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Shock waves and reaction-diffusion equations, by Joel Smoller, A Series of Comprehensive Studies in Mathematics, Vol. 258, Springer-Verlag, New York, 1983, xx + 581 pp., $39.00. ISBN 0-3879-0752-1

The mathematical theory of compressible media includes systems of conservation laws,

$$\partial_t u + \sum_{j=1}^{m} \partial_{x_j} f^j(x, t, u) = 0, \tag{1}$$

of hyperbolic type in which the state of the medium at position x and time t is characterized by a dependent variable $u = u(x, t)$ in R^n. Field equations of the form (1) express the basic laws of physics associated with the conservation of mass, momentum and energy and provide mathematical models for a variety of nonlinear wave motion. The special systems arising in fluid dynamics, elasticity, combustion theory, shallow water theory, petroleum reservoir engineering, etc. serve as the prototypes for the general development of the subject.

The analytical theory of conservation laws was initiated nearly a century ago during a period beginning with the work of Hugoniot, Rankine, and Riemann and has progressed since then through many important contributions to both the formal and rigorous sides of the subject. We shall restrict our attention here mainly to some work from the last decade dealing with the rigorous theory for systems of equations, as they relate to the sections of the book under review dealing with shock waves.

Much of the recent progress on conservation laws has involved systems in one space dimension,

$$\partial_t u + \partial_x f(u) = 0, \qquad -\infty < x < \infty, \tag{2}$$

and has developed in two directions connected with the geometric and functional analytic branches of the subject. The geometric theory has been concerned primarily with the action of the solution operator in the strong topology and has treated various problems dealing with existence and qualitative behavior of solutions, in particular, the development and propagation of singularities. The associated analysis employs observations of the solution structure using various singular measures in the x-t plane, for the purpose of resolving and organizing the fine scale features in terms of elementary waves and their interactions. In contrast, the functional analytic theory has involved a study of the solution operator in the weak topology and has dealt mainly with the development and propagation of oscillations. The related analysis appeals to observations of the solution structure using nonsingular measures in an effort to analyze the basic fluctuations in terms of averaged quantities and their correlations.

The geometric theory for systems (2) stems from three seminal articles in the period 1950–1970: the article by Lax [**26**] containing the construction of the fundamental solution with small data via elementary waves, the article by Glimm [**17**] on global existence of solutions with small data and the monograph of Glimm and Lax [**18**] on generalized characteristic curves and large time decay of solutions with small data. It has developed in several stages since the beginnings in the early fifties. The book under review is divided into four sections, the third of which is concerned with the subject of conservation laws and deals primarily with the geometric theory in one space dimension as it stood prior to the fundamental monograph of Glimm and Lax [**18**] in the late sixties. In the paragraphs below, several references are provided to survey articles and lecture notes that discuss some of the developments that have taken place since 1970 and topics of current research interest.

The functional analytic theory for systems (2) arises from basic work in the middle and late seventies by Tartar and Murat on nonlinear functional analysis [**36, 41, 42**], specifically on the theory of compensated compactness. We shall comment on just a few aspects of each branch, beginning with the geometric theory.

1. Geometric theory. The geometry theory for systems of conservation laws (2) addresses itself to the general problem of determining how local laws for wave interaction influence global qualitative behavior, specifically the formation and propagation of coherent waves and the large time decay to equilibrium. The laws for weak wave interactions were derived in the form of an asymptotic expansion in the fundamental paper of Glimm [**17**] which initiated the geometric theory for systems in one space dimension. These laws of interaction asymptotically characterize conservative motion at the local level in one space dimension: in each channel, the magnitudes of elementary waves are conserved up to linear terms; the deviation from linearity is measured (modulo cubic terms) by a reduced section of the general quadratic form associated with the Taylor expansion that corresponds to the physically approaching waves.

The structure of the linear and quadratic terms of the expansion expresses several of the basic physical stabilizing mechanisms for the wave interaction process, for example, the cancellation of elementary waves of different signature in the same channel during interaction and the recession of waves of arbitrary signature in different channels after interaction. Knowledge of the expansion up to cubic terms and of the elementary group structure expressed by translation and dilation invariance serves as the starting point for the study of regularity and asymptotic behavior.

2. Existence. Many of the developments in the geometric theory for systems (2) have involved the random choice method introduced by Glimm [**17**] in 1965. The random choice method has provided the main tool for the construction and analysis of solutions: it generates solutions of the Cauchy problem as limits of finite difference approximations and presents a detailed picture of wave interactions which is instrumental in the study of singularities and asymptotic behavior. The local component of this semianalytical method consists of the Lax solution of the Riemann problem [**26**] and is employed in a time-marching fashion to generate an approximate solution from piecewise constant initial data.

In [17] Glimm established global existence of solutions to the Cauchy problem with small data by deriving uniform bounds on the amplitude and distributional derivatives of the random choice approximations $u(x, t; \Delta x)$ measured in the spatial L^∞ norm and total variation norm respectively:

$$|u(\cdot, t; \Delta x)|_\infty \leqslant \text{const}|u_0|_\infty, \tag{3}$$

$$\text{TV}\, u(\cdot, t; \Delta x) \leqslant \text{const}\,\text{TV}\, u_0, \tag{4}$$

where the constants are independent of the data u_0 and the mesh length Δx. A classical compactness argument yields a subsequence that converges to a globally defined distributional solution $u = u(x, t)$ whose spatial L^∞ and TV norms are bounded uniformly in time:

$$|u(\cdot, t)|_\infty \leqslant \text{const}|u_0|_\infty, \tag{5}$$

$$\text{TV}\, u(\cdot, t) \leqslant \text{const}\,\text{TV}\, u_0. \tag{6}$$

The central chapter in the third section of the book under review discusses the proof of the stability bounds (3)–(4) for the random choice approximations, a basic topic in the geometric theory for systems of equations. In connection with the exposition, it appears to be appropriate to comment at some length on the technical analysis and underlying physics.

From the physical point of view, the spatial total variation norm of the solution u at time t measures the sum of magnitudes of all elementary waves in u at time t: all shock waves, rarefaction waves and compression waves. The basic difficulty in proving uniform stability bounds of the type (5)–(6) arises from the fact that many processes of local wave interaction increase individual wave magnitudes and thus eliminate the possibility of simple maximum principles and Gronwall inequalities for the solution operator in the standard metrics defining the strong topology. In [17] Glimm identified two primary mechanisms that compensate for local wave amplification and induce temporary uniform bounds of the form (5)–(6), namely, wave cancellation during interaction and wave recession after interaction. The cancellation process for waves in the same channel leads to the absorption of small waves by large waves and a corresponding reduction in the contribution to the total variation norm. This process is reflected in the structure of the linear terms of the laws for wave interaction. The recession process for waves in different channels guarantees that the cumulative effect of local amplification is finite: primary weak waves radiate quadratically small secondary waves, which in turn generate cubically small third generation waves, etc. This process is reflected in the structure of the quadratic terms of the laws for wave interaction.

The stabilizing effects of both compensating mechanisms are captured analytically by a nonlocal quadratic functional Q introduced by Glimm [17]. The functional Q is defined on random choice approximations and measures the potential that a global wave configuration in $u(\cdot, t; \Delta x)$ has, at time t, for interactions in the future. The functional Qu is decreasing with time and compensates for the growth in TV u (during the initial and intermediate stages of the evolution) to the extent that the sum

$$F(t) = \mathrm{TV}\, u(\cdot, t, \Delta x) + Qu(\cdot, t, \Delta x)$$

is decreasing with time, if the total variation of the initial data is sufficiently small. Local weak wave interactions produce two simultaneous effects: one is to alter individual wave magnitudes, the other is to reduce the potential for future interaction by an amount that exceeds the local amplification. The form and analysis of the Glimm functional F are motivated by the derivation and structure of the laws for weak wave interactions.

Unfortunately, most of the intuition behind the stability mechanisms associated with the functionals F and Q is dispersed throughout the research literature of the last eighteen years. A thorough discussion, accessible to a general audience, remains to be organized in a fashion that coordinates the technical proofs with the underlying physics and includes both a derivation of the weak local laws and an overview of their impact for the global behavior of the solution.

The potential Q plays a central role in analyzing the qualitative behavior of the solution. The ultimate source for the large time decay and asymptotic shape of the solution, as well as the local structure and propagation of singularities, stems from the fact that the total amount of wave interaction in the x-t plane is finite. At the technical level, this fact is expressed by a finite measure Q^* on the x-t plane which is associated with the potential Q and which assigns to each subset the amount of wave interaction occurring therein. The measure Q^* is one of a list of several singular measures introduced by Glimm and Lax [**18**] for the purpose of organizing the special features of the global solution structure relevant to the large time decay to equilibrium. Extensions and refinements of the techniques associated with these measures serve as the starting point for the work in the seventies on singularities and large time asymptotic behavior.

3. **Qualitative behavior.** Among the various avenues of introduction to the subject of conservation laws in one space dimension, one may reasonably focus attention on one of the three main temporal stages in the evolution of the solution from smooth data: The initial stage of wave formation and interaction, the second stage of developed wave propagation and interaction, and the final stage of asymptotic wave decay. As one representative, we shall briefly discuss the intermediate stage in terms of the geometric structure of the solution, remark on a few results, and reference several articles that may serve as an introduction to the general subject.

In the absence of forcing terms, it is observed under general circumstances that shock waves arise spontaneously out of an initially smooth state, increase and decrease in magnitude in the presence of various interactions, and finally decay at a slow algebraic rate as an equilibrium state is approached at large times. On one hand, compressive discontinuities (shock waves) develop from an initially smooth state. On the other hand, expansive discontinuities are instantly resolved as continuous fronts advancing at sonic speed (rarefaction waves). Both the deregularizing and regularizing phenomena stem from the nonlinear dependence of the wave speeds on the amplitude of medium.

The associated geometric structure of the motion in one space dimension is captured by the random choice method in the following way. The random choice method generates solutions $u(x, t)$ in the space BV of functions with bounded variation in the sense of Tonelli-Cesari: each of the first order distributional derivatives u_x and u_t forms a Borel measure with locally finite total mass. The theory of BV functions **[10, 16, 45]** provides preliminary information on the geometric structure of the solution. The domain of definition of an arbitrary BV function on R^n decomposes into the union of three disjoint sets: (1) a set A of points of approximate continuity in the Lebesgue sense, (2) a rectifiable set J of points of approximate jump discontinuity, and (3) a sparse set I of irregular points having zero $(m - 1)$-dimensional Hausdorff measure. The set A is reminiscent of the "continuous" regions of flow representing the rarefaction waves and compression waves. The set J recalls the family of shock fronts. The lower-dimensional set I is reminiscent of the set of points of wave interaction: shock collisions, points of shock formation, centers of rarefaction waves, etc.

A basic problem for conservation laws in one space dimension concerns the regularity of weak solutions in BV. Can all of the singularities of the BV category be realized in a BV solution? Does the special conservation structure convert Lebesgue averaged limits to classical pointwise limits? How large is the residual set I? A complete regularity theory for conservation laws in one space dimension has been developed which answers these and related questions for the solutions $u(x, t)$ generated by the random choice method. It turns out that the solution u is substantially more regular than an arbitrary BV function and presents a picture of interacting waves which is not so very far from the classical engineering conception of a piecewise smooth flow consisting of a finite number of shocks moving through a smooth background of rarefaction and compression waves. We refer the reader to **[12]** for a treatment of systems with nondegenerate wave speeds such as isentropic gas dynamics and to T.-P. Liu **[32]** for a treatment of systems with linearly degenerate wave speed such as nonisentropic gas dynamics. Both analyses employ techniques from the monograph of Glimm and Lax **[18]** which develops a theory of generalized characteristics for solutions constructed by the random choice method.

Several remarks are in order regarding historical background. The space BV of functions of bounded variation of several variables first appeared in the theory of conservation laws in an article by E. Conway and J. Smoller **[3]** which established existence of solutions to scalar equations in several space dimensions using the Lax-Friedrichs finite difference scheme and a BV compactness argument. The earliest variational estimates were derived by Oleinik in the form of bounds on the spatial total variation for solutions to scalar conservation laws in one space dimension; see **[31]** for example. The geometric structure of BV functions was introduced into the theory by Vol′pert **[45]**. Overall, the technical tools provided by the BV theory have played an important role in studying uniqueness, stability and asymptotic behavior **[11, 14, 45]**.

We remark that it remains an open problem to develop an a priori theory of regularity for systems of equations. It is only in the setting of a scalar equation with a convex flux,

$$u_t + f(u)_x = 0, \qquad f'' > 0, \tag{7}$$

that an a priori analysis of the solution has been carried out. A definitive treatment of regularity and decay for (7) is contained in the work of Dafermos [**6, 9**] on generalized characteristics. As an introduction to the geometrical side of the subject with appropriate reference to the basic physics, we would like to recommend the article by Lax [**28**] on the formation and decay of shock waves in solutions to scalar equations, the article by Dafermos on generalized characteristics and the C.I.M.E. lecture notes by Majda [**35**] on systems in several space dimensions.

The lecture notes of Majda provide a stimulating introduction to the theory of conservation laws in one and in several space dimensions, with special attention to the underlying mechanics. Chapter I discusses the basic structural hypothesis for systems of equations and the role which primitive scalar laws, such as Burger's equation [**22**],

$$u_t + (u^2/2)_x = 0,$$

play as quantitative models for special solutions to systems of equations. Chapter II discusses local existence and continuation of smooth solutions. Chapter III discusses shock wave formation. Chapter IV presents the first rigorous results on existence of solutions to systems in several space dimensions in the form of the small-time construction of stable multidimensional shock fronts. Each of the chapters includes comments on interesting and tractable open problems for future work. A version of the notes will appear in Springer-Verlag's Applied Mathematics Series.

At the present time no comprehensive treatment of the rigorous analytical theory of conservation laws is available even for systems in one space dimension. Certain aspects of the subject are contained in the books by Courant and Friedrichs [**4**], Rozhdestvensky and Yanenko [**40**], A. Jeffrey [**23**] and the book under review by Smoller. The survey articles by Lax [**27**] and Dafermos [**7, 8**] provide an excellent introduction to the subject. The earlier articles by Dafermos [**8**] and Lax [**27**] deal mainly with developments between 1950 and 1970. The article by Dafermos [**7**] contains a discussion of several developments which have occurred since 1970 concerning the entropy condition, uniqueness, regularity and large time behavior [**11, 12, 32, 47**]. A forthcoming monograph of T.-P. Liu will contain a discussion of regularity and asymptotic behavior that uses, among other things, the important wave-partitioning technique which he introduced in [**32**]. The list of references below presents only a small selection of articles connected with the geometric side of the theory of systems in one space dimension. As a small sample from the numerical side dealing with the analysis of high resolution finite difference schemes, we refer the reader to the work of Engquist, Harten, Hyman, Lax, van Leer, Majda, Osher, and Ralston [**15, 21, 33, 34, 44**] and to the references cited therein. Regarding the important developments stemming from the Chinese and Russian schools, we refer to the aforementioned survey articles.

4. Some open problems in the geometric theory. A general problem remains to derive local interaction laws for large amplitude waves in either approximate or exact form and deduce the global consequences. A specific goal is to establish uniform bounds on the L^∞ norm and TV norm of solutions with large data, either by deriving uniform bounds on the random choice approximations or by developing an a priori system of estimates. Is the solution operator bounded in the total variation norm for the systems of mechanics by virtue of their special symmetries in the physical domain and state space? Can a uniform L^∞ and TV estimate be established using just the primary mechanisms of wave cancellation, recession, and spreading? To what extent is it necessary to analyze the global structure of wave configurations in order to describe the qualitative behavior of solutions with large data? The long range goal is, of course, to understand the development, propagation, and interaction of coherent waves in several space dimensions. One of the current goals is to establish global existence of solutions for general systems (2) of conservation laws with large data.

A second general problem concerns the derivation of a priori estimates on the solution in the regimes of small and/or large data. Virtually all of the estimates presently available for solutions to systems (2) have been established from discrete versions on the corresponding random choice approximations and passage to the limit. One approach to the problem of a priori estimates is to introduce an appropriate analogue of the Glimm functional defined on solutions in space $L^\infty \cap \mathrm{BV}$. The associated technical problems in this situation are concerned with the development of an efficient calculus. The hope is that an efficient operational method at the hyperbolic level in the BV category will provide the key to both the special problem of deriving uniform L^∞ and TV bounds for associated parabolic regularizations

$$(8) \qquad \partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 u, \qquad u = u(x, t, \varepsilon), \quad \varepsilon \to 0,$$

and conservative finite difference schemes,

$$(9) \qquad D_t u + D_x f(u) = 0, \qquad u = u(x, t; \Delta x), \quad \Delta x \to 0,$$

and the general problem of treating the fine scale features at alternative levels of description. A basic problem remains to establish convergence of the viscosity method (8) as the diffusion parameter ε vanishes and of classical finite difference schemes as the mesh length Δx vanishes, in the setting of general systems of n equations. We shall remark on some progress in direction below.

Because of the lack of a priori estimates the theory for systems of conservation laws has not yet yielded to treatment using the standard tools of functional analysis, which require some form of a priori control on both the amplitude and derivatives of the solution. In contrast, the theory for scalar equations enjoys a nearly complete set of estimates, based on maximum principles in L^p, and contains on the function analytic side an elegant treatment of existence with nonlinear semigroup theory by M. Crandall [5]. A new functional analytic approach to several problems in conservation laws has recently been developed with the aid of the theory of compensated compactness. The approach requires a priori control only on the amplitude of the solution and involves a study of averaged quantities and the weak topology rather than wave interactions and the strong topology. The theory of compensated compactness originates in the work of L. Tartar and F. Murat [**36, 41, 42**]. With the aid of compensated compactness, the first convergence results for the viscosity method (8) and for classical finite difference schemes (9) have been established in the setting of systems of two equations [**13, 46**]. As a corollary one obtains the first large data existence results for isentropic gas dynamics and elasticity.

5. Functional analytic theory. In the general setting of nonlinear equations, a basic difficulty arises from the fact that nonlinear maps are not continuous in the weak topology, and expresses itself in a variety of ways. With regard to existence problems, for example, one of the classical strategies is to introduce a sequence of approximate solutions and attempt to extract a subsequence that converges in the strong topology. The approximating sequence may be generated in a multitude of ways corresponding to regularizations, discretization, extremization, penalization, etc., but in all cases the classical compactness arguments require some type of uniform control, in pointwise or average form, on both the amplitude and derivatives of the sequence in question, in order to select a subsequence that converges in the strong topology, i.e. in the topology L^1_{loc} or one of its variants.

In many cases, linear and nonlinear, uniform a priori bounds on the amplitude of exact and approxiamte solutions are available through maximum principles and energy estimates. Thus one may usually generate a sequence of approximations and extract a subsequence that converges in the weak topology, i.e. in the sense of distributions. In the nonlinear setting, classical functional analysis requires uniform bounds on all partial derivatives in order to convert weak convergence to strong convergence and pass to the limit in a general nonlinear map. For this reason, the use of the weak topology has been restricted for a long time either to linear problems or to mildly nonlinear problems with monotonic structure. In contrast the theory of compensated compactness developed by Tartar and Murat provides an opportunity to pass to the limit in highly nonlinear problems using only the derivative control presented by the special linear combinations (of nonlinear functions) that define the equations themselves.

It is not our purpose in this brief book review to provide an overview of recent developments in nonlinear p.d.e. stemming from compensated compactness or even to provide an overview of recent results in the theory of conservation laws using compensated compactness. The techniques associated with compensated compactness are relevant to a broad range of problems including hyperbolic conservation laws. The subject was in fact stimulated by problems in homogenization and by work in nonlinear elasticity by J. Ball. With regard to a discussion of the physical motivation, theory and application we refer the reader to [**13, 36, 41, 42, 46**].

The book under review is divided into four parts entitled linear theory, reaction-diffusion equations, the theory of shock waves and the Conley index.

The author suggests in the introduction that the material of each section is suitable for presentation in a one-semester graduate course on partial differential equations. Section three concerns the theory of conservation laws and contains an exposition of selected results in the geometric theory of conservation laws in one space dimension. The author presents the early treatments of uniqueness and asymptotic behavior of solutions to a scalar conservation law in one space dimension that were available prior to the definitive work of Vol′pert [**45**] and Kruzkov [**25**] on uniqueness of a scalar equation in the early seventies, and the definitive work of Dafermos [**7**] on regularity and decay for a scalar equation in the late seventies. In the setting of systems of equations in one space dimension, the author presents the constructive argument of Lax [**26**] on the fundamental solution via elementary waves and discusses weak wave interactions for general systems of n equations and strong wave interactions for special systems of two equations. These topics provide background for the author's discussions of the random choice method and for some of the work carried out since 1968 on large data existence for special systems of two conservation laws by H. Bakhrarov, J. Greenberg, T.-P. Liu, T. Nishida, J. Smoller, B. Temple and others [**1, 19, 20, 29, 31, 37, 38, 43**]. This line of research stems from the stimulating paper by Nishida [**37**] on large data existence for the equations of isothermal gas dynamics using the random choice method.

6. Reaction-diffusion equations. The mathematical theory of reaction-diffusion processes, as represented by semilinear parabolic systems

$$\partial_t u = D\Delta u + f(u),$$

has recently received a great deal of attention stimulated by both formal and rigorous work on a variety of biological, chemical and ecological models. Perhaps the deepest scientific problems in this setting concern the accuracy of the models, in various regimes, as a mathematical description of natural phenomena and require tools in formal perturbation theory and numerical analysis for their solution. Part II of the book under review provides a lucid description of several rigorous results in the analytical theory which deal with existence and solution-set-structure and which follow from the o.d.e. viewpoint using maximum principles comparison arguments and bifurcation theory. Part II also contains a discussion of relevant background material on linearization, topological methods, and bifurcation theory. Part IV provides an introduction to the Conley index, a new topological tool. As just one representative achievement of the Conley index we mention the proof of existence of structure for magnetohydrodynamic shock waves [**2**], which is presented in the last chapter of Part IV dealing with existence of traveling waves.

As a whole, the book contains a wide variety of material, and a rather extended report would be required to provide the appropriate historical remarks for all of the special topics which are discussed together with the references to alternative presentations and points of view, recent work, etc. In summary, the book by J. Smoller contains a clear and well-written account of several fundamental topics and is a very valuable contribution to the expository literature on shock waves and reaction-diffusion equations.

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A concrete approach to division rings, by John Dauns, Research and Education in Mathematics, Vol. 2, Heldermann, Verlag, Berlin, 1982, xx + 417 pp., $78.00. ISBN 3-8853-8202-4

Skew fields, by P. X. Draxl, London Mathematical Society Lecture Note Series 81, Cambridge University Press, New York, 1982, 182 pp., $19.95. ISBN 0-5212-7274-2

In 1843 Hamilton set down the generators and relations for the algebra of quaternions. This was the first nontrivial division algebra or skew field. That is,