_m/2) = \pm 1/(2\sqrt{2}),\ \sin(t_k/2) = \pm\sqrt{5/8}, \quad m,k \in \mathbb{N}.$$

Proposition 3.3. *The Minkowski question mark function supports a Rajchman measure on* $[0,1]$ *if and only if it supports a Rajchman measure on* $[0,\infty[$, *i.e.* $\widehat{\mu_\psi}(t) \to 0,\ |t| \to \infty$, *if and only if* $\widehat{\nu_\psi}(t) \to 0,\ |t| \to \infty$.

Proof. The conclusion follows immediately from the equation (3.6), which yields the inequality

$$\frac{1}{2}\,|\widehat{\nu_\psi}(t)| \le |\widehat{\mu_\psi}(t)| \le \frac{3}{2}|\widehat{\nu_\psi}(t)|,\ t \in \mathbb{R}. \tag{3.9}$$

□

Letting $\varphi(x) = \psi\left(\frac{1}{x}\right)$ in Theorem 2.1 and Corollary 2.2, we get a simpler looking reformulation of the Salem problem.

Corollary 3.4. *The problem "does*

$$\int_0^1 e^{ixt} d\psi(x) \to 0,\ |t| \to \infty?\text{''}$$

is equivalent to the question: "does

$$t\int_0^\infty e^{itx}\psi\left(\frac{1}{x}\right)dx \to 2i,\ |t| \to \infty?\text{''}$$

Proof. In fact, the Minkowski question mark function $\psi\left(\frac{1}{x}\right)$ satisfies the assumptions of Corollary 2.2. Hence

$$\int_0^\infty e^{ixt} d\psi\left(\frac{1}{x}\right) \to 0,\ |t| \to \infty$$

if and only if

$$\lim_{|t|\to\infty} t\int_0^\infty e^{itx}\psi\left(\frac{1}{x}\right)dx = 2i.$$

By (3.3)

$$\int_0^\infty e^{ixt} d\psi\left(\frac{1}{x}\right) = -\widehat{\nu_\psi}(t)$$

and by (3.9) $\widehat{\mu_\psi}(t) \to 0, |t| \to \infty$. Hence Proposition 3.3 completes the proof. □

Finally, we generalize Salem's problem, proving

Theorem 3.5. *Let* $k \in \mathbb{N}_0$. *If an answer on Salem's question is affirmative, then*

$$f^{(k)}(t) = \int_0^1 (ix)^k e^{itx}\, d\psi(x) = o(1),\ |t| \to \infty. \tag{3.10}$$

Proof. It is easily seen that the Fourier- Stieltjes transform of the Minkowski question mark function over $(0,1)$ is infinitely differentiable and so for any $k \in \mathbb{N}_0$ we have the first equality in (3.10). Suppose that $f^{(k)}$ does not tend to zero as $|t| \to \infty$. Then we can find a sequence $\{t_m\}_{m=1}^\infty$, $|t_m| \to \infty$ such that

$$|f^{(k)}(t_m)| = \left|\int_0^1 x^k e^{it_m x}\, d\psi(x)\right| \ge \delta > 0.$$

Let $\frac{t_m}{2\pi} = n_m + \beta_m$, where n_m is an integer and $0 \le \beta_m < 1$. One may suppose that β_m tends to a limit β, we can always do it choosing again subsequence from $\{t_m\}$ if necessary. Thus

$$\int_0^1 e^{2\pi i\beta x} x^k e^{2\pi i n_m x}\, d\psi(x) - f^{(k)}(t_m)$$

$$= \int_0^1 \left(e^{2\pi i\beta x} - e^{2\pi i\beta_m x}\right) x^k e^{2\pi i n_m x}\, d\psi(x) \to 0,\ m \to \infty.$$

Therefore

$$\left|\int_0^1 e^{2\pi i\beta x} x^k e^{2\pi i n_m x}\, d\psi(x)\right| \ge \delta > 0.$$

But this contradicts to Salem's lemma [11], p. 38, because $\widehat{\mu_\psi}(2\pi n) \to 0,\ n \to \infty$ via assumption of the theorem and the Riemann -Stieltjes integral

$$\int_0^1 e^{2\pi i\beta x} x^k\, d\psi(x)$$

converges for any $k \in \mathbb{N}_0$. $\square$

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