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Fourier Transform and Compactness in $(L^q,\ l^p)^\alpha$ and $M^{p,\,\alpha}$ Spaces

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Abstract

The spaces $(L^q,\ l^p)^\alpha$ and $M^{p,\,\alpha}$ are closely related to classical problems in Harmonic Analysis: properties of multiplier and Fourier multiplier from a Lebesgue space to another, finite (1,p)-energy measures. We characterize the Fourier transforms of their elements and establish criteria of compactness in these spaces.

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1 Introduction

Let us fix an integer d. The space $\mathbb{R}^d$ is endowed with its usual scalar product $(x,\ \xi) \mapsto x \cdot \xi$, euclidean norm $|\cdot|$ and Lebesgue measure.

For $1 \leq p \leq \infty$, we denote by $\|\cdot\|_p$ the usual norm on the classical Lebesgue space $L^p = L^p(\mathbb{R}^d)$ and by p' the conjugate of p ($\frac{1}{p} + \frac{1}{p'} = 1$).

Let $C_c = C_c(\mathbb{R}^d)$ denote the space of complex valued continuous functions on $\mathbb{R}^d$, with compact support.

Let $C_0 = C_0(\mathbb{R}^d)$ denote the space of complex valued continuous functions vanishing at infinity on $\mathbb{R}^d$, endowed with the sup norm $f \mapsto \|f\|_\infty = \sup\limits_{x \in \mathbb{R}^d} |f(x)|$.

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We denote by M the space of Radon measures on $\mathbb{R}^d$. The total variation of an element μ of M is denoted by $|\mu|$ and $M^1 = \{\mu \in M : \|\mu\| := |\mu|(\mathbb{R}^d) < \infty\}$ is the space of bounded Radon measures on $\mathbb{R}^d$.

We are interested in the Fourier transform defined as follows.

Definition 1.1. When μ and f are respectively in M^1 and L^1, their Fourier transforms $\widehat{\mu}$ and $\widehat{f}$ are defined by

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\, d\mu(x), \quad \xi \in \mathbb{R}^d, \tag{1.1}$$

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx, \quad \xi \in \mathbb{R}^d.$$

Let us recall some well known, elementary and nevertheless very important properties of the Fourier transform.

Proposition 1.2. *a) For $\mu \in M^1$, $\|\widehat{\mu}\|_\infty \leq \|\mu\|$.*

b) $\{\widehat{f} : f \in L^1\}$ is a dense subset of C_0.

This leads to the two following problems:

(I) given a subset E of M^1, find a nice characterization of the Fourier transforms of the elements of E.

(II) investigate the connection between local regularity of an element of M^1 and the decay at infinity of its Fourier transform.

The goal of this paper is to investigate analogs of these two problems in the case where E is the Banach space $\left(M^{p,\alpha}, \|\cdot\|_{p,\alpha}\right)$ or its Banach subspace $\left((L^q, l^p)^\alpha, \|\cdot\|_{q,p,\alpha}\right)$ (see section 2 for definitions).

These spaces deserve attention for several reasons.

a) For $1 \leq q \leq \alpha \leq p \leq \infty$, it is proved in [4] that $(L^q, l^p)^\alpha$ always contains L^α and for some values of α, the two spaces are equal (see Proposition2.4).

b) For $1 \leq q \leq \alpha \leq p \leq 2$ and $\frac{1}{\gamma} = \frac{1}{q} - \frac{1}{p}$, it is established in [6] that if m is a Fourier multiplier from L^q to L^p, that is there exists a bounded linear map T_m from L^q to L^p satisfying

$$\widehat{T_m f} = m\widehat{f}, \quad f \in L^q,$$

then $m \in (L^{p'}, l^\infty)^\gamma$.

c) In [9] it was proved that for $1 \leq q \leq p \leq \infty$ and $\frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p}$, if μ is a nonnegative Radon measure on $\mathbb{R}^d$ which represents a multiplier from L^q to L^p, that is $f \mapsto \mu * f$ is a bounded linear map from L^q to L^p, then $\mu \in M^{p,\alpha}$ ($*$ denotes the convolution product).

d) In [2] it was showed that for $0 < \gamma < \frac{1}{\alpha}, \frac{1}{p} = \frac{1}{\alpha} - \gamma$, if μ is a nonnegative Radon measure on $\mathbb{R}^d$ which Riesz potential

$$x \mapsto I_\gamma\mu(x) = \int_{\mathbb{R}^d} |x-y|^{\gamma-1} d\mu(y)$$

belongs to L^p, then $\mu \in M^{p,\alpha}$.

2 Notations and results

An early answer to Problem (I) is the following result of Schoenberg.

Proposition 2.1. *[11] Given an element F of L^0, the following assertions are equivalent:*

(i) $F = \widehat{\mu}$ *for some* $\mu \in M^1$;

(ii) there exists a real constant C such that

$$\left| \int_{\mathbb{R}^d} F(x)g(x)dx \right| \leq C\|\widehat{g}\|_\infty, \qquad g \in C_c.$$

Notice that if μ is unbounded then the integral (1.1) does not exist in the usual sense. However, if μ defines a tempered distribution then its Fourier transform exists in the distributional sense. This is the case when μ belongs to the space M^p defined as follows.

Definition 2.2. Let us consider a real number $r > 0$.

a) We define $I_k^r = \prod_{j=1}^{d} \left[k_j r, \left(k_j + 1\right) r\right), \quad k = \left(k_j\right)_{1 \leq j \leq d} \in \mathbb{Z}^d.$

b) For any $\mu \in M$,

$$_r\|\mu\|_p = \begin{cases} \left(\sum_{k \in \mathbb{Z}^d} |\mu|\left(I_k^r\right)^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \sup_{k \in \mathbb{Z}^d} |\mu|\left(I_k^r\right) & \text{if } p = \infty. \end{cases}$$

c) We define $M^p = \left\{ \mu \in M \ : \ _1\|\mu\|_p < \infty \right\}$.

These spaces of measures are closely related to the Wiener amalgams spaces $(L^q,\ l^p)$ defined as follows.

Definition 2.3. Suppose that $1 \leq q,\ p \leq \infty$.

a) Let L^q_{loc} denote the space of equivalent classes, modulo equality Lebesgue almost everywhere, of complex valued measurable functions on $\mathbb{R}^d$ which are locally in L^q.

b) For any real number $r > 0$ and any element f of L^q_{loc}

$$_r\|f\|_{q,\,p} = \begin{cases} \left(\sum_{k \in \mathbb{Z}^d} \left(\|f\chi_{I_k^r}\|_q\right)^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \sup_{k \in \mathbb{Z}^d} \left\|f\chi_{I_k^r}\right\|_q & \text{if } p = \infty. \end{cases}$$

where χ_A stands for the characteristic function of the subset A of $\mathbb{R}^d$.

c) We define $(L^q,\ l^p) = \{f \in L^q_{loc} \ : \ _1\|f\|_{q,\,p} < \infty\}$ and $(C_0,\ l^p) = C_0 \cap (L^\infty,\ l^p)$.

Notice that for any element f of L^1_{loc}, the map $g \mapsto \int_{\mathbb{R}^d} g(x)f(x)dx$ of C_c into $\mathbb{C}$ defines a Radon measure μ_f ($d\mu_f(x) = f(x)dx$) satisfying

$$|\mu_f|(A) = \mu_{|f|}(A) = \int_A |f(x)|dx, \qquad A \subset \mathbb{R}^d$$

and consequently

$$_r\|\mu_f\|_p = \ _r\|f\|_{1,\, p}, \qquad r > 0 \quad 1 \leq p \leq \infty.$$

So, by identifying f to μ_f, we may (and do) consider $\left(L^1,\, l^p\right)$ as a subspace of M^p. It is also clear that $(L^q,\, l^p)$ is included in $\left(L^1,\, l^p\right)$ for $1 \leq q,\ p \leq \infty$. Holland, who initiated the systematic study of these spaces, has obtained the following result.

Proposition 2.4. *[7] Suppose that* $1 \leq q,\ p \leq 2$. *The following assertions hold :*

a) $\{\widehat{\mu}\ :\ \mu \in M^p\} \subset (L^{p'},\, l^\infty)$ *and there is a real constant C such that*

$$_1\|\widehat{\mu}\|_{p',\,\infty} \leq C\ _1\|\mu\|_p, \qquad \mu \in M^p;$$

b) $\{\widehat{f}\ :\ f \in (L^q,\, l^p)\} \subset (L^{p'},\, l^{q'})$ *and there is a real constant C such that*

$$_1\|\widehat{f}\|_{p',\, q'} \leq C\ _1\|f\|_{q,\, p}, \qquad f \in (L^q,\, l^p);$$

c) $\{\widehat{f}:\ f \in (L^q, l^1)\} \subset (C_0, l^{q'})$.

This result led Holland (for $q = 1$) and Torrès de Squire (for $1 < q \leq 2$) to the following extension of Schoenberg criterion.

Proposition 2.5. *[8], [14] Suppose that* $1 \leq q,\ p \leq 2$ *and* $F \in L^1_{loc}$. *Then the following assertions are equivalent:*

a) there is a real constant C such that

$$\left|\int_{\mathbb{R}^d} F(x)g(x)dx\right| < C\ _1\|\widehat{g}\|_{q',\, p'}, \qquad g \in C_c$$

b) $F = \widehat{\mu}$ *for some* $\mu \in M^p$ *if* $q = 1$ *and* $F = \widehat{f}$ *for some* $f \in (L^q,\, l^p)$ *if* $1 < q$.

Fofana has introduced subspaces of M^p which are also super-spaces of Lebesgue ones, defined as follows.

Definition 2.6. Suppose that $1 \leq q \leq \alpha \leq p \leq \infty$. We define:

a) $\|\mu\|_{p,\,\alpha} = \sup\limits_{r>0} r^{d\left(\frac{1}{\alpha}-1\right)}\ _r\|\mu\|_p$, for any element μ of M;

b) $M^{p,\,\alpha} = \{\mu \in M\ :\ \|\mu\|_{p,\,\alpha} < \infty\}$;

c) $\|f\|_{q,\,p,\,\alpha} = \sup\limits_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\ _r\|f\|_{q,\,p}$, for any element f of L^1_{loc};

d) $(L^q,\, l^p)^\alpha = \{f \in L^1_{loc} \,:\, \|f\|_{q,\,p,\,\alpha} < \infty\}$.

The space $(L^q, l^p)^\alpha$ is related to L^α as follows.

Proposition 2.7. *[4] Suppose that $1 \leq q \leq \alpha \leq p \leq \infty$. The following assertions hold:*

i) $(L^q, l^p)^\alpha$ is a linear subset of L^1_{loc} and a complex Banach space when endowed with the norm $f \mapsto \|f\|_{q,p,\alpha}$

ii) L^α is continuously included in $(L^q, l^p)^\alpha$

iii) $(L^q, l^p)^\alpha = (L^q, l^\alpha)^\alpha = L^\alpha$

iv) if $q < \alpha < p$ then the weak Lebesgue space $L^{\alpha,\infty}$ (see [12] for the definition of this space) is continuously included in $(L^q,\, l^p)^\alpha$.

The following properties of the Fourier transform in the spaces M^p and $M^{p,\alpha}$ are known.

Proposition 2.8. *[6] Assume that $1 \leq \alpha \leq p \leq 2$. Then*

(i)

$$\left|\int_{\mathbb{R}^d} \widehat{\mu}(rx) g(x) dx\right| \leq {}_{\frac{1}{r}}\|\mu\|_p \; {}_1\|\widehat{g}\|_{\infty,\,p'}, \quad \mu \in M^p, \quad g \in C_c, \quad r > 0$$

and

$$\left|\int_{\mathbb{R}^d} \widehat{\mu}(rx) g(x) dx\right| \leq \|\mu\|_{p,\,\alpha} \; {}_1\|\widehat{g}\|_{\infty,\,p'} \, r^{-\frac{d}{\alpha'}}, \quad \mu \in M^{p,\,\alpha}, \quad g \in C_c, \quad r > 0,$$

(ii) there exists a real constant C such that

$${}_r\|\widehat{\mu}\|_{p',\,\infty} \leq C \; {}_{\frac{1}{r}}\|\mu\|_p \, r^{d\left(1-\frac{1}{p}\right)}, \quad \mu \in M^p, \quad r > 0$$

$${}_r\|\widehat{f}\|_{p',\,q'} \leq C \; {}_{\frac{1}{r}}\|f\|_{q,\,p} \, r^{d\left(\frac{1}{q}-\frac{1}{p}\right)}, \quad f \in (L^q,\, l^p), \quad r > 0$$

$$\|\widehat{\mu}\|_{p',\,\infty,\,\alpha'} \leq C\, \|\mu\|_{p,\,\alpha}, \quad \mu \in M^{p,\alpha},$$

and

$$\|\widehat{f}\|_{p',\,q',\,\alpha'} \leq C\, \|f\|_{q,\,p,\,\alpha}, \quad f \in (L^q,\, l^p)^\alpha.$$

In this paper we will prove the following criterion on functions which are Fourier transforms of elements of $M^{p,\alpha}$ or $(L^q,\, l^p)^\alpha$.

Theorem 2.9. *Let F be an element of L^1_{loc}.*
a) If $1 \leq \alpha \leq p \leq 2$ and $1 < p$ then the following assertions are equivalent:

(i) $F = \widehat{\mu}$ for some $\mu \in M^{p,\,\alpha}$,

(ii) there is a real constant C such that

$$\left|\int_{\mathbb{R}^d} F(rx) g(x) dx\right| \leq C r^{-\frac{d}{\alpha'}} \; {}_1\|\widehat{g}\|_{\infty,p'}, \quad r > 0,\; g \in C_c.$$

b) If $1 < q \leq \alpha \leq p \leq 2$ *then the following assertions are equivalent:*

(i) $F = \widehat{f}$ *for some* $f \in (L^q, l^p)^\alpha$,

(ii) there is a real constant C such that

$$\left| \int_{\mathbb{R}^d} F(rx)g(x)dx \right| \leq Cr^{-\frac{d}{\alpha'}}{}_1 \|\widehat{g}\|_{q',p'}, \quad r > 0,\ g \in C_c.$$

By Proposition 2.7, the following result follows immediately from Theorem 2.9 b).

Corollary 2.10. *Let F be an element of* L^1_{loc} *and* $1 < p \leq 2$. *Then F is the Fourier transform of some element of* L^p *if and only if there is a real constant C such that*

$$\left| \int_{\mathbb{R}^d} F(rx)g(x)dx \right| \leq Cr^{-\frac{d}{p'}}{}_1 \|\widehat{g}\|_{\infty,p'}, \quad r > 0,\ g \in C_c.$$

This result has been established already in [6].

As far as analogs of Problem (II) are concerned, the following result has been obtained by Pego.

Proposition 2.11. *[10] Suppose that* $1 \leq p \leq 2$ *and H is a bounded subset of* L^p. *Then*

i) $\lim\limits_{\rho \to \infty} \sup\limits_{f \in H} \left\| f - f\chi_{J_0^\rho} \right\|_p = 0 \Rightarrow \lim\limits_{u \to 0} \sup\limits_{f \in H} \left\| \widehat{f} - t_u \widehat{f} \right\|_{p'} = 0,$

ii) $\lim\limits_{u \to 0} \sup\limits_{f \in H} \|f - t_u f\|_p = 0 \Rightarrow \lim\limits_{\rho \to \infty} \sup\limits_{f \in H} \left\| \widehat{f} - \widehat{f}\chi_{J_0^\rho} \right\|_{p'} = 0$

where t_u *stands for the translation operator with translation vector u and*

$$J_0^\rho = \left(-\frac{\rho}{2}, \frac{\rho}{2} \right)^d, \quad \rho > 0.$$

The above result may be extended to the spaces $M^{p,\alpha}$ and $(L^q, l^p)^\alpha$ as follows.

Theorem 2.12. *Suppose that* $1 \leq \alpha \leq p \leq 2$ *and K is a bounded subset of* $M^{p,\alpha}$.*Then*

$$\lim_{\rho \to \infty} \sup_{\mu \in K} \left\| \mu - \chi_{J_0^\rho} \mu \right\|_{p,\alpha} = 0 \Rightarrow \lim_{u \to 0} \sup_{f \in K} \left\| \widehat{\mu} - t_u \widehat{\mu} \right\|_{p',\infty,\alpha'} = 0.$$

Theorem 2.13. *Suppose that* $1 \leq q \leq \alpha \leq p \leq 2$ *and H is a bounded subset of* $(L^q, l^p)^\alpha$. *Then*

i) $\lim\limits_{\rho \to \infty} \sup\limits_{f \in H} \left\| f - f\chi_{J_0^\rho} \right\|_{q,p,\alpha} = 0 \Rightarrow \lim\limits_{u \to 0} \sup\limits_{f \in H} \left\| \widehat{f} - t_u \widehat{f} \right\|_{p',q',\alpha'} = 0,$

ii) $\lim\limits_{u \to 0} \sup\limits_{f \in H} \|f - t_u f\|_{q,p,\alpha} = 0 \Rightarrow \lim\limits_{\rho \to \infty} \sup\limits_{f \in H} \left\| \widehat{f} - \widehat{f}\chi_{J_0^\rho} \right\|_{p',q',\alpha'} = 0.$

Clearly Theorem 2.13 contains Proposition 2.11 as a special case ($q = \alpha = p$). From a compactness criterion for subsets of translation invariant Banach function spaces due to Feichtinger (see Proposition 4.2), we get the following result:

Theorem 2.14. *Suppose that* $1 \leq q \leq \alpha \leq p \leq \infty$ *with* $\alpha < \infty$ *and* H *is a closed subset of* $(L^q, l^p)^\alpha$ *such that*

(i) $\sup_{f\in H} \|f\|_{q,p,\alpha} < \infty$,

(ii) $\lim_{u\to 0} \sup_{f\in H} \|f - t_u f\|_{q,p,\alpha} = 0$,

(iii) $\lim_{\rho\to\infty} \sup_{f\in H} \left\| f - f\chi_{J_0^\rho} \right\|_{q,p,\alpha} = 0$.

Then H *is compact in* $(L^q, l^p)^\alpha$.

The following result follows from Theorem 2.9 and Theorem 2.14.

Theorem 2.15. *Suppose that* $1 \leq \alpha \leq p \leq 2$ *with* $1 < p$ *and* K *is a closed subset of* $M^{p,\alpha}$ *such that*

(i) $\sup_{\mu\in K} \|\mu\|_{p,\alpha} < \infty$,

(ii) $\lim_{\rho\to\infty} \sup_{\mu\in K} \left\| \mu - \chi_{J_0^\rho}\mu \right\|_{p,\alpha} = 0$,

(iii) $\lim_{R\to\infty} \sup_{\mu\in K} \left\| \widehat{\mu} - \chi_{J_0^R}\widehat{\mu} \right\|_{p',\infty,\alpha'} = 0$.

Then K *is vaguely compact in* $M^{p,\alpha}$.

The rest of the paper is organized as follows. Section 3 contains the proof of Theorem 2.9 and Section 4 is dedicated to Theorems 2.12, 2.13 and 2.14, and related results.

3 Characterization of Fourier transforms in $(L^q, l^p)^\alpha$ spaces

The following result will be used in the proof of Theorem 2.9.

Lemma 3.1. *[1],[13] Suppose that* $1 \leq p \leq \infty$. *Let* $\phi_p = \{\varphi \in C_c \,:\, \widehat{\varphi} \in (L^\infty, l^p)\}$ *and* $\widehat{\phi_p} = \{\widehat{\varphi} \,:\, \varphi \in \phi_p\}$. *Then :*

- $\widehat{\phi_p} = C_0$ *for* $2 \leq p \leq \infty$
- $\widehat{\phi_p}$ *is dense in* (C_0, l^s) *for* $1 \leq s \leq \infty$.

Corollary 3.2. *Suppose that* $1 \leq p \leq 2$. *Then* $\{\widehat{g} \,:\, g \in \left(L^p, l^1\right)\}$ *is a dense subset of* $(C_0, l^{p'})$.

Proof. By lemma 3.1 we have $\phi_{p'} = C_c$, and clearly $C_c \subset (L^p, l^1)$. So, by Proposition2.2, we have

$$\widehat{\phi_{p'}} \subset \{\widehat{g} \,:\, g \in \left(L^p,\, l^1\right)\} \subset (C_0,\, l^{p'})$$

and the result follows from Lemma 3.1. □

Lemma 3.3. *[7], [1] Suppose that* $1 \leq q,\ p \leq \infty$.

a) *If* $1 \leq q,\ p < \infty$ *then the dual space of* $(L^q,\, l^p)$ *is* $(L^{q'},\, l^{p'})$.

b) *If* $1 \leq p < \infty$ *then the dual space of* $(C_0,\, l^p)$ *is* $M^{p'}$.

Actually it is easy to show that if $1 \leq q,\ p < \infty$ then for all $r > 0$,

$$_r\|f\|_{q',p'} = \sup\left\{\left|\int_{\mathbb{R}^d} f(x)g(x)dx\right| \,:\, g \in (L^q, l^p),\ _r\|g\|_{q,p} \leq 1\right\},\ \ f \in (L^{q'}, l^{p'}) \tag{3.1}$$

and

$$_r\|\mu\|_{p'} = \sup\left\{\left|\int_{\mathbb{R}^d} g(x)d\mu(x)\right| \,:\, g \in (C_0,\, l^p),\ _r\|g\|_{\infty,\,p} \leq 1\right\}, \quad \mu \in M^{p'}. \tag{3.2}$$

Proof of Theorem 2.9 a). 1) Suppose that i) is true. Then by Proposition 2.8, ii) is satisfied with $C = \|\mu\|_{p,\,\alpha}$.

2) Suppose that ii) is true: there is a real constant C such that

$$\left|\int_{\mathbb{R}^d} F(rx)g(x)dx\right| \leq C\ _1\|\widehat{g}\|_{\infty,\,p'}\, r^{-\frac{d}{\alpha'}}, \qquad r > 0, g \in C_c.$$

From this and Proposition 2.5 it follows that there is μ in M^p such that $\widehat{\mu} = F$. Let g be an element of $\left(L^p,\, l^1\right)$ and r a positive real.

(α) Clearly there is a sequence $(g_n)_{n\geq 1}$ in C_c which converges to g in $\left(L^p,\, l^1\right)$. By Proposition 2.4 and Lemma 3.3 , $F = \widehat{\mu}$ belongs to $(L^{p'},\, l^\infty)$ which is the dual space of $\left(L^p,\, l^1\right)$ and $(\widehat{g_n})_{n\geq 1}$ converges to $\widehat{g}$ in $(L^\infty,\, l^{p'})$. Therefore we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} F(rx)g(x)dx\right| &= \lim_{n\to\infty}\left|\int_{\mathbb{R}^d} F(rx)g_n(x)dx\right| \\
&\leq \lim_{n\to\infty} C\ _1\|\widehat{g_n}\|_{\infty,\,p'}\, r^{-\frac{d}{\alpha'}} \\
&= C\, r^{-\frac{d}{\alpha'}}\ _1\|\widehat{g}\|_{\infty,\,p'}.
\end{aligned}$$

(β) Set $h(x) = g(r^{-1}x)$ for all $x \in \mathbb{R}^d$.
We have

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} \widehat{g}(x)d\mu(x)\right| &= \left|\int_{\mathbb{R}^d} F(x)g(x)dx\right| \\
&= \left|\int_{\mathbb{R}^d} F(r^{-1}x)h(x)dx\right| r^{-d} \\
&\leq C\ _1\|\widehat{h}\|_{\infty,\,p'}\, r^{d(\frac{1}{\alpha'}-1)} \\
&= C\, r^{d(1-\frac{1}{\alpha})}\ _r\|\widehat{g}\|_{\infty,\,p'}.
\end{aligned}$$

By Corollary 3.2, $\{\widehat{g} \,:\, g \in \left(L^p,\, l^1\right)\}$ is dense in $(C_0, l^{p'})$. In addition, μ belongs to M^p which is the dual space of $(C_0, l^{p'})$. (Lemma 3.3 b)). Therefore

$$\left|\int_{\mathbb{R}^d} \varphi(x)d\mu(x)\right| \le C\, r^{d(1-\frac{1}{\alpha})}\ {}_r\|\phi\|_{\infty,p'}, \qquad r > 0, \varphi \in (C_0, l^{p'}).$$

Thus, by (3.2), we have

$$_r\|\mu\|_p \le C\, r^{d(1-\frac{1}{\alpha})}, \qquad r > 0$$

that is

$$\|\mu\|_{p,\,\alpha} \le C < \infty \quad \text{and} \quad \mu \in M^{p,\,\alpha}.$$

□

For the proof of Theorem 2.9 b) we will need the following result:

Lemma 3.4. *Suppose that* $1 \le p \le 2,\ 1 \le q \le \infty$. *Then for all* $r > 0$ *and all* $g \in C_c$,

$$\left|\int_{\mathbb{R}^d} \widehat{f}(rx)g(x)d(x)\right| \le r^{-\frac{d}{q'}}\ {}_{\frac{1}{r}}\|f\|_{q,p}\ \ {}_1\|\widehat{g}\|_{q',\,p'}, \qquad f \in (L^q, l^p).$$

Proof. Let $(f,\ g,\ r)$ be an element of $(L^q, l^p) \times C_c \times \mathbb{R}^d_+$. We have

$$\begin{aligned}
\int_{\mathbb{R}^d} \widehat{f}(rx)g(x)d(x) &= \int_{\mathbb{R}^d} \widehat{f}(x)r^{-d}g(r^{-1}x)dx \\
&= \int_{\mathbb{R}^d} f(x)\widehat{g}(rx)dx.
\end{aligned}$$

Thus

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} \widehat{f}(rx)g(x)dx\right| &\le \int_{\mathbb{R}^d} |f(x)|\left|\widehat{g}(rx)\right| dx \\
&= \sum_{k \in \mathbb{Z}^d} \int_{I_k^{\frac{1}{r}}} |f(x)|\left|\widehat{g}(rx)\right| dx.
\end{aligned}$$

Using successively Holder inequality for integrals and Holder inequality for series we get

$$\left|\int_{\mathbb{R}^d} \widehat{f}(rx)g(x)d(x)\right| \le\ {}_{\frac{1}{r}}\|f\|_{q,p}\ \ {}_{\frac{1}{r}}\|h_r\|_{q',\,p'} \qquad f \in (L^q, l^p),$$

where $h_r(x) = \widehat{g}(rx)$. A change of variables ends the proof. □

From the above lemma, we deduce immediately the following result:

Corollary 3.5. *Suppose that* $1 \le q \le \alpha \le p \le 2$. *Then for all* $r > 0$ *and all* $g \in C_c$,

$$\left|\int_{\mathbb{R}^d} \widehat{f}(rx)g(x)d(x)\right| \le r^{-\frac{d}{\alpha'}}\ \|f\|_{q,\,p,\,\alpha}\ \ {}_1\|\widehat{g}\|_{q',\,p'}, \qquad f \in (L^q, l^p)^{\alpha}.$$

Proof of Theorem 2.9 b). An argument similar to the proof of Theorem 2.9 a) will do. We need only to use Corollary 3.5 and equality (3.1) in place of Proposition 2.8 and equality (3.2) respectively. □

As a consequence of Theorem 2.9 we obtain the following result:

Corollary 3.6. *Suppose that* $1 \leq q \leq \alpha \leq p \leq 2$. *Let* $(f_n)_{n\geq 1}$ *be a sequence in* $(L^q, l^p)^\alpha$ *and* F *an element of* L^1_{loc} *such that*

(i) $\sup_{n\leq 1} \|f_n\|_{q,\,p,\,\alpha} = L < \infty$,

(ii) $\lim_{n\to\infty} \int_{\mathbb{R}^d} \widehat{f_n}(x)g(x)dx = \int_{\mathbb{R}^d} F(x)g(x)dx, \quad g \in C_c.$

Then there is an element f *of* $(L^q, l^p)^\alpha$ *such that* $\widehat{f} = F$.

Proof. By Corollary 3.5,

$$\left|\int_{\mathbb{R}^d} \widehat{f_n}(rx)g(x)d(x)\right| \leq r^{-\frac{d}{\alpha'}} \|f_n\|_{q,\,p,\,\alpha} \;{}_1\|\widehat{g}\|_{q',\,p'}, \quad r > 0,\ g \in C_c,\ n \geq 1.$$

Therefore

$$\left|\int_{\mathbb{R}^d} F(rx)g(x)d(x)\right| \leq r^{-\frac{d}{\alpha'}} L \;{}_1\|\widehat{g}\|_{q',\,p'}, \quad r > 0,\ g \in C_c$$

and the claim follows from Theorem 2.9 b). □

4 Compactness criterion and Fourier transform in $(L^q, l^p)^\alpha$ and $M^{p,\,\alpha}$

Proof of Theorem 2.12. Suppose that

$$A = \sup_{\mu\in K} \|\mu\|_{p,\,\alpha} < \infty \quad \text{and} \quad \lim_{\rho\to\infty} \sup_{\mu\in K} \|\mu - \chi_{J_0^\rho}\mu\|_{p,\,\alpha} = 0.$$

Consider an arbitrary real $\varepsilon > 0$. There is a real $\rho > 0$ such that

$$\|\mu - \chi_{J_0^\rho}\mu\|_{p,\,\alpha} < \varepsilon, \qquad \mu \in K.$$

Let u be an element of $\mathbb{R}^d$ and μ an element of K. We have

$$\|\widehat{\mu} - t_u\widehat{\mu}\|_{p',\,\infty,\,\alpha'} = \|\widehat{\mu - e^{iu\cdot}\mu}\|_{p',\,\infty,\,\alpha'} \leq C\|\mu - e^{iu\cdot}\mu\|_{p,\,\alpha},$$

where C is a real constant not depending on μ and u (see Proposition 2.8).

$$\begin{aligned}
\|\widehat{\mu} - t_u\widehat{\mu}\|_{p',\,\infty,\,\alpha'} &\leq C\left\{\|(1 - e^{iu\cdot})(\mu - \chi_{J_0^\rho}\mu)\|_{p,\,\alpha} + \|(1 - e^{iu\cdot})(\chi_{J_0^\rho}\mu)\|_{p,\,\alpha}\right\} \\
&\leq C\left\{2\|\mu - \chi_{J_0^\rho}\mu\|_{p,\,\alpha} + \sup_{x\in J_0^\rho}\left|1 - e^{iu\cdot x}\right| \|\mu\|_{p,\,\alpha}\right\} \\
&\leq C\left\{2\varepsilon + A\sup_{x\in J_0^\rho}\left|1 - e^{iu\cdot x}\right|\right\}.
\end{aligned}$$

Since J_0^ρ is relatively compact, there is a real $\delta > 0$ such that

$$|u| < \delta \Rightarrow \sup_{x \in J_0^\rho} \left|1 - e^{iu\cdot x}\right| < \varepsilon.$$

Thus

$$|u| < \delta \Rightarrow \sup_{\mu \in K} \|\widehat{\mu} - t_u\widehat{\mu}\|_{p\prime,\, \infty,\, \alpha\prime} < C(2 + A)\varepsilon.$$

□

Lemma 4.1. *Let $\psi \in L^1$ satisfying $0 \leq \psi$ and $\int_{\mathbb{R}^d} \psi(x)dx = 1$. Define*

$$\psi_R(x) = R^d \psi(Rx), \quad R > 0, x \in \mathbb{R}^d.$$

Then, for $1 \leq q \leq \alpha \leq p \leq \infty$ and for all $R > 0$, we have

$$\|f - \psi_R * f\|_{q,\, p,\, \alpha} \leq \left\{ \int_{\mathbb{R}^d} \|f - t_{\frac{y}{R}} f\|^p_{q,\, p,\, \alpha}\, \psi(y)dy \right\}^{\frac{1}{p}}, \qquad f \in (L^q,\, l^p)^\alpha,\; p < \infty$$

$$\|f - \psi_R * f\|_{q,\, \infty,\, \alpha} \leq \int_{\mathbb{R}^d} \|f - t_{\frac{y}{R}} f\|_{q, \infty,\, \alpha}\, \psi(y)dy, \qquad f \in (L^q,\, l^\infty)^\alpha.$$

Proof. Let us consider $f \in (L^q,\, l^p)^\alpha$ and a real $R > 0$.

a) First case: $p < \infty$. From the hypothesis on ψ, we have for almost every $x \in \mathbb{R}^d$

$$f(x) - \psi_R * f(x) = \int_{\mathbb{R}^d} \left(f(x) - t_{\frac{y}{R}} f(x)\right)\psi(y)dy. \tag{4.1}$$

Using Minkowsky inequality for integrals, we obtain , for any real $r > 0$ and any integer k

$$\left\|(f - \psi_R * f)\chi_{I_k^r}\right\|_q \leq \int_{\mathbb{R}^d} \left(\int_{I_k^r} \left|f(x) - t_{\frac{y}{R}} f(x)\right|^q dx \right)^{\frac{1}{q}} \psi(y)dy. \tag{4.2}$$

Thus, for any real $r > 0$

$$_r\|f - \psi_R * f\|_{q,\, p} \leq \left\{ \sum_{k \in \mathbb{Z}^d} \left[\int_{\mathbb{R}^d} \left(\int_{I_k^r} \left|f(x) - t_{\frac{y}{R}} f(x)\right|^q dx \right)^{\frac{1}{q}} \psi(y)dy \right]^p \right\}^{\frac{1}{p}}$$

and therefore, by Holder inequality and equality $\int_{\mathbb{R}^d} \psi(y)dy = 1$

$$_r\|f - \psi_R * f\|_{q,\, p} \leq \left\{ \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \left(\int_{I_k^r} \left|f(x) - t_{\frac{y}{R}} f(x)\right|^q dx \right)^{\frac{p}{q}} \psi(y)dy \right\}^{\frac{1}{p}}$$

$$_r\|f - \psi_R * f\|_{q,\, p} \leq \left\{ \int_{\mathbb{R}^d} {_r\|f - t_{\frac{y}{R}} f\|^p_{q,\, p}}\, \psi(y)dy \right\}^{\frac{1}{p}}.$$

The result follows.

b) Second case: $p = \infty$. The result is obtained easily from inequality (4.2) if $q < \infty$ and (4.1) if $q = \infty$. □

Proof of Theorem 2.13. a)Proof of ii) Let us suppose that $\lim_{u\to 0}\sup_{f\in H}\|f-t_uf\|_{q,\,p,\,\alpha}=0$. We define

$$\psi(x)=(2\pi)^{-\frac{d}{2}}e^{-\frac{|x|^2}{2}},\qquad x\in\mathbb{R}^d.$$

Then ψ satisfies the hypothesis of Lemma 4.1 which notations will be used throughout. We notice that

$$\widehat{\psi}(x)=(2\pi)^{-\frac{d}{2}}e^{-\frac{|x|^2}{2}},\qquad x\in\mathbb{R}^d,$$

$$\widehat{\psi_R}(x)=\widehat{\psi}\left(\frac{x}{R}\right),\qquad R>0,\ x\in\mathbb{R}^d,$$

$$\frac{1}{2}\leq 1-\widehat{\psi_R}(x),\quad R>0,\ |x|\geq 2R.$$

From the above inequality and Proposition 2.8, we have, for any $f\in(L^q,\ l^p)^\alpha$ and real $R>0$

$$\frac{1}{2}\|\widehat{f}\chi_{\mathbb{R}^d\setminus J_0^{4R}}\|_{p\prime,\,q\prime,\,\alpha\prime}\leq\|\widehat{f}\left(1-\widehat{\psi_R}\right)\|_{p\prime,\,q\prime,\,\alpha\prime}\leq C\|f-\psi_R*f\|_{q,\,p,\,\alpha}$$

where C is a real constant not depending on f and R. So, by Lemma 4.1, we have

$$\sup_{f\in H}\|\widehat{f}\chi_{\mathbb{R}^d\setminus J_0^{4R}}\|_{p\prime,\,q\prime,\,\alpha\prime}\leq 2C\left\{\int_{\mathbb{R}^d}\left(\sup_{f\in H}\|f-t_{\frac{y}{R}}f\|_{q,\,p,\,\alpha}\right)^p\psi(y)dy\right\}^{\frac{1}{p}},\quad R>0.$$

Therefore, using the hypothesis and Lebesgue dominated convergence theorem, we get

$$\lim_{\rho\to 0}\sup_{f\in H}\|\widehat{f}\chi_{\mathbb{R}^d\setminus J_0^{\rho}}\|_{p\prime,\,q\prime,\,\alpha\prime}.=0$$

b) (i) is proved as Theorem 2.12. □

The classical criterion of Kolmogorov-Riesz for compactness of subsets of L^p $(1\leq p<\infty)$ has been extended by Feichtinger to translation invariant Banach function spaces. His result contains the proposition stated below.

Proposition 4.2. *[3] Let $(B,\ \|\cdot\|_B)$ be a Banach space such that*

a) B is included in L^1_{loc} and the canonical injection is continuous, that is for any compact subset K of $\mathbb{R}^d$ there is a real $C_K>0$ such that

$$\int_K|f(x)|dx\leq C_K\|f\|_B,\qquad f\in B,$$

b) $C_c\cap B$ is dense in C_c,

c) B is translation invariant and

$$\lim_{u\to 0}\|t_uf-f\|_B=0,\qquad f\in B.$$

Suppose that M is a closed subset of B which is *i) bounded:*

$$\sup_{f\in M}\|f\|_B \le \infty,$$

ii) equicontinuous: for any real $\varepsilon > 0$ *there is* $k \in C_c$ *such that*

$$\sup_{f\in M}\|k * f - f\|_B < \varepsilon,$$

iii) tight: for any real $\varepsilon > 0$ *there is* $h \in C_c$ *such that*

$$\sup_{f\in M}\|hf - f\|_B < \varepsilon.$$

Then M *is a compact subset of* B*.*

It is easy to verify that $(L^q, l^p)^\alpha$ $(1 \le q \le \alpha \le p \le \infty)$ is a Banach space satisfying conditions a) and b) of the above proposition. In addition, it is translation invariant. But it does not fulfill condition c). So, let us consider its subspace $(L^q, l^p)^\alpha_c$ defined as follows.

Definition 4.3. For $1 \le q \le \alpha \le p \le \infty$,

$$(L^q, l^p)^\alpha_c = \left\{ f \in (L^q, l^p)^\alpha \;:\; \lim_{u\to 0}\|t_u f - f\|_{q,p,\alpha} = 0 \right\}.$$

Let us quote the following result obtained by Fofana.

Proposition 4.4. *[4] Suppose that* $1 \le q \le \alpha \le p \le \infty$*. Then* $(L^q, l^p)^\alpha_c$ *is a closed subspace of* $(L^q, l^p)^\alpha$ *and* $L^\alpha \subset (L^q, l^p)^\alpha_c$ *if* $\alpha < \infty$*.*

It is clear from the above that, if $1 \le q \le \alpha \le p \le \infty$ and $\alpha < \infty$, then $B = (L^q, l^p)^\alpha_c$ endowed with the norm $\|\cdot\|_{q,p,\alpha}$ satisfies all conditions of Proposition 4.2. We are now in position to prove Theorem 2.14.

Proof of Theorem 2.14. We notice that H is actually a closed and bounded subset of $(L^q, l^p)^\alpha_c$. In addition, it is easy to see that Lemma 4.1 and hypothesis (i) and (ii) implie that for any real $\varepsilon > 0$, there is $k \in C_c$ such that $\sup_{f\in H}\|k * f - f\|_{q,p,\alpha} < \varepsilon$. Furthermore hypothesis (i) and (iii) implie that for any real $\varepsilon > 0$ there is $h \in C_c$ such that $\sup_{f\in H}\|hf - f\|_{q,p,\alpha} < \varepsilon$. Thus, by Proposition 4.2, H is compact in $(L^q, l^p)^\alpha_c$ and subsequently in $(L^q, l^p)^\alpha$. □

Proof of Theorem 2.15. Set $H = \{\widehat{\mu} \;:\; \mu \in K\}$. a) It is clear that from the hypothesis, Proposition 2.8 and Theorem 2.12 that

i) $\sup_{h\in H}\|h\|_{p',\infty,\alpha'} < \infty$,

ii) $\lim_{u\to 0}\sup_{h\in H}\|h - t_u h\|_{p',\infty,\alpha'} = 0$,

iii) $\lim_{\rho\to\infty}\sup_{h\in H}\|h\chi_{\mathbb{R}^d\setminus J_0^\rho}\|_{p',\infty,\alpha'} = 0$.

Thus, by Theorem 2.14, H is a relatively compact subset of $(L^{p'}, l^\infty)^{\alpha'}$.

b) Let $(\mu_n)_{n\geq 1}$ be a sequence in K. From a) it follows that $(\widehat{\mu_n})_{n\geq 1}$ has a subsequence $(\widehat{\mu_{i(n)}})_{n\geq 1}$ which converges in $(L^{p'}, l^\infty)^{\alpha'}$ to an element we denote h. So, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\widehat{\mu_{i(n)}}(x)g(x)dx = \int_{\mathbb{R}^d} h(x)g(x)dx, \qquad g\in\left(L^p,\, l^1\right)$$

and

$$\left|\int_{\mathbb{R}^d}\widehat{\mu_{i(n)}}(rx)g(x)dx\right| \leq C\left\|\mu_{i(n)}\right\|_{p,\,\alpha\quad 1}\left\|\widehat{g}\right\|_{\infty,\,p'} r^{-\frac{d}{\alpha'}}, \qquad r>0,\ g\in C_c,\ n\geq 1,$$

where C is a real constant not depending on the sequence $(\mu_n)_{n\geq 1}$ (see Proposition 2.8). Thus

$$\left|\int_{\mathbb{R}^d} h(rx)g(x)dx\right| \leq C\sup_{\mu\in K}\|\mu\|_{p,\,\alpha\quad 1}\left\|\widehat{g}\right\|_{\infty,\,p'} r^{-\frac{d}{\alpha'}}, \qquad r>0,\ g\in C_c$$

and therefore, by Theorem 2.9, $h=\widehat{\mu}$ for some $\mu\in M^{p,\,\alpha}$. Notice that, for any $g\in\left(L^p,\, l^1\right)$, we have

$$\int_{\mathbb{R}^d}\widehat{g}(x)d\mu(x) = \int_{\mathbb{R}^d}\widehat{\mu}(x)g(x)d(x) = \lim_{n\to\infty}\int_{\mathbb{R}^d}\widehat{\mu_{i(n)}}(x)g(x)d(x) = \lim_{n\to\infty}\int_{\mathbb{R}^d}\widehat{g}(x)d\mu_{i(n)}(x).$$

In addition, by Corollary 3.2, $\left\{\widehat{g}\ :\ g\in\left(L^p,\, l^1\right)\right\}$ is a dense subset of $(C_0,\, l^{p'})$ which contains C_c. Thus

$$\int_{\mathbb{R}^d}\varphi(x)d\mu(x) = \lim_{n\to\infty}\int_{\mathbb{R}^d}\varphi(x)d\mu_{i(n)}(x), \qquad \varphi\in C_c.$$

□

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