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The Actions Of Subgroups Of $SL_2(\mathbb{R})$ For The Clifford Algebra In EPH Cases

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Abstract

We study the action of each subgroup A, N and K of the group $SL_2(\mathbb{R})$ for the Clifford algebra $\mathcal{C}\ell(a)$ and calculate their vector fields, using the derived representation of the Lie algebra sl_2.

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1 Introduction

Geometry brings life to any subject including algebra which supplies tools for any manipulation [7, 8, 9, 15]. The idea of classifying the different branches of geometry in accordance with the classes of transformations considered, is addressed in the "Erlangen Program" [7, 12, 13, 15]. Further, the classification of conic sections: elliptic, parabolic and hyperbolic, is generally abbreviated as EPH - classification. We lay down foundations for all three (including parabolic!) EPH-types of analytic function theories in the paper [11]. Here, we study the actions of subgroups of $SL_2(\mathbb{R})$ for the Clifford algebra in EPH cases.

2 Preliminaries

A Clifford algebra $\mathcal{C}\ell(a)$ as a 4-dimensional linear space is spanned by 1, e_1, e_2, e_1e_2 with *non-commutative* multiplication defined by the identities, in Lounesto [13]:

$$e_1^2 = -1, \qquad e_2^2 = \begin{cases} -1, & \text{for } \mathcal{C}\ell(e)\text{—elliptic case} \\ 0, & \text{for } \mathcal{C}\ell(p)\text{—parabolic case} \\ 1, & \text{for } \mathcal{C}\ell(h)\text{—hyperbolic case} \end{cases}, \qquad e_1e_2 = -e_2e_1. \tag{2.1}$$

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It contains both the planes $\mathbb{E}$ and the vector plane $\mathbb{R}^a$ so that $\mathbb{R}^a$ is spanned by e_1 and e_2, $\mathbb{E}$ is spanned by 1 and e_{12}. The projection on axes of coordinates u, v [4] as $E_1(ue_1 + ve_2) = u$ and $E_2(ue_1 + ve_2) = v$. The group $SL_2(\mathbb{R})$ (Lang [12]) consists of 2×2 matrices whose determinant is one [6]. By the Iwasawa decomposition for semisimple Lie groups [12, p. 39], $SL_2(\mathbb{R})$ can be decomposed as a product of certain closed subgroups (not normal) in the form

$$SL_2(\mathbb{R}) = ANK, \tag{2.2}$$

where A, N, K are defined in [3, 4, 6]. The Möbius transformation T is defined in [2, 4]. Details of a Lie group with mapping and Lie algebra with derivative are in [14]. In our terminology, the derived representation or Lie derivative of a vector field X is given by

$$d\rho(X)(ue_1 + ve_2) = \frac{\partial}{\partial t}\rho\left(e^{tX}(ue_1 + ve_2)\right)\Big|_{t=0}, \tag{2.3}$$

where $e^{tX} \in SL_2(\mathbb{R})$, $X \in \mathfrak{sl}_2$ (Lie algebra of $SL_2(\mathbb{R})$) and $ue_1 + ve_2 \in \mathbb{R}^a$. Isomorphic realisation of $SL_2(\mathbb{R})$ in EPH cases, is included in [1, 4] which contains a realisation of the *Iwasawa decomposition* for semisimple Lie groups (as in Lang [12, § III.1]) as shown in equation (2.2). Geometric and algebraic conditions for circle, parabola and hyperbola are considered in [3, 5]. (Non)-Invariance of the upper half plane under Möbius transformations in EPH cases, is in [1, 4].

3 Actions of subgroups

In all three EPH cases, the subgroups A and N act through Möbius transformations uniformly:

Lemma 3.1. *For any type of the Clifford algebra $\mathcal{C}\ell(a)$:*

(1) The subgroup N defines shifts $ue_1 + ve_2 \mapsto (u+\chi)e_1 + ve_2$ along the "real" axis U by χ. The vector field of the derived representation is $dN_a(u,v) = (1,0)$.

(2) The subgroup A defines dilations $ue_1 + ve_2 \mapsto \alpha^{-2}(ue_1 + ve_2)$ by the factor α^{-2} which fixes origin $(0,0)$. The vector field of the derived representation is $dA_a(u,v) = (2u, 2v)$.

Orbits and vector fields corresponding to the derived representation *[10, § 6.3], [12, Chap. VI] of the Lie algebra $\mathfrak{sl}_2$ for subgroups A and N are shown in [2, Figure 2].*

(3) By contrast the actions of the subgroup K is significantly different between the EPH cases and correlates with names chosen for $\mathcal{C}\ell(e)$, $\mathcal{C}\ell(p)$, $\mathcal{C}\ell(h)$ [2, Figure3]: The vector fields of the derived representation are:

$$\begin{array}{rcll} dK_e(u,v) & = & (1+u^2-v^2, & 2uv) \\ dK_p(u,v) & = & (1+u^2, & 2uv) \\ dK_h(u,v) & = & (1+u^2+v^2, & 2uv). \end{array}$$

These vector fields can be obtained, by using the formula of the Lie derivative by equation (2.3)(see Section 2). The actions of the subgroup K in three cases are as follows:

Lemma 3.2. *(1) For $\mathcal{C}\ell(e)$ the orbits of K are circles. A circle with centre at $(0,(v+v^{-1})/2)$ passes through two points $(0,v)$ and $(0,v^{-1})$. The vector field of the derived representation is $dK_e(u,v) = (u^2 - v^2 + 1, 2uv)$.*

(2) The curvature of a K-orbit at a point $(0,v)$ in $\mathbb{R}^e$ is equal to $\kappa = \frac{2v}{1-v^2}$.

Proof. 1. Suppose for $\mathcal{C}\ell(e)$, an orbit of the subgroup K intersects the V-axis at the point $(0,v)$. To find the other point of intersection. An element of the subgroup K looks like $k = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$. To find the action of the subgroup K on an element of the orbit. For the elliptic case $e_2^2 = -1$ and the corresponding Möbius mapping [4, equation (2.5)] is

$$\begin{pmatrix} \cos t & e_1 \sin t \\ e_1 \sin t & \cos t \end{pmatrix} : ve_2 \longrightarrow \frac{\cos t (ve_2) + e_1 \sin t}{e_1 \sin t (ve_2) + \cos t}, \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in SL_2(\mathbb{R}).$$

From the previous results [4, equation (3.3)],

$$\frac{\cos t (ve_2) + e_1 \sin t}{e_1 \sin t (ve_2) + \cos t} = \left(\frac{L}{D}\right) e_1 + \left(\frac{V}{D}\right) e_2,$$

where $L = \dfrac{(1-v^2)}{2} \sin 2t$ and $D = \big(1 - (1-v^2)\sin^2 t\big)$ and, the components are

$$x(t) = \frac{L}{D} \quad \text{and} \quad y(t) = \frac{V}{D}. \tag{3.1}$$

From the equation (3.1), we conclude that the image of a point ve_2 under the K-action belongs to the imaginary axis if and only if $\sin 2t = 0$, that is $t = k\frac{\pi}{2}$ for an integer k. We observe that at time $t = 0$, we have $(x(t), y(t)) = (0, v)$ and at time $t = \pi/2$, we get the other point of intersection as $(x(t), y(t)) = (0, v^{-1})$. Next to show that the K-orbit is a circle. For this, we define $v_0 = (v + v^{-1})/2$. Then the radius vector $\vec{r}$ is $\vec{r} = (x(t), y(t) - v_0)$. Also the vector field $V_{(x,y)}$ at the position $(x(t), y(t))$ is given by $V_{(x,y)} = \big(1 + \{x(t)\}^2 - \{y(t)\}^2, 2x(t)y(t)\big)$. To show that $\vec{r} \perp V_{(x,y)}$, we take the dot product [5, equation (8)],

$$\vec{r} \cdot V_{(x,y)} = \frac{L}{D}\left(1 + \frac{L^2}{D^2} - \frac{v^2}{D^2}\right) + 2v\frac{L}{D^2}\left(\frac{v}{D} - v_0\right).$$

Therefore,

$$\vec{r} \cdot V_{(x,y)} = \frac{2L}{D^3}\left[\frac{(1-v^2)^2}{8}\sin^2 2t - \frac{(1-v^2)^2 \sin^2 t \cos^2 t}{2}\right] = 0.$$

Hence $\vec{r} \perp V_{(x,y)}$. In other words the radius vector is perpendicular to the vector field at any arbitrary point $(x(t), y(t))$ on the K-orbit. As a result the K-orbits on $\mathcal{C}\ell(e)$ are circles which pass through two points $(0,v)$ and $(0,v^{-1})$, having centre as $v_0 = (v + v^{-1})/2$, see Figures 3(K_e) in [2, 4] and 1.

2. Differentiating $x(t)$ and $y(t)$ in equation (3.1) twice with respect to t, we get at $t = 0$,

$$\dot{x}(0) = 1 - v^2, \quad \dot{y}(0) = 0, \quad \ddot{x}(0) = 0 \quad \text{and} \quad \ddot{y}(0) = 2v(1 - v^2).$$

Therefore the curvature (κ) at time $t=0$ is given by

$$\kappa|_{t=0} \;=\; \left.\frac{|\ddot{x}\dot{y}-\ddot{y}\dot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}}\right|_{t=0} \;=\; \frac{2v}{(1-v^2)}.$$

Hence the radius of curvature (ρ) at time $t=0$, is given by

$$\rho \;=\; \frac{1}{\kappa} \;=\; \frac{(1-v^2)}{2v}.$$

In case of a circle, we know that the radius $p=\rho$ (radius of curvature). Therefore, $p=\frac{(1-v^2)}{2v}$ is the radius of the circle passing through the points $(0,v)$ and $(0,v^{-1})$ whose diameter is given by $\frac{(1-v^2)}{v}$.

□

Lemma 3.3. *(1) For $\mathcal{C}\ell(p)$ the orbits of K are parabolas with the vertical axis V. A parabola passing through $(0,v/2)$ has its horizontal directrix passing through $(0,(v-v^{-1})/2)$ and focus at $(0,(v+v^{-1})/2)$. The vector field of the derived representation is $dK_p(u,v)=(u^2+1,2uv)$.*

(2) The curvature of a K-orbit at a point $(0,v/2)$ in $\mathbb{R}^p$ is equal to $\kappa=v$.

Proof. 1. We suppose for $\mathcal{C}\ell(p)$, an orbit of the subgroup K intersects the V-axis at the point $(0,v/2)$. For the parabolic case $e_2^2=0$ and as earlier, the Möbius mapping is

$$\begin{pmatrix}\cos t & e_1\sin t\\ e_1\sin t & \cos t\end{pmatrix} : \frac{v}{2}e_2 \longrightarrow \frac{\cos t\left(\frac{v}{2}e_2\right)+e_1\sin t}{e_1\sin t\left(\frac{v}{2}e_2\right)+\cos t},\; \begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix} \in SL_2(\mathbb{R}).$$

Therefore the components of the vector field at time t (see equation (3.4), [4]) are

$$x(t)=\tan t \quad\text{and}\quad y(t)=\frac{v}{2}\sec^2 t, \tag{3.2}$$

and at time $t=0$, $(x(t),y(t))=(0,v/2)$. Next to show that the K-orbit is a parabola, we take $v_0=(v+v^{-1})/2$. Then the radius vector $\vec{r}=(x(t),y(t)-v_0)$ and the vector field $V_{(x,y)}$ at the position $(x(t),y(t))$ is $V_{(x,y)}=\big(1+\{x(t)\}^2,2x(t)y(t)\big)$. Here, $(u,v)=\Big(\tan t,\left(\frac{v}{2}\sec^2 t-v_0\right)\Big)$ and $(u',v')=\big(\sec^2 t, v\tan t\sec^2 t\big)$, we have

$$\begin{aligned} u'u-v'\left(\sqrt{u^2+v^2}-v\right) &= \tan t\sec^2 t\left[1-v\left\{\sqrt{\left(\frac{v}{2}\tan^2 t+\frac{v^{-1}}{2}\right)^2}-\left(\frac{v}{2}\tan^2 t-\frac{v^{-1}}{2}\right)\right\}\right]\\ &= 0. \end{aligned}$$

that is, condition being satisfied (Lemma 3.1, [5]), the K-orbits on $\mathcal{C}\ell(p)$ are parabolas. Therefore $v_0=(v+v^{-1})/2$ is the focus of the parabola passing through the point $(0,v/2)$. The horizontal directrix then passes through the point $(0,(v-v^{-1})/2)$, see Figure $3(K_p)$ in [2].

2. On differentiating $x(t)$ and $y(t)$ in equation (3.2) twice and computing their values at $t=0$, as in the earlier case, the curvature at time $t=0$, is $\kappa|_{t=0} = v$ and the radius of curvature at $t=0$ is $\rho = \frac{1}{v}$. In case of a parabola, the focal length (distance between the focus and the vertex) being $p=\frac{1}{2}\rho$, therefore

$$p = \frac{1}{2}\left(\frac{1}{v}\right) = \frac{1}{2v} = \frac{1}{4(v/2)},$$

is the focal length of the parabola which passes through the point $(0, v/2)$, see Figure 1.

□

Lemma 3.4. *(1) For $\mathcal{C}\ell(h)$ the orbits of K are hyperbolas with asymptotes parallel to lines $u=\pm v$. The vector field of the derived representation is $dK_h(u,v) = (u^2+v^2+1, 2uv)$.*

(2) Also for $\mathcal{C}\ell(h)$ the orbits of K are rectangular hyperbolas. In other words, the angle between the asymptotes of the hyperbolas is a right angle.

(3) The curvature of a K-orbit at a point $(0,v)$ in $\mathbb{R}^h$ is equal to $\kappa = \frac{2v}{1+v^2}$. A hyperbola passing through the point $(0,v)$ has the focal distance between foci $2p$, where $p = \frac{v^2+1}{\sqrt{2}v}$ and the upper focus is located at $(0,f)$ with:

$$f = \begin{cases} p - \sqrt{\frac{p^2}{2} - 1}, & \textit{for } 0 < v < 1; \textit{ and} \\ p + \sqrt{\frac{p^2}{2} - 1}, & \textit{for } v \geq 1. \end{cases}$$

Proof. 1. We consider for $\mathcal{C}\ell(h)$, an orbit of the subgroup K intersects the V-axis at the point $(0,v)$. To find the other point in which it intersects the V-axis, we proceed as Lemma 3.2 above, with the exception that $e_2^2 = 1$ for the hyperbolic case. From the previous results (see equation (3.5), [4]),

$$\frac{\cos t(ve_2) + e_1 \sin t}{e_1 \sin t(ve_2) + \cos t} = \left(\frac{M}{D'}\right) e_1 + \left(\frac{v}{D'}\right) e_2.$$

The components of the vector field at time t are

$$x(t) = \frac{M}{D'} \quad \text{and} \quad y(t) = \frac{v}{D'}, \tag{3.3}$$

where, $M = \frac{(1+v^2)}{2}\sin 2t$ and $D' = \left(1-(1+v^2)\sin^2 t\right)$. As before at time $t=0$, $(x(t),y(t)) = (0,v)$ and at time $t=\pi/2$, $(x(t),y(t)) = (0,-v^{-1})$. To show that the K-orbit is a hyperbola, we define $v_0 = (v-v^{-1})/2$. The radius vector $\vec{r}$ and the vector field $V_{(x,y)}$ at the position $(x(t),y(t))$ as earlier are

$$\vec{r} = (x(t), y(t)-v_0) = \left(\frac{M}{D'}, \frac{v}{D'} - v_0\right) \text{ and } V_{(x,y)} = \left(1 + \frac{(M^2+v^2)}{D'^2}, \frac{2vM}{D'^2}\right).$$

To show that the condition $u'u - v'v = 0$ is satisfied (see Lemma 3.2, [5]) for it to be a hyperbola. Here, $(u,v) = \vec{r}$ and $(u',v') = V_{(x,y)}$. We have

$$u'u - v'v = \frac{2M}{D'^3}\left[\frac{(1+v^2)^2}{8}\sin^2 2t - \frac{(1+v^2)^2 \sin^2 t \cos^2 t}{2}\right] = 0.$$

As a result, the K-orbits on $\mathcal{C}\ell(h)$ are hyperbolas. Hence $v_0 = (v - v^{-1})/2$ is the centre of the hyperbola. The equation of the hyperbola is given by

$$u^2 - (v - v_0)^2 = -\frac{(v + v^{-1})^2}{4}.$$

The equations of the asymptotes are given by

$$u^2 - (v - v_0)^2 = 0 \quad \Rightarrow \; u = \pm(v - v_0),$$

which are parallel to the lines $u = \pm v$, see Figure 3(K_h)in [2].

2. The components of the vector field (Lemma 3.1) for $\mathcal{C}\ell(h)$ are

$$u_1 = 1 + u^2 + v^2 \quad \text{and} \quad v_1 = 2uv,$$

where (u, v) is any point on the hyperbola. As $(u, v) \to +\infty$, the asymptote (given by the equation $u = v$) approaches the tangent at that point, see Figure 2. Therefore the slope of the tangent as given by

$$\tan\theta_1 \;=\; \lim_{(u,v)\to+\infty} \frac{v_1}{u_1} \;=\; \lim_{(u,v)\to+\infty} \frac{2uv}{(1 + u^2 + v^2)},$$

is on the line $u = v$,

$$\tan\theta_1 = \lim_{u\to+\infty} \frac{2u^2}{(1 + 2u^2)} \left(\text{of } \frac{\infty}{\infty} \text{ form}\right).$$

Using L' Hôpital's rule, we get

$$\tan\theta_1 \;=\; 1 \Rightarrow \theta_1 \;=\; \pi/4.$$

Similarly as $(u, v) \to -\infty$, the asymptote (given by the equation $u = -v$) tends to the tangent at that point and as before,

$$\tan\theta_2 \;=\; -1 \Rightarrow \theta_2 = -\pi/4.$$

Hence the angle between the asymptotes $u = \pm v$ is given by

$$\theta \;=\; |\theta_1| + |\theta_2| \;=\; |\pi/4| + |-\pi/4| \;=\; \pi/2,$$

which is a right angle. As a result the K-orbits are rectangular hyperbolas.

3. Differentiating the components $(x(t), y(t))$ in equation (3.3) twice with respect to t and obtaining their values at $t = 0$ as before, the curvature (κ) at time $t = 0$ is $\kappa|_{t=0} \;=\; \dfrac{2v}{(1 + v^2)}$ and the radius of curvature (ρ) is $\rho \;=\; \dfrac{(1 + v^2)}{2v}$.

In case of a hyperbola, the focal length (distance between the focus and the centre) $p = \sqrt{2}\rho$ where ρ is the radius of curvature. Therefore

$$p \;=\; \frac{(v^2 + 1)}{\sqrt{2}v}, \tag{3.4}$$

where $2p$ is the focal distance between foci of the hyperbola passing through the point $(0,v)$.

At $v=1$, $f=p=\sqrt{2}$, as it is a rectangular hyperbola. Here the hyperbola passes through the point $(0,v)$, $2p$ is the focal distance between foci and the upper focus is located at $(0,f)$, see Figure 3.

At $v>1$, $f=p+x$, $x>0$, and at $v<1$, $f=p-x$, $x>0$, see Figure 4. The focal distance between foci $2p$ is from equation (3.4)

$$\begin{aligned} v^2-\sqrt{2}vp+1 &= 0. \\ \text{Therefore} \quad v &= \left(\frac{p}{\sqrt{2}}+\sqrt{\frac{p^2}{2}-1}\right), \left(\frac{p}{\sqrt{2}}-\sqrt{\frac{p^2}{2}-1}\right). \end{aligned}$$

For the case $v>1$, we know from the property of a rectangular hyperbola (Figure 4) that

$$p \;=\; \sqrt{2}(v-x) \;\Rightarrow\; x \;=\; \frac{1}{\sqrt{2}}(\sqrt{2}v-p) = \sqrt{\frac{p^2}{2}-1}$$

(rejecting the other value of v as $x>0$). Therefore

$$f \;=\; p+x \;=\; p+\sqrt{\frac{p^2}{2}-1}.$$

Similarly, for the case $v<1$, we get

$$p \;=\; \sqrt{2}(v+x) \;\Rightarrow x \;=\; \frac{1}{\sqrt{2}}(p-\sqrt{2}v) = \sqrt{\frac{p^2}{2}-1}$$

Hence,

$$f \;=\; p-x \;=\; p-\sqrt{\frac{p^2}{2}-1}.$$

Thus the upper focus located at $(0,f)$ is given by

$$f=\begin{cases} p-\sqrt{\dfrac{p^2}{2}-1} & \text{for } 0<v<1\text{; and} \\ p+\sqrt{\dfrac{p^2}{2}-1} & \text{for } v\geq 1. \end{cases}$$

□

Remark 3.5. 1. The values of all three vector fields dK_e, dK_p and dK_h coincide on the "real" U-axis ($v=0$), i.e. they are three different extensions into the domain of the same boundary condition.

2. The hyperbola passing through the point $(0,1)$ has the shortest focal length $\sqrt{2}$ among all other hyperbolic orbits; two hyperbolas passing through $(0,v)$ and $(0,-v^{-1})$ have the same focal length as

$$p \equiv \frac{(-v^{-1})^2+1}{\sqrt{2}(-v^{-1})} = \frac{-(v^2+1)}{\sqrt{2}v},$$

which has the same expression as in equation (3.4) except for a negative sign. These hyperbolas are related to each other as explained in Remark [11].

3. An alternative proof of Lemma 3.4(2) can be presented by the parametric representation (equation (3.3)).

4 Conclusion

Here, we have calculated the vector fields for the three subgroups A, N and K, using the formula for the derived representation. Then we study the actions of the subgroups of $SL_2(\mathbb{R})$. We took an isomorphic realisation of $SL_2(\mathbb{R})$ for studying the actions of the subgroups A, N and K. In drawing the figures, we have employed MetaPost software package.

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