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STRUCTURE THEORY AND REFLEXIVITY OF CONTRACTION OPERATORS

B. CHEVREAU, G. EXNER, AND C. PEARCY

1. Introduction. Let $\mathscr{H}$ be a separable, infinite-dimensional, complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. The purpose of this note is to announce several new, and rather general, sufficient conditions that a contraction T in $\mathscr{L}(\mathscr{H})$ be reflexive, and, at the same time, to give various characterizations of the class of those contractions that possess an analytic invariant subspace (definition given below). Complete proofs and other results will appear in [**7**]. The principal new idea involved is a considerable improvement of the main construction of §3 of [**9**]. The new reflexivity theorems also depend on techniques from [**9, 3, 1,** and **4**], and yield, in particular, the following improvement of the main result of [**4**].

THEOREM 1.1. *If T is a contraction in $\mathscr{L}(\mathscr{H})$ such that the spectrum $\sigma(T)$ of T contains the unit circle $\mathbf{T}$, then either T is reflexive or T has a nontrivial hyperinvariant subspace.*

If $T \in \mathscr{L}(\mathscr{H})$ we denote by $\mathscr{A}_T$ the dual algebra generated by T (i.e., $\mathscr{A}_T$ is the smallest unital subalgebra of $\mathscr{L}(\mathscr{H})$ containing T that is closed in the weak* topology (which accrues to $\mathscr{L}(\mathscr{H})$ by virtue of its being the dual space of the Banach space $\mathscr{C}_1(\mathscr{H})$ of trace-class operators)). It follows that $\mathscr{A}_T$ is the dual space of $Q_T = \mathscr{C}_1(\mathscr{H})/{}^{\perp}\mathscr{A}_T$, where ${}^{\perp}\mathscr{A}_T$ is the preannihilator of $\mathscr{A}_T$ in $\mathscr{C}_1(\mathscr{H})$, under the pairing

$$\langle A, [L]\rangle = \mathrm{tr}(AL), \qquad A \in \mathscr{A}_T,\ L \in \mathscr{C}_1(\mathscr{H}),$$

where $[L]$ denotes the element of the quotient space Q_T containing the trace-class operator L. Thus, if x and y are vectors in $\mathscr{H}$, then $[x \otimes y]$ denotes the element of Q_T containing the rank-one operator $x \otimes y$. The dual algebra $\mathscr{A}_T$ is said to have property $(\mathbf{A}_{1,\aleph_0})$ if for any sequence $\{[L_j]\}_{j=1}^{\infty}$ of elements from Q_T there exist vectors x and $\{y_j\}_{j=1}^{\infty}$ in $\mathscr{H}$ satisfying

$$[L_j] = [x \otimes y_j], \qquad j = 1, 2, \dots. \tag{1}$$

If, moreover, there exists $\rho \geq 1$ (independent of the family $\{[L_j]\}$) with the property that for every $s > \rho$, the vectors $\{x\}$ and $\{y_j\}$ satisfying (1) can also be chosen to satisfy

$$\|x\| \leq \left(s \sum_{k=1}^{\infty} \|[L_k]\|\right)^{1/2}, \qquad \|y_j\| \leq (s\|[L_j]\|)^{1/2}, \qquad j = 1, 2, \dots,$$

then we say that $\mathscr{A}_T$ has property $(\mathbf{A}_{1,\aleph_0}(\rho))$.

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Recall that if T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, and $H^\infty(\mathbf{T})$ is the usual Hardy algebra of functions on $\mathbf{T}$, then the Sz.-Nagy–Foias functional calculus $\Phi_T\colon H^\infty(\mathbf{T}) \to \mathcal{A}_T$ is a weak* continuous algebra homomorphism with range weak* dense in $\mathcal{A}_T$. The class $\mathbf{A} = \mathbf{A}(\mathcal{H})$ is defined to be the set of all those absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which Φ_T is an isometry; in other words, the set of such T for which $\|f(T)\| = \|f\|_\infty$ for every f in $H^\infty(\mathbf{T})$. Various sufficient conditions for an absolutely continuous contraction T to belong to $\mathbf{A}$ are known [**2**]. One such is that $\sigma(T) \cap \mathbf{D}$ is dominating for $\mathbf{T}$, where $\mathbf{D}$ is the open unit disc in $\mathbf{C}$. The class $\mathbf{A}_{1,\aleph_0}$ [resp. $\mathbf{A}_{1,\aleph_0}(\rho)$] is defined to consist of those T in $\mathbf{A}(\mathcal{H})$ for which $\mathcal{A}_T$ has property $(\mathbf{A}_{1,\aleph_0})$ [resp. $(\mathbf{A}_{1,\aleph_0}(\rho)$].

2. Analytic invariant subspaces. It turns out that another concept plays a central role in the derivation of our results—namely, the notion of an analytic invariant subspace (cf. [**10, 3**]). If T is a contraction in $\mathcal{L}(\mathcal{H})$, $\mathcal{M} \in \operatorname{Lat}(T)$, and there exists a nonzero conjugate analytic function $e\colon \lambda \to e_\lambda$ from $\mathbf{D}$ into $\mathcal{M}$ such that

$$(T|\mathcal{M} - \lambda)^* e_\lambda = 0, \qquad \forall \lambda \in \mathbf{D},$$

then $\mathcal{M}$ is said to be an *analytic invariant subspace* for T. If, in addition, $\bigvee_{\lambda\in\mathbf{D}} e_\lambda = \mathcal{M}$, then $\mathcal{M}$ is said to be a *full analytic invariant subspace* for T.

If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma_p(T)$, $\sigma_r(T)$, and $\sigma_e(T)$ for the point spectrum, right spectrum and essential (Calkin) spectrum of T respectively. Moreover, following [**8**], we write $\mathcal{F}'_+(T)$ for the set of all λ in $\mathbf{C}$ for which $T - \lambda$ is a Fredholm operator with (strictly) positive index. Recall also that a subspace $\mathcal{K}$ of $\mathcal{H}$ is said to be *semi-invariant* for T if $\mathcal{K} = \mathcal{M} \ominus \mathcal{N}$, where $\mathcal{M}, \mathcal{N} \in \operatorname{Lat}(T)$ and $\mathcal{M} \supset \mathcal{N}$; we denote the set of all semi-invariant subspaces for T by $\mathcal{SI}(T)$. (Of course, $\mathcal{H}$ itself and all elements of $\operatorname{Lat}(T)$ belong to $\mathcal{SI}(T)$.) As usual, if $\mathcal{K} \in \mathcal{SI}(T)$, we write $T_{\mathcal{K}}$ for the compression of T to $\mathcal{K}$.

THEOREM 2.1. *If T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, the following statements are equivalent:*

(a) *T has an analytic invariant subspace.*
(b) *T has a full analytic invariant subspace.*
(c) $T \in \mathbf{A}_{1,\aleph_0}$.
(d) $T \in \mathbf{A}_{1,\aleph_0}(\rho)$ *for some* $\rho \geq 1$.
(e) *There exists $\mathcal{K} \in \mathcal{SI}(T)$ such that $\sigma_p(T^*_{\mathcal{K}}) = \mathbf{D}$.*
(f) *There exists $\mathcal{K} \in \mathcal{SI}(T)$ such that $T_{\mathcal{K}} \in \mathbf{A}$ and*

$$(\sigma_r(T_{\mathcal{K}}) \cap \mathbf{D}) \cup (\mathbf{D} \backslash \mathcal{F}'_+(T_{\mathcal{K}}))$$

is dominating for $\mathbf{T}$.

Some of the implications in this "wheel of equivalences" are easy; the deeper ones depend on additional, more technical, characterizations of the class $\mathbf{A}_{1,\aleph_0}$ in terms of certain properties $E^r_{\theta,\gamma}$ and $F^r_{\theta,\gamma}$ which appear in [**9** and **7**], as well as on techniques and results from [**8, 4** and **5**].

3. Results on reflexivity. Recall that an operator T in $\mathscr{L}(\mathscr{H})$ is said to be *reflexive* if every operator S in $\mathscr{L}(\mathscr{H})$ such that $\mathrm{Lat}(S) \supset \mathrm{Lat}(T)$ belongs to $\mathscr{W}_T$, the closure of $\mathscr{A}_T$ in the weak operator topology. If T is a contraction, we denote by T_a the direct summand of T that is the absolutely continuous part of T (i.e., T_a is the direct sum of the completely nonunitary part of T and the absolutely continuous part of the unitary part of T).

THEOREM 3.1. *Each of the following is a sufficient condition that an arbitrary contraction T in $\mathscr{L}(\mathscr{H})$ be reflexive:*

(A) T *(or T^*) satisfies any one of the conditions* (a)–(f) *of Theorem* 2.1.
(B) T_a *(or T_a^*) satisfies* (c) *or* (d) *of Theorem* 2.1.
(C) $T_a \in (C_{0\cdot} \cup C_{\cdot 0}) \cap \mathbf{A}$.
(D) $T_a \in (C_{1\cdot} \cup C_{\cdot 1}) \cap \mathbf{A}$.
(E) T *is hyponormal and* $T_a \in \mathbf{A}$.

Theorem 1.1 follows from Theorem 3.1(C) via the fact that any contraction T with $\sigma(T) \supset \mathbf{T}$ not in the class $(C_{0\cdot} \cup C_{\cdot 0}) \cap \mathbf{A}$ has nontrivial hyperinvariant subspaces (cf. [**2**, Theorem 4.3]), and on the basis of Theorem 3.1 we make the following conjectures.

CONJECTURE 3.2 [**6**]. *Every T in $\mathbf{A}$ is reflexive.*

CONJECTURE 3.3. *Every hyponormal operator is reflexive.*

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U. E. R. DE MATHÉMATIQUES, UNIVERSITÉ DE BORDEAUX I, 33405 TALENCE, FRANCE

DEPARTMENT OF MATHEMATICS, OBERLIN COLLEGE, OBERLIN, OHIO 44070

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48109