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ROCHLIN INVARIANTS, THETA MULTIPLIERS AND HOLONOMY

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In this note (M^{8k+2}, w) is a compact closed smooth spin manifold of dimension $8k+2$ with torsion free middle integral homology $H_{4k+1}(M^{8k+2}; \mathbf{Z})$. Moreover, for convenience, we assume that the quadratic map

$$q_w : H_{4k+1}(M^{8k+2}; \mathbf{Z}) \to \mathbf{Z}/2\mathbf{Z}$$

associated to the spin structure w (see E. H. Brown [**2**]) has Arf invariant zero.

Let $F = (f, b)$ be a spin automorphism of (M^{8k+2}, w). Under favorable conditions, which are always satisfied if k equals 0 or 1, we may define an invariant of F, the Rochlin invariant $R(M, w, F)$, which is an integer mod 16. We announce a way of computing this invariant mod 8.

Further, for a suitable pair of spin automorphisms F_i of (M, w_i), $i = 1, 2$, we express the complex number $\exp((2\pi i/8)[R(M, w_1, F_1) - R(M, w_2, F_2)])$ as a quotient of theta multipliers (the eighth roots of unity entering into the transformation law for theta functions). When M is a Riemann surface V, this number is the inverse of the holonomy of the flat determinant line bundle $(\det \bar{\partial}_{w_1}) \otimes (\det \bar{\partial}_{w_2})^{-1}$. Similarly, the holonomy of the flat bundle $(\det \bar{\partial}_w \otimes (\Delta^+)^h) \otimes (\det \bar{\partial}_w)^{3h^2-1}$ (with Δ^+ the $+$ chirality spin bundle) is completely determined. The motivation for this work has been twofold—to interpret theta multipliers in topological terms and to answer certain questions in physics. In the physics terminology we have calculated the "global anomaly" for these coupled fields [**9**].

1. The Rochlin invariants. By a spin structure on N^n we mean an oriented manifold N^n together with a reduction of its tangent bundle to

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Spin(n). A spin automorphism $F = (f, b)$ of (M^{8k+2}, w) is a diffeomorphism $f\colon M \to M$ together with a bundle map $b\colon w \to w$ covering f. Given F we may form the mapping torus of f

$$(M \times_f S^1) = M \times [0,1]/(x,1) \sim (f(x),0)$$

and from the bundle map b we obtain a spin structure w' on $M \times_f S^1$ in a natural manner. Under the favorable condition that $(M \times_f S^1, w')$ is the boundary of a spin manifold (X^{8k+4}, w'') we define the *Rochlin invariant* $R(M, w, F)$ via

$$R(M, w, F) = (\text{signature of } X^{8k+4}) \qquad (\operatorname{mod} 16).$$

By theorems of Rochlin and Ochanine [**6**] this is a well-defined invariant if (X, w'') exists (as is always the case if $k = 0, 1$).

We now indicate how $R(M, w, F)$ (mod 8) can be computed in terms of the induced map f_* on $H_{4k+1}(M; \mathbf{Z})$, the quadratic map q_w, and the intersection pairing $I\colon H_{4k+1}(M; \mathbf{Z}) \otimes H_{4k+1}(M; \mathbf{Z}) \to \mathbf{Z}$.

Denote by A the subgroup of x such that $nx = (1 - f_*)(y)$ for some y in $H_{4k+1}(M; \mathbf{Z})$ and nonzero integer n. Then the torsion subgroup of $H_{4k+1}(M \times_f S^1; \mathbf{Z})$ is isomorphic to $T = A/\operatorname{Image}(1 - f_*)$ and the linking pairing $B\colon T \otimes T \to \mathbf{Q}/\mathbf{Z}$ is given by

$$B([x_1], [x_2]) = (1/n) I(x_1, y_2).$$

Here $[x_j]$ lifts to x_j in A and $n \cdot x_2 = (1 - f_*)(y_2)$.

THEOREM 1. *Suppose $(M \times_f S^1, w')$ is a spin boundary. Define $Q[q_w]\colon T \to \mathbf{Q}/\mathbf{Z}$ by the formula*

$$Q[q_w]([t]) = (1/2)(B(t,t) + q_w(t)) \quad \text{in } \mathbf{Q}/\mathbf{Z}$$

where $[t]$ lifts to t in A. Then $Q[q_w]$ is a quadratic refinement of the linking pairing B and the Gaussian sum, $\sum_t \exp(2\pi i \cdot Q[q_w](t))$, has nonzero modulus and its argument is given by $(-2\pi/8) \cdot R(M, w, F)$.

2. Relationship to theta multipliers. Via a symplectic basis, we identify $H_{4k+1}(M; \mathbf{Z})$ with the free abelian group $\mathbf{Z}^{2g}$. For any spin automorphism $F = (f, b)$, the map f_* lies in the theta group $\Gamma(m)$ consisting of integral symplectic matrices L such that $q_w(L(x)) = q_w(x)$ for all x. Here $m = (m_1', \ldots ; m_1'', \ldots)$, the characteristic associated to q_w, is given by the formula $q_w(x_1, \ldots ; y_1, \ldots) = \sum_j [(x_j \cdot y_j) + (m_j' \cdot x_j) + (m_j'' \cdot y_j)]$. By assumption the Arf invariant of q_w, $\sum_j (m_j' \cdot m_j'')$ (mod 2), is zero; hence, m is called an even characteristic.

The law of transformation of the theta function θ_m of characteristic m is recorded by a homomorphism $\tilde{\lambda}(m)\colon \tilde{\Gamma}(m) \to \mu_8$ into the eighth root of unity. (See Selberg [**7**, p. 86] and Weil [**8**].) Here $\tilde{\Gamma}(m)$ is the $(\mathbf{Z}/2\mathbf{Z})$-central extension of $\Gamma(m)$ obtained from pulling back to the theta group $\Gamma(m)$ the metaplectic double covering $\operatorname{Mpl}(2g, R) \to \operatorname{Spl}(2g, R)$. For even characteristics m_1, m_2 the quotient and square of these maps induce homomorphisms which we denote by

$$(\lambda(m_1)/\lambda(m_2))\colon \Gamma(m_1) \cap \Gamma(m_2) \to \mu_8; \qquad (\lambda(m))^2\colon \Gamma(m) \to \mu_8$$

since the nontrivial element of the center maps to (-1) under all the homomorphisms $\tilde{\lambda}(m)$. Easy to use and explicit formulas for these maps have been given by Igusa [**4**] and Johnson and Millson [**5**].

THEOREM 2. *Let $F_j = (f, b_j)$ be spin automorphisms of (M^{8k+2}, w_j) covering the same diffeomorphism f for $j = 1, 2$. Let $q(w_j)$ have even characteristic m_j. Suppose that $R(M, w_j, F_j)$ is defined for $j = 1, 2$ (as is always the case if $k = 0$ or 1). Then*

$$(\lambda(m_2)/\lambda(m_1))(f_*) = \exp((2\pi i/8)[R(M, w_1, F_1) - R(M, w_2, F_2)])$$

and these are both fourth roots of unity.

3. Determinant line bundles. We now consider the moduli spaces $\mathcal{M}_g$ of Riemann surfaces V of genus g with some extra structures such as one or more spin structures w_j and extrinsic bundles E_j. Over these moduli spaces there are the family of universal Riemann surfaces $\mathcal{U} \to \mathcal{M}_g$ and also families of coupled Dirac-Weyl operators $\partial\!\!\!/_{w_j} \otimes (\Delta^+)^h \otimes E_j$. Here E_j is an extrinsic bundle and $(\Delta^+)^h$ is the hth power of the + chirality spin bundle for the spin structure w_j. Moreover, under some natural assumptions on our setting (e.g., E has a connection), by the work of Bismut and Freed [**1**] the determinant line bundle $(\det \partial\!\!\!/_{w_j} \otimes (\Delta^+)^h \otimes E_j)$ has a natural unitary connection. Three important cases in which *flat* unitary bundles arise are

$$L(w_1, h) = (\det \partial\!\!\!/_{w_1} \otimes (\Delta^+)^h) \otimes (\det \partial\!\!\!/_{w_1})^{3h^2-1},$$
$$L(w_1, E) = (\det \partial\!\!\!/_{w_1} \otimes E) \otimes (\det \partial\!\!\!/_{w_1})^{-1}$$

where E is a flat line bundle, and

$$L(w_1, w_2) = (\det \partial\!\!\!/_{w_1}) \otimes (\det \partial\!\!\!/_{w_2})^{-1}.$$

The holonomy of these flat unitary bundles is specified as follows. Let $\mathcal{M}_g$ be the moduli space for two spin structures w_j, $j = 1, 2$, (perhaps the same) and one extrinsic bundle E (perhaps trivial). For convenience, assume that w_j has associated quadratic refinement q_{w_j} and characteristic m_j which is even for $j = 1, 2$.

THEOREM 3. *Let γ be a loop in $\mathcal{M}_g$ represented by a pair of spin automorphisms $F_1 = (f, b_1)$, $F_2 = (f, b_2)$ covering the same diffeomorphism f for the spin structures w_1, w_2 respectively. Then*

(a) *The holonomy of the flat bundle $L(w_1, h)$ around γ is $+1$ if h is even and is the fourth root of unity $(\lambda(m_1))^2(f_*)$ if h is odd.*

(b) *If E is a flat line bundle, then the holonomy of $L(w_1, E)$ around γ is $\exp(2\pi i \cdot Q[q_{w_1}](z))$ where $Q[q_{w_1}]$ is the quadratic refinement of the linking pairing of the mapping torus $(V \times_f S^1)$ specified by F_1 as above and z is the Poincaré dual of the first Chern class of $E/(V \times_f S^1)$.*

(c) *The holonomy of the flat bundle $L(w_1, w_2)$ around γ is always a fourth root of unity and is given by*

$$(\lambda(m_1)/\lambda(m_2))(f_*) = \exp((2\pi i/8)[R(V, w_2, F_2) - R(V, w_1, F_1)]).$$

REMARKS. The proof of Theorem 1 above is based on a combination of the work of Brown, Brumfiel, and Morgan [**2, 3**]. As for Theorems 2 and 3, they rely on a detailed knowledge of the abelianization of the spin mapping class group, the work of Igusa [**4**] on theta functions and of Johnson and Millson on the theta group [**5**], and the holonomy formula of Bismut and Freed [**1**]. A thorough treatment of all these topics will appear in a future paper.

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