BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 6, Number 1, January 1982

ON THE OSCILLATION THEORY OF $f'' + Af = 0$ WHERE A IS ENTIRE[1]

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This paper is concerned with the distribution of zeros of solutions of differential equations of the form,

$$f'' + A(z)f = 0, \tag{1}$$

where $A(z)$ is an entire function. More specifically, we are concerned with determining the exponent of convergence (which we will denote by $\lambda(f)$) of the zero-sequence of a solution $f \not\equiv 0$ of (1). In this paper new techniques are developed which yield strong, new results concerning $\lambda(f)$.

We first consider the case where $A(z)$ is a nonconstant polynomial.

THEOREM 1. *Let $A(z)$ be a nonconstant polynomial of degree n, and let f_1 and f_2 be two linearly independent solutions of* (1). *Then, at least one of f_1, f_2 has the property that the exponent of convergence of its zero-sequence is* $(n + 2)/2$.

REMARK 1. If n is odd, all solutions $f \not\equiv 0$ of (1) have the property that $\lambda(f) = (n + 2)/2$ because the Wiman-Valiron theory (see [**5**, Chapter 4] or [**6**, Chapter 1] or [**7**, p. 281]) shows that the order of growth of f is $(n + 2)/2$ which is not a positive integer if n is odd.

REMARK 2. In the case when n is even, there may exist solutions of (1) having no zeros. (E.g., if $k = (n + 2)/2$, then $f_1(z) = \exp(z^k/k)$ satisfies $f'' - (z^n + (n/2)z^{(n-2)/2})f = 0$.) On the other hand, there are examples where all solutions $f \not\equiv 0$ satisfy $\lambda(f) = (n + 2)/2$. For example, the equation $f'' - z^nf = 0$ can be shown to have this property using the Wiman-Valiron theory.

REMARK 3. In the case when n is even, and f is a solution of (1) satisfying $\lambda(f) < (n + 2)/2$, it can be shown using a theorem of Pöschl (see [**6**, p. 70]) that f can have only finitely many zeros.

Received by the editors July 14, 1981.

1980 *Mathematics Subject Classification*. Primary 34A20, 30D35; Secondary 34C10, 34A30.

[1]The work of both authors was supported in part by the National Science Foundation (MCS 78-02188 and MCS 80-02269). The work of the second author was also supported in part by a research grant from the Finnish Academy.

0002-9904/82/0000-0146/$02.00

PROOF OF THEOREM 1. If we set $E = f_1 f_2$, then a straightforward calculation shows that E satisfies the relation,

$$E^2 = c^2/((E'/E)^2 - 2(E''/E) - 4A), \tag{2}$$

where the nonzero constant c is the Wronskian of f_1 and f_2. The relation (2) shows that E cannot be a polynomial since $A(z) \longrightarrow \infty$ as $z \longrightarrow \infty$. From the Nevanlinna theory (see [3]) it now follows from (2) that the Nevanlinna characteristic $T(r, E)$ of E satisfies an estimate of the form,

$$T(r, E) = O(\overline{N}(r, 1/E) + T(r, A) + \log r), \tag{3}$$

as $r \longrightarrow \infty$ outside a set of finite measure, where $\overline{N}(r, 1/E)$ denotes the counting function for the distinct zeros of E. However, relation (2) can be rewritten,

$$c^2 - (E')^2 + 2EE'' + 4AE^2 = 0, \tag{4}$$

and when the Wiman-Valiron theory is applied to (4), we see that the order of growth of E is $(n + 2)/2$. From (3), it now follows that $\lambda(E) = (n + 2)/2$ which proves Theorem 1 since $E = f_1 f_2$.

In the case when $A(z)$ is transcendental, the situation concerning the zeros of solutions of (1) can be far different than in the polynomial case. It is possible for (1) to possess two linearly independent solutions each having no zeros. To prove this, let $\varphi(z)$ be any nonconstant entire function, and let h denote a primitive of e^{φ}. Set $g = -(\varphi + h)/2$. Then $f_1 = e^g$ and $f_2 = e^{g+h}$ are linearly independent solutions of (1) where $A = -((h')^2 + (\varphi')^2 - 2\varphi'')/4$.

However, a result in the positive direction is the following:

THEOREM 2. *Let $A(z)$ be an entire transcendental function of finite order σ, where σ is not a positive integer. Let f_1 and f_2 be two linearly independent solutions of* (1). *Then, if $\sigma \geqslant 1/2$, at least one of f_1, f_2 has the property that the exponent of convergence of its zero-sequence is at least σ. If $\sigma < 1/2$, at least one of f_1, f_2 has the property that the exponent of convergence of its zero-sequence is ∞.*

PROOF OF THEOREM 2. We again set $E = f_1 f_2$ so (2)–(4) hold. We consider first the case $\sigma \geqslant 1/2$. If we assume $\lambda(E) < \sigma$, then from (3) we see that the order of E is at most σ. However, solving (4) for A, it follows that the order of E is also at least σ, and so E is of order σ. Since σ is not a positive integer, we have $\lambda(E) = \sigma$ contradicting our assumption. Thus $\lambda(E) \geqslant \sigma$, and the first part is proved. For the case $\sigma < 1/2$, we apply the Wiman-Valiron theory to (4). Hence there is a set D in $[1, \infty)$ of finite logarithmic measure such that if $r \notin D$ and z is a point on $|z| = r$ at which $|E(z)| = M(r, E)$ (where $M(r, E)$ is the maximum modulus of E), then

$$2|A(z)| \leqslant (\nu(r)/r)^2, \tag{5}$$

where $v(r)$ denotes the central index of E (i.e., if $E(z) = \Sigma_{n=0}^{\infty} a_n z^n$, and $m(r) = \max\{|a_n|r^n : n = 0, 1, \ldots\}$, then $v(r)$ is the largest index n for which $|a_n|r^n = m(r)$). However, since $\sigma < \frac{1}{2}$, it follows from a theorem of P. Barry [2, p. 294] that there is a sequence $\{r_n\} \longrightarrow \infty$ such that $r_n \notin D$ and the minimum modulus of $A(z)$ on $|z| = r_n$ is at least $M(r_n, A)^\epsilon$ for some fixed $\epsilon > 0$. In view of (5), it follows that $\{v(r_n)/r_n^\alpha\} \longrightarrow \infty$ as $n \longrightarrow \infty$ for every $\alpha > 0$, and so E is of infinite order (see [5, p. 34]). Hence from (3), we have $\lambda(E) = \infty$ and the theorem follows.

In the case where the order σ of $A(z)$ is either a positive integer or ∞, we obtain the following very strong result if $\lambda(A) < \sigma$.

THEOREM 3. *Let $A(z)$ be an entire function of order σ, where $0 < \sigma \leqslant \infty$, and assume that $\lambda(A) < \sigma$. Then for any solution $f \not\equiv 0$ of* (1), *we have $\lambda(f) \geqslant \sigma$.*

The proof of Theorem 3 is lengthy, and hence we will give only a sketch of it. (However, complete details will appear elsewhere.) We denote the order of a meromorphic function F by $\sigma(F)$. We assume that the conclusion of Theorem 3 fails to hold, so that $\lambda(f) < \sigma$. We can write $f = Qe^g$, where g is entire, and Q is a canonical product for which $\sigma(Q) < \sigma$. From (1) we have

$$Q'' + 2Q'g' + Q(g')^2 + Qg'' = -AQ. \tag{6}$$

This shows that $\sigma(g) \geqslant \sigma$. Solving (6) for $(g')^2$, and applying a variant of a lemma of Clunie [1, Lemma 1], we obtain $\sigma(g') \leqslant \sigma$ so $\sigma(g) = \sigma$. We now set $b = \max\{\lambda(f), \lambda(A)\}$ so that $b < \sigma$. We assume first that $\sigma < \infty$, and we solve (6) for $A(z)$. Applying a simple modification of a result of Clunie [3, p. 69, Theorem 3.9], it follows that $-A = (g' + \alpha)^2$ where α is an entire function of order at most b. Setting $H = 1/(g' + \alpha)$, we obtain an equation,

$$FH^2 = H' - 2((Q'/Q) - \alpha)H, \tag{7}$$

where F is a meromorphic function of order at most b, which is not identically zero. Again applying a variant of a result of Clunie [3, p. 68, Lemma 3.3], it follows that for any $\epsilon > 0$, we have as $r \longrightarrow \infty$, $m(r, H) = O(r^{b+\epsilon})$, and thus $m(r, 1/A) = O(r^{b+\epsilon})$. Since $\lambda(A) \leqslant b$, we obtain $\sigma(A) \leqslant b$ which contradicts the fact that $b < \sigma = \sigma(A)$. In the case $\sigma = \infty$, we obtain a similar contradiction.

The result of Theorem 3 allows us to obtain the following quantitative improvement of a result of W. Hayman [4, Theorem 8].

THEOREM 4. *Let f be an entire transcendental function of order σ, where $0 < \sigma \leqslant \infty$, and let $\varphi = f' - cf^2$ where c is a nonzero constant. Then $\lambda(\varphi) = \sigma$. (Hayman showed that φ has infinitely many zeros.)*

The proof of Theorem 4 proceeds as follows. We first show that $\sigma(\varphi) = \sigma$ by using variants of the Tumura-Clunie theory. We then set $y = e^{-cg}$, where g

is a primitive of f. Then y solves equation (1), where $A = c\varphi$. Since y has no zeros, it follows from Theorem 3 that we must have $\lambda(A) = \sigma(A)$, and so $\lambda(\varphi) = \sigma$.

We briefly mention two related results in the case when $A(z)$ is transcendental. First, examples can be constructed to show that the case where (1) has a solution having no zeros can occur for any order of $A(z)$. Secondly, the conclusion of the first part of Theorem 2 holds even for σ being a positive integer or ∞ provided that the exponent of convergence of the sequence of distinct zeros of $A(z)$ is less than σ.

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