BIFURCATION AND SYMMETRY BREAKING
IN APPLIED MATHEMATICS

BY D. H. SATTINGER[1]

I. Introduction. Bifurcation theory is the study of branch points in nonlinear equations, that is, of singular points of the equations where several solutions come together. It is important in applications because bifurcation phenomena typically accompany the transition to instability when a characteristic parameter passes through a critical value. To state the situation more precisely, suppose the states of a physical system are determined as solutions of a functional equation

$$G(\lambda, u) = 0 \tag{1}$$

where $\lambda \in \Lambda$ is a parameter, u is an element of a Banach space $\mathcal{E}$, and G is a mapping from $\Lambda \times \mathcal{E}$ to another Banach space $\mathcal{F}$. Let $\mathcal{S}$ be the zero set of G in $\Lambda \times \mathcal{E}$ and suppose γ is a smooth curve in $\mathcal{S}$. Then a branch point (λ_c, u_c) is a point of γ such that for any neighborhood of U of (λ_c, u_c) in $\Lambda \times \mathcal{E}$, $(U \setminus \gamma) \cap \mathcal{S} \neq \varnothing$. Some typical "bifurcation diagrams" are pictured schematically in Figure 1.
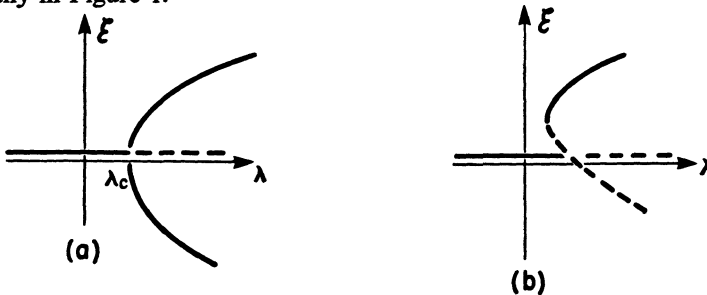

FIGURE 1.
Schematic diagram of bifurcation at $(\lambda_c, 0)$; one nontrivial
branch. The vertical axis represents a Banach space $\mathcal{E}$.

Closely tied to the phenomenon of bifurcation is the property of *stability*. Suppose the solutions of (1) represent equilibrium solutions for a dynamical system which evolves according to the time dependent equations $u_t = G(\lambda, u)$. An equilibrium solution u_0 is stable if small perturbations from it remain close to u_0 as $t \to \infty$; u_0 is asymptotically stable if small perturbations decay to zero in time. When the parameter λ is varied one solution may persist but become unstable as λ crosses a critical value λ_c, and it is at such a transition point that new solutions may bifurcate from the known solution. In Figure 1 unstable solutions are represented by dashed lines.

A very simple nonlinear partial differential equation which exhibits the

transition phenomena depicted in Figure 1(a) is

$$\frac{\partial u}{\partial t} = \Delta u + \lambda u + u^3, \qquad u|_{\partial D} = 0$$

where D is a smoothly bounded domain in $\mathbf{R}^n$. The equilibrium states are given by solutions of the time independent equations (set $\partial u/\partial t = 0$). One solution valid for all λ is clearly $u = 0$; this solution becomes unstable at $\lambda = \lambda_1$, the first eigenvalue of the Laplacian on D: $\Delta \varphi_1 + \lambda_1 \varphi_1 = 0$, $\varphi_1|_{\partial D} = 0$. For $\lambda > \lambda_1$ there are (at least) three solutions of the nonlinear equilibrium equation. The structure of the solution set in the vicinity of $(\lambda_1, 0)$ is given in Figure 1(a); the new bifurcating solutions are stable.

The Laplacian has a series of eigenvalues $\lambda_1 < \lambda_2 < \ldots$ tending to infinity, and all of these eigenvalues are potential bifurcation points. The investigation of the branch points $(\lambda_j, 0)$ for $j > 1$ is, however, largely academic, since all of these later solutions must be unstable.

The area of fluid mechanics is a rich source of instability and bifurcation phenomena and the subject has always stimulated the development of mathematical analysis. It has the advantage that accurate, tractable mathematical models are known (for example, the Navier-Stokes equations), for which many careful experimental studies have been made.

One problem in fluid mechanics which has attracted much interest in recent years is the so-called Bénard problem, named after H. Bénard who first performed his series of experiments at the turn of the century. In these experiments a layer of fluid is heated from below, causing an instability to develop in the fluid layer when the temperature drop across the layer exceeds a certain critical value. The resulting instability takes the form of convective motions in the fluid. One of the most striking features of the experiment is the regular cellular, in fact crystallographic structure exhibited by these fluid motions.
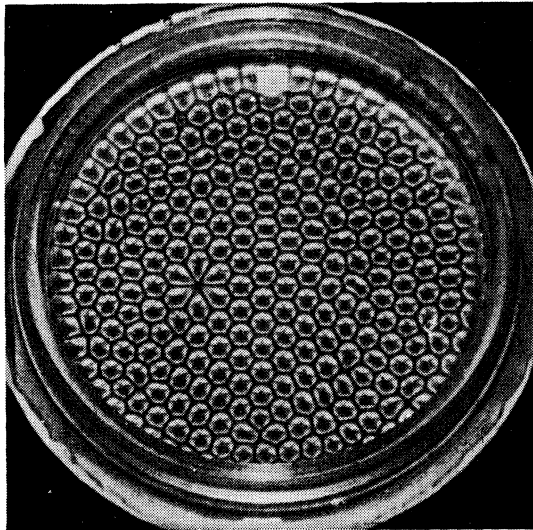

FIGURE 2

Hexagonal Cells in the Bénard problem; E. L. Koschmieder [49]

The formation of convection cells in the Bénard problem furnishes an excellent example of what is called a "symmetry breaking instability". Prior to the onset of instability the solution is invariant under the entire group of rigid motions (in the idealized infinite plane layer model; see §III.4) whereas the bifurcating convective motions are invariant only under a crystallographic subgroup. Symmetry is broken "spontaneously", because the symmetry group of the equations is unchanged, while the bifurcating solutions have a smaller symmetry group.

The appearance of the Bénard cells is not an isolated example, but rather typifies a broad range of phenomena in nature. Bifurcation, and symmetry breaking instabilities in particular, plays an important role in a diversity of physical disciplines, including buckling problems in elasticity, pattern formation in reaction-diffusion problems, convective flows in geophysical phenomena, neurobiology, physical chemistry, and statistical physics, to name a few. The formation of order in dissipative structures is a subject currently of interest not only to physicists, but also to biologists endeavoring to explain the mechanisms of morphogenesis and pattern formation in evolving biological organisms. The thesis that order-disorder transitions in biological structures can be modeled by systems of partial differential equations describing the processes of reaction and diffusion was advanced by A. M. Turing in 1952 [96] and is today the basis of much of the work in theoretical biology and morphogenesis. (See Fife [19].)

I should like to present, in this article, a summary of the principal mathematical techniques of bifurcation theory, an account of some of the physical problems in which bifurcation and symmetry breaking play an important role, and some open mathematical problems whose resolution I believe is germane to future significant progress. The principal tool for dealing with symmetry breaking bifurcations is group representation theory; this aspect of the subject will be discussed in detail in Chapter III.

II. Basic techniques of bifurcation theory.

1. *Nonlinear functional calculus.* A nonlinear mapping F from a Banach space $\mathscr{E}$ to $\mathscr{F}$ is said to be *Fréchet* differentiable at a point u provided there is a bounded linear operator A from $\mathscr{E}$ to $\mathscr{F}$ such that the quantity $R(u; h) = F(u + h) - F(u) - Ah$ is $o(h)$ as $h \to 0$; that is

$$\lim_{\|h\| \to 0} \frac{\|R(u; h)\|}{\|h\|} = 0.$$

We denote the Fréchet derivative of F at u by $F'(u)$ or by F_u; when it exists it may be computed by the usual formula

$$F'(u)h = \lim_{t \to 0} \frac{F(u + th) - F(u)}{t}.$$

The functional calculus familiar in the theory of finite-dimensional mappings from $\mathbf{R}^n$ to $\mathbf{R}^m$ carries over directly to the infinite-dimensional case with no difficulty. The chain rule, Taylor's theorem, and the implicit function theorem in a Banach space are all valid, and are extremely useful in nonlinear

analysis. The following version of the implicit function theorem is adequate for most applications:

THEOREM 1: IMPLICIT FUNCTION THEOREM. *Let* Λ, $\mathfrak{E}$, $\mathfrak{F}$ *be Banach spaces and let G be a Fréchet differentiable mapping from a domain $U \subset \Lambda \times \mathfrak{E}$ to $\mathfrak{F}$. Suppose $G(\lambda_0, u_0) = 0$ and $G_u(\lambda_0, u_0)$ is an isomorphism from $\mathfrak{E}$ to $\mathfrak{F}$. Then locally, for $|\lambda - \lambda_0|$ sufficiently small, there is a differentiable mapping $u(\lambda)$ from Λ to $\mathfrak{E}$, with $(\lambda, u(\lambda)) \subset U$, such that $G(\lambda, u(\lambda)) = 0$. Furthermore, in a sufficiently small neighborhood $U' \subset U$, $(\lambda, u(\lambda))$ is the only solution of $G = 0$. If G is C^k then u is C^k. If Λ, $\mathfrak{E}$, and $\mathfrak{F}$ are complex Banach spaces and G is Fréchet differentiable, then G is analytic and u is analytic in λ.*

This theorem is proved by a contraction mapping argument. See Dieudonné [16] and also [82] for some details of the analytic case in infinite dimensions.

From Theorem 1 it follows that if G vanishes at (λ_0, u_0) and G_u is invertible there, then there is locally a smooth curve of solutions $u(\lambda)$ through (λ_0, u_0). Bifurcation may occur, however, when $G_u(\lambda_0, u_0)$ is not invertible. In this article we restrict ourselves to problems in which G_u is a Fredholm operator of index zero–that is, G_u has a closed range $\mathfrak{R} \subset \mathfrak{F}$ and finite-dimensional kernel $\mathfrak{N}$, with dim $\mathfrak{N}$ = codim $\mathfrak{R}$. The assumption that G_u is a Fredholm operator is satisfied in many applied problems where G is typically an elliptic system or a completely continuous integral operator. In some cases, for example when working on unbounded domains, it is the choice of function space which determines whether G_u is Fredholm or not; for example, by imposing some kind of periodicity conditions it may be possible to ensure that G_u be Fredholm.

2. *Principle of linearized stability*. Suppose now that the dynamics of a physical system are governed by the evolution equations

$$\frac{\partial u}{\partial t} = G(\lambda, u). \tag{2}$$

Let $U(t; f)$ denote the solution of this equation with initial data $U(0; f) = f$, and suppose that $u_0 = u_0(\lambda)$ is an equilibrium solution, that is, $G(\lambda, u_0(\lambda)) = 0$. The equilibrium u_0 is stable if given any $t > 0$ there is a $\delta > 0$ such that $\|u(t; f) - u_0\| < t$ for all $t > 0$ whenever $\|f - u_0\| < \delta$. Furthermore, u_0 is asymptotically stable if in addition $u(t) \to u_0$ as $t \to \infty$. If (2) is a system of ordinary differential equations, that is, if the Banach space $\mathfrak{E}$ is finite dimensional, then Lyapounov's first theorem states that u_0 is asymptotically stable if all eigenvalues of the $G_u(\lambda, u_0)$ have negative real parts, and u_0 is unstable if some eigenvalues of $G_u(\lambda, u_0)$ have positive real parts.

Lyapounov's theorem has been extended to dissipative systems of nonlinear partial differential equations; for example, to parabolic systems and to the Navier-Stokes equations, which govern the dynamics of a viscous incompressible fluid [25], [35], [43], [47], [68], [79]. By the principle of linearized stability we mean that the stability of an equilibrium solution u_0 is determined formally by the spectrum of the linearized operator $G_u(\lambda, u_0)$. The phrase "linearized stability" refers to the fact that the full nonlinear equations for the perturbations are replaced by the linearized equations $v_t = G_u(\lambda, u_0)v$, and it

is assumed that the stability of these linearized equations determines that for the full nonlinear equations. This principle is generally accepted as valid in the applied literature, and stability is determined formally by solving the linear eigenvalue problem $G_u(\lambda, u_0)\varphi = \sigma\varphi$.

Now suppose we have a known solution $u(\lambda)$ of the equilibrium equations $G(\lambda, u(\lambda)) = 0$ and let $L(\lambda) = G_u(\lambda, u(\lambda))$. Suppose that as λ crosses λ_0 one or more eigenvalues of $L(\lambda)$ cross the imaginary axis from the left to the right-half plane. This is precisely the situation when $u(\lambda)$ becomes unstable. Then $L(\lambda_0)$ has eigenvalues on the imaginary axis. If some of these eigenvalues lie on the origin, then $(\lambda_0, u(\lambda_0))$ is a possible bifurcation point, for in this case the implicit function theorem can no longer be invoked to guarantee the existence of a unique solution curve through $(\lambda_0, u(\lambda_0))$.

3. *Bifurcation at a simple eigenvalue.* When $u(\lambda)$ loses stability by virtue of a simple eigenvalue crossing the origin a fairly general result is available.

THEOREM 2. *Let $G(\lambda, u)$ be an analytic mapping and suppose there is a known solution $u(\lambda)$ of (1) which becomes unstable by virtue of a simple isolated eigenvalue $\sigma(\lambda)$ of $G_u(\lambda, u)$ crossing the origin as λ crosses λ_0, and let G_u be a Fredholm operator. Assume $\sigma(\lambda_0) = 0$ and $\sigma'(\lambda_0) > 0$. Then there is a smooth solution curve $(\lambda(\varepsilon), u(\varepsilon))$ bifurcating from the given solution $u(\lambda)$ at $(\lambda_0, u(\lambda_0))$. The bifurcating solutions are stable when they appear supercritically $(\lambda > \lambda_0)$ and unstable subcritically $(\lambda < \lambda_0)$.*

The three possible situations are depicted below in Figure 3.

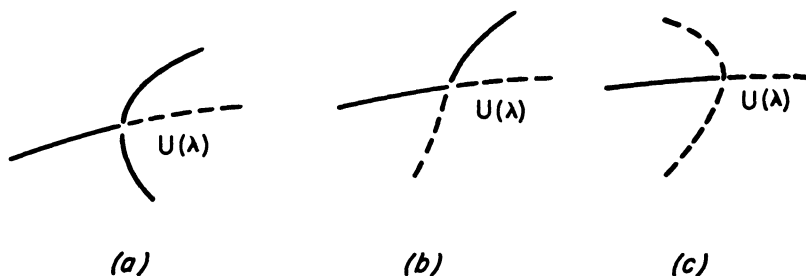(a)   (b)   (c)

FIGURE 3.

Bifurcation at a simple eigenvalue, dashed lines denote unstable branches.

Theorem 2 was first stated for ordinary differential equations by E. Hopf in his 1942 article [28], but with certain restrictive assumptions on $\lambda(\varepsilon)$, namely that $\lambda'(0)$ and $\lambda''(0)$ are not both zero. The indicated method was to compute the perturbation series for the critical eigenvalue along the bifurcating branch. The general infinite-dimensional result, without the restrictive assumptions on $\lambda(\varepsilon)$, was proved using a topological degree argument [80]. (See also [14] for a perturbative proof.) H. Weinberger [100] has generalized these results to the case where $\sigma'(\lambda_0) = 0$. In that case the bifurcation consists in general of several smooth curves intersecting in a point, with the stability of each branch alternating as one goes consecutively from branch to branch. Stability results based on topological degree arguments have also been obtained by T. B. Benjamin [4].

4. *Bifurcation of periodic solutions.* If $u(\lambda)$ loses stability by virtue of a pair of complex conjugate eigenvalues crossing the imaginary axis then, under suitable but reasonable technical conditions, one may prove the existence of bifurcating time periodic solutions of the evolution equation $u_t = G(\lambda, u)$. Let $L(\lambda) = G_u(\lambda, u(\lambda))$ and suppose $\gamma(\lambda)$ is the critical eigenvalue of $L(\lambda)$; thus $L(\lambda)\varphi(\lambda) = \gamma(\lambda)\varphi(\lambda)$. For real equations (that is, if G preserves the space of $\mathfrak{S}$ of real functions) $L(\lambda)$ computes with complex conjugation, so $\bar{\gamma}(\lambda)$ is also an eigenvalue with eigenvector $\bar{\varphi}(\lambda)$. We have

THEOREM 3. *Let $G(\lambda, u)$ be real analytic and suppose that $L(\lambda)$ generates an analytic semigroup. Suppose furthermore that $\gamma(\lambda_c) = i\omega_0$, $\mathrm{Re}\, \gamma'(\lambda_c) > 0$, and $ni\omega_0$ is not in the spectrum of $L(\lambda_c)$ for $n = 2, 3, \ldots$. Then there exists a one-parameter analytic family $(\lambda(\varepsilon), \omega(\varepsilon), u(s, \varepsilon))$ such that*
   (i) $\lambda(0) = \lambda_c$, $\omega(0) = \omega_0$, $u(s, 0) = u(\lambda_c)$,
   (ii) $u(s, \varepsilon)$ *is 2π-periodic in s,*
   (iii) $\lambda(\varepsilon)$, $u(\omega(\varepsilon)t, \varepsilon)$ *is a solution of the time dependent equations.*
   This bifurcating time periodic family is unique up to phase shifts and occurs on one side of criticality only. If $\lambda''(0) \neq 0$, the branching solutions are stable if they appear supercritically and unstable if they appear subcritically.

E. Hopf first proved the result in this generality for the finite-dimensional case in 1942 [28] and also discussed the possibility of bifurcation of time periodic solutions for the Navier-Stokes equations. The first proofs of the theorem for the Navier-Stokes equations were given by Iudovic [38], Iooss [31], and myself [81], and an improved version was subsequently obtained by D. Joseph and myself [42]. A careful treatment of the problem, under fairly general hypotheses, has been given in a recent paper by Crandall and Rabinowitz [15]. The phenomenon of time periodic oscillations is important not only in fluid mechanics, but also in any dynamical system; for example in reaction-diffusion problems in chemical kinetics, in biological and ecological systems, and lasers [23], [26] to name a few. An alternative proof of the Hopf bifurcation theory, based on the center manifold theorem in a Banach space, has been developed by Marsden and McCracken [58].

The stability statements in Theorem 3 refer to the behavior of the Floquet exponents as one continues along the bifurcating solutions. Rigorous proofs that the Floquet exponents determine the stability for the nonlinear equations have been given by Iooss [31] and Iudovich [37].

5. *Hysteresis.* This section must necessarily take on somewhat of a metaphysical flavor, for I want to discuss a phenomenon which is quite common in applications, but, because it pertains to the structure of the solution set in the large, is often beyond the reach of rigorous mathematical treatment. Consider the diagram in Figure 4. Again, stable solutions are indicated by solid lines and unstable solutions by dashed lines. According to Theorem 2 subcritical solutions bifurcating at a simple eigenvalue are unstable and supercritical solutions are stable. At multiple eigenvalues, however, this need no longer be the case, and a transcritical branch may be unstable on both sides of criticality. Suppose, however, that the subcritical branch has one unstable mode, and that this subcritical branch "bends back" as shown in the figure. Precisely at the point (λ_0, u_0), G_u must have a nontrivial null space.

For, let $(\lambda(\varepsilon), u(\varepsilon))$ be a regular parametrization of the solution curve. Then at (λ_0, u_0), $d\lambda/d\varepsilon = 0$ and $du/d\varepsilon \neq 0$, and differentiating $G(\lambda(\varepsilon), u(\varepsilon))$ along the curve one obtains that $G_u u' = 0$. If dim ker $G_u = 1$ then it can be shown that an eigenvalue of $G_u(\lambda(\varepsilon), u(\varepsilon))$ crosses the origin as we pass through (λ_0, u_0). If the eigenvalue which crosses is the unstable one associated with the lower branch, then the solution will regain stability as we move to the upper branch. If the solution structure is as in Figure 4 then the following effects would be observed. As λ is increased past λ_c the basic solution becomes unstable and the system makes a rapid transition to the new, nontrivial solution, as indicated by the vertical line (a); as λ is decreased the system moves along (b) until the point λ_0 is reached, when it drops back to the basic state (c). This behavior is detected in many physical systems, and in fact the nontrivial branch sometimes extends very deeply into the subcritical region. The effect associated with Figure 4 may be called *hysteresis* in analogy with similar effects observed in ferromagnets, namely the transition path from state A to B is different than that from B to A.
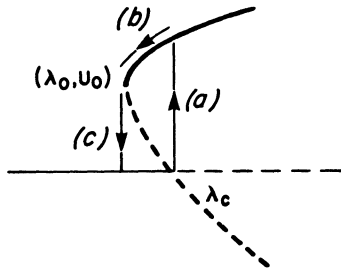

FIGURE 4

6. *Lyapounov-Schmidt procedure.* We now discuss a general method for reducing an infinite-dimensional bifurcation problem to a finite-dimensional, algebraic one. We assume that $L(\lambda) = G_u(\lambda, u(\lambda))$ has, at $\lambda = \lambda_c$, an n-fold eigenvalue at the origin, that is, that dim ker $L(\lambda_c) = n$. For simplicity, let us assume the origin transformed so that $\lambda_c = 0$ and $u(\lambda_c) = 0$, and write $L_0 = G_u(0, 0)$. It is very natural to make the assumption in applications that $\mathscr{E} \subseteq \mathscr{F}$ (for example if G is a second-order elliptic operator then typically $\mathscr{E} = C_{2,\alpha}$ and $\mathscr{F} = C_\alpha$), and this assumption somewhat simplifies the arguments. Let $\mathfrak{N} = $ ker $L_0 = [\varphi_1, \ldots, \varphi_n]$ and let P be the projection onto this linear space which commutes with L_0. Then P must take the form

$$Pu = \sum_{j=1}^{n} \langle u, \varphi_j^* \rangle \varphi_j$$

where the $\varphi_j^* \in \mathscr{F}^* \subset \mathscr{E}^*$ are null functions of the adjoint operator L_0^* and $\langle \varphi_i, \varphi_j^* \rangle = \delta_{ij}$. P is a linear operator from $\mathscr{F}$ to $\mathscr{F}$, hence can be regarded as a mapping from $\mathscr{E}$ to itself as well, and $Q = I - P$ is a projection onto the range of L_0 in $\mathscr{F}$. Using P and Q the equation $G(\lambda, u) = 0$ can be decomposed into the system of equations

$$QG(\lambda, v + \psi) = 0, \qquad PG(\lambda, v + \psi) = 0$$

where $v = Pu$ and $\psi = Qu$. The first equation is solved for $\psi = \psi(\lambda, v)$ by

applying the implicit function theorem, and then this result is substituted into the second equation, thus obtaining the bifurcation equations

$$F(\lambda, v) \equiv PG(\lambda, v + \psi(\lambda, v)) = 0.$$

Solutions of the bifurcation equation are in one-to-one correspondence with solutions of the original system sufficiently close to the bifurcation point.

The procedure we have just described is known as the Lyapounov-Schmidt method; the solution of the infinite-dimensional problem is reduced to an algebraic problem. In addition to finding all solutions of the bifurcation equations we shall want to determine their stability properties. A second method for reducing an infinite-dimensional problem to a finite one at the critical point is by way of the center manifold theorem. This method will be described in §8.

7. *Reduced bifurcation equations.* Having derived the bifurcation mapping F, let us expand this mapping in a power series in v:

$$F(\lambda, v) = A(\lambda)v + B_2(\lambda; v, v) + B_3(\lambda; v, v, v) + \cdots \tag{3}$$

where $A(\lambda)$ is a linear mapping from $\mathfrak{N}$ to $\mathfrak{N}$, B_2 is a quadratic mapping, and so forth. By scaling the variables λ and v as follows

$$v = \varepsilon^\alpha \xi, \qquad \lambda = \varepsilon^\beta \tau$$

with an appropriate choice of the exponents α and β F takes the form

$$F(\lambda, v) = F(\varepsilon^\beta \tau, \varepsilon^\alpha \xi) = \varepsilon^\gamma Q(\xi, \tau) + \varepsilon^{\gamma+1} Q_1(\xi, \tau, \varepsilon).$$

Dividing by ε^γ and letting $\gamma \to 0$ we arrive at the *reduced bifurcation equations* $Q(\xi, \tau) = 0$. The choice of the exponents α and β can be determined from a Newton diagram analysis of F [77], [83], [98]. One would like to obtain as much information as possible from an analysis of the reduced bifurcation equations (3). If (ξ_0, τ_0) is a solution of the reduced bifurcation equations and $Q_\xi(\xi_0, \tau_0)$ is invertible, then a solution to the full equations can be obtained by an application of the implicit function theorem. In fact, this solution takes the form

$$v = \varepsilon^\alpha(\xi_0 + \varepsilon, \xi_1 + \cdots), \qquad \tau = \varepsilon^\beta \tau_0. \tag{4}$$

On the other hand, in applications where the problem is invariant under the action of a continuous symmetry group, Q_ξ will in general be singular, and the standard form of the implicit function theorem can no longer be applied. In particular cases (e.g. the Bénard problem [83]) solutions of the reduced bifurcation equations can be extended to solutions of the full equations, even in the presence of a continuous transformation group, but the general case is an open problem.

8. *Stability of bifurcating solutions.* The stability of the bifurcating solutions can sometimes be determined, in the neighborhood of the branch point, by computing the eigenvalues of the Jacobian $Q_\xi(\tau_0, \xi_0)$. If $(\lambda(\varepsilon), u(\varepsilon))$ is the bifurcating branch let $L(\varepsilon) = G_u(\lambda(\varepsilon), u(\varepsilon))$. $L(0)$ has an n-fold eigenvalue at the origin, and, according to standard perturbation theory for an eigenvalue of finite multiplicity, the spectrum of $L(\varepsilon)$ in the neigborhood of the origin is given by the eigenvalues of an $n \times n$ matrix $B(\varepsilon)$.

THEOREM 4 [84]. $B(\varepsilon) = \varepsilon^{\gamma-\alpha}Q_\xi(\tau_0, \xi_0) + O(\varepsilon^{\gamma-\alpha+1})$ *where v and τ are scaled as in (4).*

In particular, if $F(\lambda, v) = \lambda Av + B_k(v) + O(|\lambda|^2 + \|v\|^{k+1})$ then $\alpha + \beta = \gamma$ and $B(\varepsilon) = \varepsilon^\beta Q_\xi(\tau_0, \xi_0) + O(\varepsilon^{\beta+1})$. If β is even, bifurcation is one sided and supercritical (subcritical) solutions are obtained by solving the reduced bifurcation equations for $\tau_0 = 1$ ($\tau_0 = -1$). When β is odd the bifurcation is transcritical and we may take $\tau_0 = 1$.

COROLLARY 5 [77], [86]. *When $F_{\lambda v} = A$ is nonsingular*

$$B(\varepsilon) = \varepsilon^\beta Q_\xi(\tau_0, \xi_0) + O(\varepsilon^{\beta+1}).$$

For even β unstable modes of the branching solutions are in one-to-one correspondence with the positive eigenvalues of Q_ξ. For odd β bifurcation is transcritical and unstable modes of the supercritical (subcritical) branch are in one-to-one correspondence with the positive (negative) eigenvalues of Q_ξ.

9. *Center manifold theorem.* The center manifold theorem in a Banach space is an alternative method for reducing an infinite-dimensional problem to a finite-dimensional one. The theorem in the case of smooth diffeomorphisms was given by O. E. Lanford [52].

THEOREM 6. CENTER MANIFOLD THEOREM IN A BANACH SPACE. *Let Z be a Banach space and let $\psi \in C^{k+1}(Z)$, with $\psi(0) = 0$. Let $\psi'(0)$ have spectral radius 1 and let the spectrum of $\psi'(0)$ be the union of a part on the unit circle and a part inside. (This is the case if $\psi'(0)$ is compact.) Let Y denote the generalized eigenspace of $\psi'(0)$ belonging to the spectrum of $\psi'(0)$ on the unit circle, and assume $\dim Y = d < +\infty$. Then there exists a neighborhood U of 0 in Z and a C^k submanifold M of U of $\dim d$, passing through 0 and tangent to Y at 0 such that*
(a) $\psi(M \cap U) \subset M$,
(b) *if $\psi^n(x) \in U$ for $n = 1, 2, \ldots$ then, as $n \to \infty$, $\mathrm{dist}\|\psi^n(x) - M\| \to 0$. M is called a center manifold; in general it is not unique.*

The reduction of the infinite case to the finite-dimensional case makes use of a special trick: the center manifold theorem is applied not to $\phi(x, \mu)$ but to

$$\psi(x, \mu) = (\phi(x, \mu), \mu).$$

If $\phi'(0, 0)$ has two simple complex conjugate eigenvalues of $|z| = 1$ then

$$D\psi = \begin{pmatrix} \phi_x & \phi_\mu \\ 0 & 1 \end{pmatrix}$$

has a three-dimensional space Y with spectrum on the unit circle. So by the center manifold theorem there is a three-dimensional manifold M in $x - \mu$ space; fixing μ we obtain a two-dimensional section M_μ which is locally invariant and attracting for $\Phi(0, \mu)$. Now restrict $\Phi(0, \mu)$ to M_μ and we have reduced the situation to a two-dimensional problem.

10. *Repeated branching, strange attractors, and turbulence.* We have seen that when an equilibrium solution loses stability it may bifurcate to a time-periodic solution. What happens when, at a later stage, that time periodic solution loses stability? The situation is best described in terms of the

Poincaré map, which is constructed as follows. Take a small section $\mathcal{S}$ of a hyperplane of codimension 1 which intersects a periodic orbit γ in a single point x_0. For a point x in that section let $\phi(x)$ be the point where the trajectory through x next intersects $\mathcal{S}$ going in the same direction. ϕ is thus defined on some subset $\mathcal{S}'$ of $\mathcal{S}$ and fixes x_0. Periodic orbits near γ show up as fixed points of ϕ, while subharmonic solutions of period roughly $N \times$ (per γ) correspond to fixed points of $\phi^{(n)}$, the nth iterate of ϕ. The stability of a fixed point x is determined by the spectrum of the Fréchet derivative $\phi(x)$: x is stable if the spectrum of ϕ' is contained in the disk $|\sigma| < 1$ and unstable if the spectrum of ϕ' contains points in $|\sigma| > 1$. The existence and regularity of the mapping ϕ is a consequence of the smooth dependence of the solution on the initial data. This includes the infinite-dimensional case when the differential equations are parabolic and the data is smoothed as time goes forward.
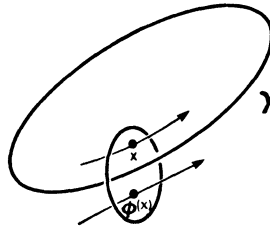

FIGURE 5.

Now let us suppose that as μ crosses zero the fixed point x_0 of ϕ loses stability by a simple eigenvalue of $\phi'_x(x_0, \mu)$ crossing through ± 1 or by a complex conjugate pair of simple eigenvalues $\lambda(\mu)$, $\bar{\lambda}(\mu)$ crossing the unit circle. The multidimensional case (even the infinite-dimensional case) can, under reasonable regularity assumptions, be reduced to the finite-dimensional case by the center manifold theorem, and when λ, $\bar{\lambda}$ cross the unit circle we have a two-dimensional bifurcation problem with the following result [52], [65], [74], [76].

THEOREM 7. *If $\lambda^m \neq 1$ for $m = 1, 2, 3, 4$ there exists a coordinate transformation after which ϕ: $\mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}^2$ takes the form $\phi(x, \mu) = N(x, \mu) + O(|x|^5)$ where, in polar coordinates,*

$$N(x, \mu): (r, \varphi) \to \big((1 + \mu)r - f_1(\mu)r^3, \varphi + \theta(\mu) + f_3(\mu)r^2\big).$$

If $f_1(0) > 0$ there exists, for small μ, an invariant attracting circle of $\phi(\cdot, \mu)$.

REMARK. If $f_1(0) < 0$ there exists an attracting invariant circle for $\phi^{-1}(\cdot, \mu)$ for $\mu < 0$. These invariant circles appear as bifurcating invariant tori for the original dynamical system. This theorem was announced by Neimark but with the condition $\lambda^m(0) \neq 0$ for $m = 1, \ldots, 4$ replaced by the condition that the origin be a weak attractor for the diffeomorphism $\phi(\cdot, 0)$ at criticality. It was proved by Sacker in 1964 but appeared only as a New York University technical report. It was proved independently by Ruelle and Takens, who also extended it to the infinite-dimensional case using the center manifold theorem. (See also Lanford [52].) The situation in the strong

resonance cases $\lambda^m = 1$ for $m = 1, 2, 3, 4$ can be dealt with in greater detail as follows.

THEOREM 8 (IOOSS AND JOSEPH [33]). *Assuming sufficient regularity for the diffeomorphism φ and that $\lambda^m(0) = 1$ for $m = 1, 2, 3, or 4$, we have*

$m = 1$: *bifurcating fixed points of φ; supercritical solutions are stable, subcritical solutions are unstable.*

$m = 2$: *bifurcating fixed points of φ ∘ φ appearing on one side of criticality only; supercritical solutions are stable, subcritical solutions unstable.*

$m = 3$: *bifurcating fixed points of φ ∘ φ ∘ φ; unstable on both sides of criticality.*

$m = 4$: *three possibilities*

(i) *one transcritical branch of fixed points of period 4; unstable on both sides of criticality,*

(ii) *two one-parameter branches of fixed points of period 4; both branches are one-sided (that is, appear on one side of criticality only) and at least one must be unstable,*

(iii) *no bifurcation at all.*

For further details and results, see Iooss' book [32].

It is not in general possible to determine the flow on the invariant torus. One possibility is that the solutions are quasi-periodic with two periods and may be represented in the form $f(\omega_1 t, \omega_2 t)$ where f is 2π periodic in each variable and ω_1/ω_2 is irrational, but this situation is nongeneric. (See the recent paper by Iooss and Joseph [34] for a more detailed discussion of this aspect of the subject.)

Laudau [51] and Hopf [29] proposed that hydrodynamic turbulence was the result of a cascade of repeated bifurcations: a periodic motion to a quasi-periodic motion with two periods, then two to three, etc. Such solutions can indeed be constructed formally [40], but the perturbation series always lead to small divisors. Moreover, it is impossible to do the calculations in general, and it sometimes is the case that the first bifurcation is subcritical and the system passes directly to a turbulent state. Chenciner and Iooss [10] have shown that under rather stringent (indeed nongeneric) conditions an n-torus will bifurcate to an $(n + 1)$-torus.

Ruelle and Takens, however, state the following proposition: *Let ω be a constant vector field on a torus T^k ($k \geqslant 4$). In every C^{k-1} small neighborhood of ω there exists an open set of vector fields possessing a strange attractor.* This result shows that systems of ordinary differential equations which exhibit a bifurcation of a 4-torus to a higher-dimensional torus are nongeneric. On this basis Ruelle and Takens disputed the Hopf-Landau view of turbulence; such cascades are nongeneric, and a dynamical system may be expected to exhibit, after only a few bifurcations, a transition to a chaotic state exhibiting no quasi-periodic behavior. This time-dependent state, though complex, should nevertheless exhibit a discernible structure in configuration space and a highly stable behavior. Thus, it should be a set invariant under the flow and which is approached asymptotically as $t \to \infty$ by all data initially sufficiently close. Furthermore it should be neither an equilibrium solution nor a periodic one. Such stable, aperiodic attracting sets are called "strange attractors".

Strange attractors are difficult to deal with analytically and rigorous proofs of their existence for a given dynamical system are rare. (See, however, Levinson [54].) Smale's "horseshoe map" [89] is an example of a diffeomorphism possessing a strange attractor (with diffeomorphisms continuous time is replaced by discrete time, and one looks at the flow under iterates of the mapping). Most of the current work in this area consists of numerical analysis of differential equations or diffeomorphisms. In 1963 E. Lorenz [55], a meteorologist interested in the problem of long-range weather preduction, analyzed numerically the very simple system

$$\dot{x} = -\sigma x + \sigma y, \qquad \dot{y} = -xz + rx - y, \qquad \dot{z} = xy - bz.$$

The solutions of these equations exhibit, for certain values of the parameters σ, r and b, chaotic behavior and sensitive dependence on initial conditions. The divergence of the vector field is everywhere equal to $-(\sigma + b + 1)$ so that volumes in $\mathbf{R}^3$ are contracted under time evolution. For

$$r > \sigma(\sigma + b + 3)(\sigma - b - 1)^{-1} > 0$$

there are three equilibrium points, none of them stable. Lanford and Ruelle [75] analyzed the solutions by studying a section map associated with the flow. Namely, let P be a point on the plane $z = 27$ at which $\dot{z} < 0$ and follow the trajectory through P until the orbit again crosses $z = 27$ in the downward direction; the new point is denoted by $\phi(P)$. Using a computer with a graphical output Lanford plotted the successive iterates $\phi^{(n)}(P)$. For large n the successive images tend to lie along two point sets Γ and Γ' as in the figure below.
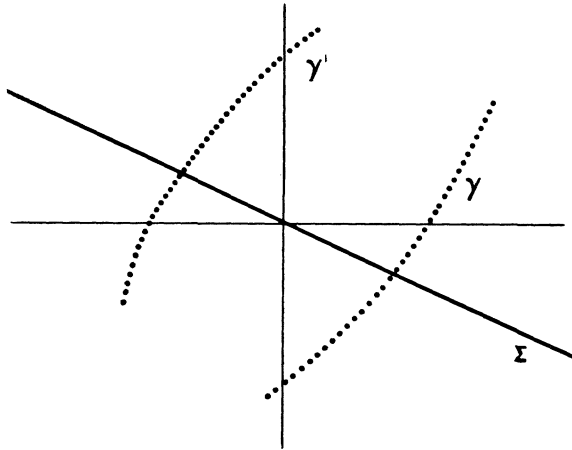

FIGURE 6.
Successive iterates of the section map for the Lorenz equations obtained numerically by Lanford.

(The line Σ represents an exceptional set of points which do not come back to cross the plane $z = 27$; trajectories through Σ tend to the unstable equilibrium point at the origin.) Under successive iterations of ϕ points move about Γ and Γ' in an erratic manner. Moreover, two points P_1 and P_2 initially close move apart and follow completely different histories under $\phi^{(n)}$; yet

these sequences nevertheless lie along Γ and Γ'. $\Gamma \cup \Gamma'$ represents the intersection of an attracting set S with the plane $z = 27$.

Other numerical studies on strange attractors have been carried out by Rössler [73] for simple systems of ordinary differential equations and by Hénon and Pomeau [24] for a simple diffeomorphism of $\mathbf{R}^2$. Iterations of nonlinear diffeomorphisms arise directly in models of population dynamics where successive generations are presumed to evolve according to a simple finite difference equation: $x_{n+1} = \phi(x_n)$. See May [59], Guckenheimer [22].

In addition to the computer experiments there are now a number of physical experiments which demonstrate the existence of aperiodic phenomena. Chaotic behavior has been observed in chemical kinetics [88] and has been measured quantitatively in recent experiments with turbulent flow. In 1975 Gollub and Swinney [20] used laser light scattering and photon correlation methods to obtain highly accurate measurements of the power spectrum of the fluid motion in a classical experiment known as the Taylor problem. Such experimental techniques allow one to obtain quantitative information about the frequency spectrum of the flow, and Gollub and Swinney (later with Fenstermacher) compared the results of their experiments with the theoretical discussions of Ruelle and Takens, Joseph and Iooss.

In the Taylor experiment (first performed and analyzed theoretically by G. I. Taylor in 1923 [91]) the region between two concentric cylinders is filled with fluid, and the inner cylinder is rotated. The nondimensional bifurcation parameter is the Reynolds number $R = \Omega_i r_i (r_0 - r_i)/\nu$ where Ω_i is the angular velocity of the inner cylinder, r_i and r_0 are the radii of the inner and outer cylinders, and ν is the kinematic viscosity of the fluid. At small R the flow is purely azimuthal, that is, $v_r = v_z = 0$ and only $v_\theta = v_\theta(r)$ is nonzero. When R exceeds a critical value R_c there is a bifurcation to a stable flow with a horizontal toroidal system of vortices. This is the bifurcation which was studied in 1923 by Taylor. Under a further increase of R the Taylor vortices become unstable to a time-dependent flow, in fact a series of waves moving around the cylinder. These were experimentally studied in detail by Coles [11]. Under a still further increase of R these wavy vortices break down to a more irregular flow.

The power spectra of the radial velocity flow at various values of R/R_c are shown in Figure 7
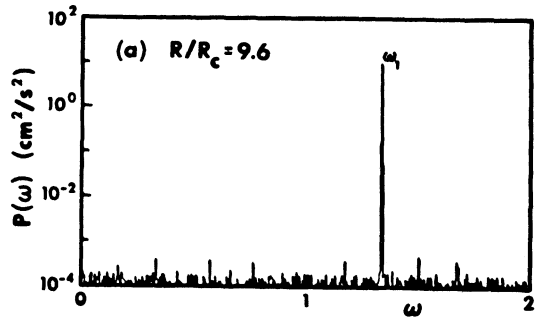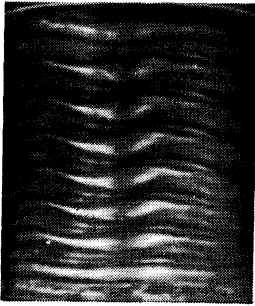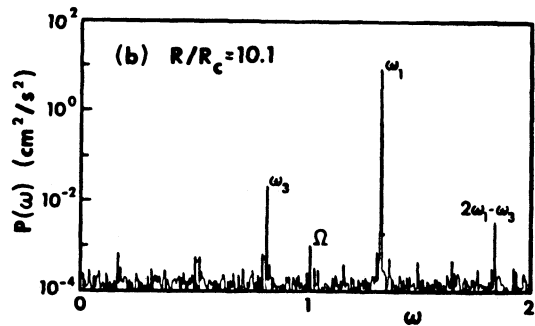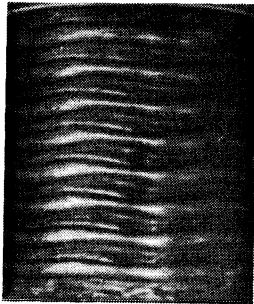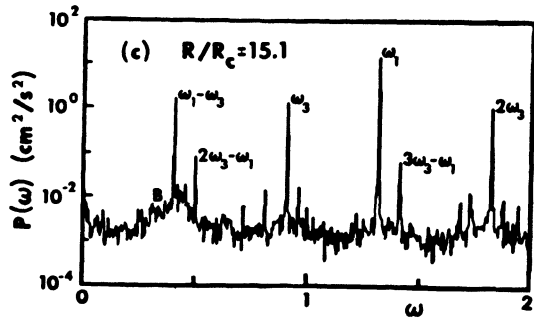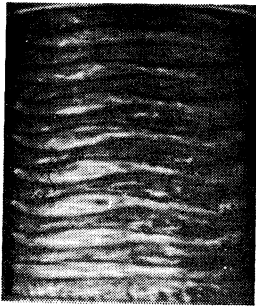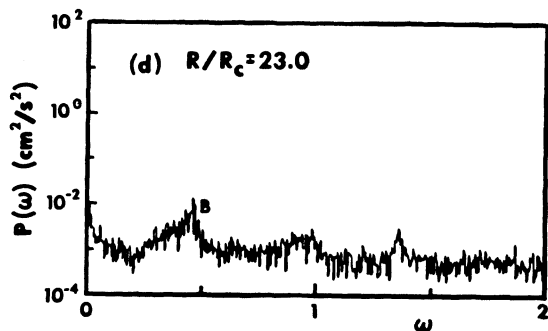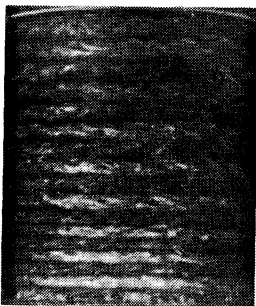
Taylor vortices at $R/R_c = 1.1$
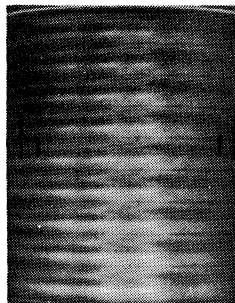
$R/R_c = 9.6$

$R/R_c = 11$

$R/R_c = 18$

$R/R_c = 26$

FIGURE 7. Power spectra of the radial velocity of the flow in the Taylor problem obtained in the experiments by Fenstermacher, *et al.* [17].

The time independent Taylor vortices exist in the range $1 < R/R_c < 1.2$. After that the wavy vortices bifurcate and persist stably to about $R/R_c =$ 10.1 ± 0.2. In this range of R there is a single spike in the power spectrum at $\omega = \omega_1$. A second component, denoted by ω_3 (a component ω_2 found in previous experiments turned out to be a transient), appears at $R/R_c = 10.1$ ± 0.2 and persists to $R/R_c = 19.8$ ± 0.4, where it disappears. In this range of R the flow is "apparently quasi-periodic, neglecting a weak broadband component" (denoted here by B).

The onset of the broadband component denoted by B is difficult to determine accurately. It exists with the sharp spikes at ω_1 and ω_3, and is the only survivor above $R/R_c = 22.4$. In this range the flow appears to be qualitatively different, "one characterized by a continuous spectrum or, equivalently, a decaying velocity auto correlation function".

These experiments confirm the view that turbulence (1) sets in after only a few bifurcations and (2) cannot be described in terms of a cascade of bifurcations. There are as yet, however, no satisfactory mathematical techniques for giving a theoretical treatment of these observations, even for so simple a system as the Lorenz equations.

11. *Imperfections and singularity theory.* Bifurcation is, in a sense, an idealized, nongeneric phenomenon; diagrams such as that in Figure 1 are structurally unstable, and small perturbations of the mapping will break the bifurcation diagram. (Consider the behavior of the zero set of $g(x, \lambda, \varepsilon) = \lambda x - x^3 + \varepsilon$ as ε varies in a neighborhood of zero.) Since imperfections are present in any physical system we should require that a description of the transition point include the effects of variations in the problem other than the bifurcation parameter. If the idealized bifurcation problem is determined by $g(x, \lambda) = \lambda x - x^3 = 0$, for example, what is the most general form of the solution structure under small perturbations of the system?

Golubitsky and Schaeffer [21] have addressed this question using the theory of singularities of mappings. Let $\mathcal{B}_n$ denote the class of germs of all C^∞ mappings of $\mathbf{R}^n \times \mathbf{R}$ into $\mathbf{R}^n$ and associate each element of $\mathcal{B}_n$ with a "bifurcation problem". $\mathcal{B}_n$ is a module over the ring of C^∞ scalar functions with respect to addition and multiplication of functions.

DEFINITION 9. *Two bifurcation problems $G, \overline{G} \in \mathcal{B}_n$ are contact equivalent if there exists a linear transformation $T_{x,\lambda}$ and orientation-preserving coordinate transformations $X(x, \lambda)$ and $\Lambda(\lambda)$, with $X(0, 0) = \Lambda(0) = 0$, such that*

$$G(x, \lambda) = T_{x,\lambda}\overline{G}(X(x, \lambda), \Lambda(\lambda)).$$

Of course, the analysis is entirely local here.

Let $G \in \mathcal{B}_n$ and let O_G be the orbit of G under contact equivalence, that is $O_G = \{\overline{G}|\overline{G}$ is contact equivalent to $G\}$. O_G may be regarded as an infinite-dimensional submanifold of $\mathcal{B}_n$ whose tangent space T_G at G may be calculated formally by differentiating along one-parameter curves in O_G through G. Thus, differentiating

$$G_t = T(x, \lambda, t)G(X(x, \lambda, t), \Lambda(\lambda, t))$$

with respect to t and setting $t = 0$, where $T(x, \lambda, 0) = I$, $X(x, \lambda, 0) = x$, and

$\Lambda(\lambda, 0) = \lambda$, we get

$$\dot{G}_0 = \dot{T}(x, \lambda, 0)G(x, \lambda) + \frac{\partial X^i}{\partial t}\frac{\partial G}{\partial x^i} + \frac{\partial G}{\partial \lambda}\dot{\Lambda}(\lambda, 0).$$

For example, when $n = 1$,

$$TG = \left\{ a(x, \lambda)G + b(x, \lambda)\frac{\partial G}{\partial x} + c(\lambda)\frac{\partial G}{\partial \lambda} \right\},$$

for arbitrary smooth functions $a(x, \lambda)$, $b(x, \lambda)$, $c(\lambda)$. TG is an ideal generated by G, G_x, and G_λ in the ring $\mathfrak{B}_1$.

DEFINITION 10. *The bifurcation problem G has codimension k if there exists a k-dimensional subspace K of $\mathfrak{B}_n$ such that $\mathfrak{B}_n = T_G \oplus K$.*

THEOREM 11. *If G has finite codimension k and K is spanned by $g_1, \ldots, g_k(x, \lambda)$, then*

$$G(x, \lambda, \alpha) = G(x, \lambda) + \sum_{i=1}^{k} \alpha_i g_i(x, \lambda)$$

is a universal unfolding of G.

This theorem means that any perturbation of G of the form

$$G(x, \lambda, \varepsilon_1, \ldots, \varepsilon_l) = G + \sum_{i=1}^{l} \varepsilon_i h_i(x, \lambda)$$

for smooth $h_1, \ldots, h_l$ is contact equivalent to F, and the coefficients α_i depend smoothy on the parameters ε_i. Therefore any basic bifurcation problem G with finite codimension can be transformed into an explicitly computable canonical form. If we restrict ourselves to real-valued functions (potentials) of codimension at most 4, we are led to the so-called "7 elementary catastrophes" of Thom.

COROLLARY 12. *Up to contact equivalence, $F(x, \lambda, p, q) = x^3 - \lambda x + px^2 + q$ is a universal unfolding of $x^3 - \lambda x$.*

Corollary 12 is obtained from Theorem 11 by some elementary manipulations with Taylor's series. Since this is an important technique in singularity theory it is worth presenting here. The tangent space is given by

$$TG = \left\{ a(x^3 - \lambda x) + b(3x^2 - \lambda) + c(\lambda)x \right\}.$$

Note that $x^3 = \frac{x}{2}(3x^2 - \lambda) - \frac{1}{2}(x^3 - \lambda x) \in TG$. We must show that $K = \mathfrak{B}^1/TG = \mathrm{sp}\{1, x^2\}$. Any C^∞ function $f(x, \lambda)$ can be expanded as

$$f(x, \lambda) = f_0(\lambda) + f_1(\lambda)x + f_2(\lambda)x^2 + f_3(\lambda, x)x^3;$$

but since $f_1(\lambda)x$ and x^3 belong to TG,

$$f(x, \lambda) \equiv f_0(\lambda) + f_2(\lambda)x^2 \;(\mathrm{mod}\; TG).$$

Furthermore $(3x^2 - \lambda) \in TG$ so $\lambda \equiv 3x^2 (\mathrm{mod}\; TG)$; therefore

$$f_0(\lambda) = f_0(0) + \lambda f_0'(0) + \lambda^2 h(\lambda)$$

$$\equiv P_0 + P_1 x^2 + 9x^3(xh(\lambda)) \;(\mathrm{mod}\; TG)$$

$$\equiv P_0 + P_1 x^2 \;(\mathrm{mod}\; TG);$$

and

$$f_2(\lambda) = f_2(0)x^2 + g(\lambda)x(\lambda x)$$

$$= f_2(0)x^2 + \frac{\lambda^2 g(\lambda)}{3} \ (\mathrm{mod}\ TG)$$

$$= Q_0 + Q_1 x^2 \ (\mathrm{mod}\ TG).$$

Finally, $f_0(\lambda) + f_2(\lambda)x^2 \equiv P_0 + \tilde{P}_1 x^2$ where P_0 and $\tilde{P}_1$ are constants. The behavior of the zero structure of F is sketched in Figure 8.
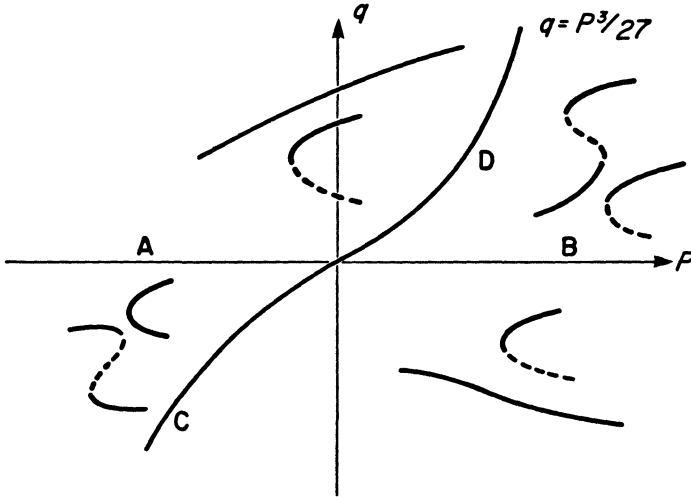
FIGURE 8.

The structure of the zero set of F changes across the lines $q = 0$ and $q = p^3/27$. Real bifurcation can occur only along the exceptional curve $q = 0$, hence is nongeneric. Hysteresis sets in upon crossing the curves C and D (Golubitsky and Schaeffer [21]).

The importance of imperfections has long been known among elasticians; see Hunt [30] and Thompson [94]. The effect of imperfections on bifurcation was first studied by J. Keener in his thesis; see Keener [45], Keener and Keller [44], and Reiss [71]. T. B. Benjamin has discussed the effects of imperfection in the bifurcation of the Taylor cells [5].

III. Group representation theory and symmetry breaking.

1. *Bifurcation at multiple eigenvalues.* Let us now suppose that at criticality, dim ker $L_0 > 1$, a situation commonly referred to as "bifurcation at a multiple eigenvalue". The Lyapounov-Schmidt method, described in II.6, is nice in theory, but in practice the computation of even the lowest order terms of the bifurcation equations is extremely complicated. Moreover, systems of n equations in n unknowns can display a complex solution structure and the algebraic problem in general can be quite complex. The results of Theorem 2 concerning the stability of bifurcating solutions are no longer valid at multiple eigenvalues.

In many problems of physical interest, however, the multiplicity of the branch point arises from an underlying symmetry of the problem. Just as in quantum mechanics, where the invariance of the Hamiltonian under a symmetry group leads to degeneracy of the energy levels, group invariance of a system of nonlinear equations leads to multiple eigenvalue branch points. The physical consequence of this situation is that the bifurcation point is characterized by a symmetry-breaking bifurcation. By applying methods of group representation theory it is possible to compute the structure of the bifurcation equations directly from a knowledge of the transformation group, thus bypassing the complicated Lyapounov-Schmidt procedure. In this way we are able to analyze and classify bifurcation phenomena on geometrical grounds, and thus build a unified mathematical theory of bifurcation at multiple eigenvalues.

2. *Covariance of the bifurcation equations.* Let T_g be a linear representation of a group $\mathcal{G}$ on the Banach spaces $\mathcal{E}$ and $\mathcal{F}$ and assume that $T_g G(\lambda, u) = G(\lambda, T_g u)$. This *covariance* of the mapping G is a natural assumption in physical theories and is a mathematical expression of the axiom that the equations of mathematical physics be independent of the observer. For example, the Navier-Stokes equations

$$\Delta u_k - \frac{\partial p}{\partial x_k} = u_j \frac{\partial u_k}{\partial x_j}, \qquad \frac{\partial u_k}{\partial u_k} = 0$$

are covariant with respect to the representation

$$Tg\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ p \end{bmatrix}(\mathbf{x}) = \left[\begin{array}{ccc|c} & & & 0 \\ & \mathbf{0} & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ p \end{bmatrix}(g^{-1}\mathbf{x})$$

of the group of rigid motions; here $g\mathbf{x} = O\mathbf{x} + \mathbf{a}$ where $\mathbf{a} \in \mathbf{R}^3$ and O is a 3×3 orthogonal matrix.

Moreover, in mathematical modeling, the introduction of a high degree of symmetry into the geometry simplifies the analysis, so that problems arising in applications are found to have a nontrivial symmetry group.

From covariance it follows that $T_g G_u(\lambda, u) = G_u(\lambda, T_g u) T_g$; so if u_0 is an invariant solution of $\mathcal{G}$ then $T_g L_0 = L_0 T_g$. Therefore $\mathfrak{N}_0 = \ker L_0$ is invariant under T_g. If $\dim \mathfrak{N}_0 < +\infty$, as is often the case in applications, $\Gamma \equiv T_{g|\mathfrak{N}_0}$ is a finite-dimensional representation of $\mathcal{G}$. Writing the bifurcation equations in the form $F(\lambda, v) = 0$, where $v \in \mathfrak{N}_0$ and $F: \mathbf{C} \times \mathfrak{N}_0 \to \mathfrak{N}_0$, we have the following [83].

THEOREM 13. *Let $G(\lambda, u)$ be an analytic operator from a complex Banach space $\mathcal{E}$ to $\mathcal{F}$, covariant with respect to a representation T_g of a compact group $\mathcal{G}$. Suppose that $G(\lambda_c, u_0) = 0$, $T_g u_0 = u_0$ for all $g \in \mathcal{G}$, and $G_u(\lambda_c, u_0)$ is a Fredholm operator of index zero with kernel $\mathfrak{N}_0$. Then $\mathfrak{N}_0$ is invariant under T_g and the bifurcation equations $F(\lambda, v)$ are covariant with respect to Γ, the restriction of T_g to $\mathfrak{N}_0$; that is $\Gamma_g F(\lambda, v) = F(\lambda, \Gamma_g v)$.*

This theorem is easily proved by following through the details of the Lyapounov-Schmidt procedure. Each term $B_k(\lambda, v, w, \dots)$ in the expansion

(3) of F is symmetric in its variables and covariant with respect to the representation Γ_g. In particular, $\Gamma_g A(\lambda) = A(\lambda)\Gamma_g$; so by Schur's lemma, if $\mathfrak{N}_0$ is irreducible $A(\lambda) = \sigma(\lambda)I$, where I is the identity transformation. The higher-order terms $B_2, B_3, \ldots$ can all be calculated, up to scalar factors, by the methods of group representation theory. For example, if $\mathcal{G} = O(3)$ and $\mathfrak{N}_0$ transforms according to the irreducible representation D^l, then the coefficients in the quadratic terms of the bifurcation equation are the ordinary Clebsch-Gordon coefficients for the rotation group. The reduced bifurcation equations are similarly covariant with respect to the representation Γ_g.

In this way, transition phenomena can be categorized according to the geometry of the problem, rather than by the particular physical mechanisms involved. From this point of view, the onset of convection in a spherical geometry, the buckling of a spherical shell, or the onset of ionic currents in a developing spherical egg, can all be given a unified mathematical treatment, even though the physical mechanisms in each of the problems may be vastly different. On the other hand, the specific physical mechanisms in the problem make themselves apparent in the determination of the critical value of λ and the transformation properties of the kernel $\mathfrak{N}_0$. In addition, the physics of the problem determine the values of the scalar parameters multiplying the covariant terms in the bifurcation equations; these parameter values determine the direction of bifurcation (supercritical or subcritical) as well as the stability of the various bifurcating solutions.

Depending on the representation Γ_g, there may be more than one covariant term of lowest degree k. In that case we would arrive at a system of reduced equations of the form

$$\lambda w = a_1 B_k^{(1)}(w) + a_2 B_k^{(2)}(w) + \cdots + a_l B_k^{(l)}(w) \tag{5}$$

where the coefficients $a_1, \ldots, a_l$ are parameters which depend on the original parameters of the systems. The multiplicity l of covariant terms of degree k can be computed directly from a knowledge of the representation Γ and does not depend on the particular structure of the equations at hand. In fact, the multiplicity l is precisely the number of times the representation Γ is contained in $(\Gamma^{\otimes k})_s$, where the subscript s denotes the restriction of $\Gamma^{\otimes k}$ to the subspace of completely symmetric tensors. When there are multiple covariant tensors, as in (5), the possibility of *selection mechanisms* arises. The stability of the various bifurcating solutions depends on the relative sizes of the parameters $a_1, \ldots, a_l$. The numerical values of the parameters $a_1, \ldots, a_l$ are determined by the physics of the problem, but in general the computation of their exact dependence on the natural physical parameters of the problem is a difficult matter (equivalent to the direct calculation of the full set of bifurcation equations). Such calculations could be carried out numerically in specific cases, but in order to classify the types of transitions which can take place, it is sufficient to consider the parameters $a_1, \ldots, a_l$ as free parameters, and investigate the possible transitions which may occur. These matters will be discussed more fully in §4.

3. *Computational methods.* The identification of covariant mappings of a given order with invariant tensors of one higher order gives us, firstly, a way

to compute the number of independent covariant mappings of a given order based on the group characters. The character of a representation is the trace: $\chi(g) = \operatorname{Tr} T(g)$. If $T(g)$ is a representation the k-fold tensor product $T^{\otimes k}(g)$ acting on the space of k-linear mappings is defined by $T^{\otimes k}(g)B(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ $= B(T(g^{-1})\mathbf{u}_1, \ldots, T(g^{-1})\mathbf{u}_k)$, and the character of this representation is χ^k. We always restrict ourselves to the subspace of mappings B which are completely symmetric in their variables, which is invariant under $T^{\otimes k}$. The restriction of the representation $T^{\otimes k}$ to the subspace of symmetric mappings is denoted by $(T^{\otimes k})_s$ and its character is denoted by $\chi_{(k)}(g)$.

THEOREM 14. *Let T be a representation on the vector space $\mathfrak{N}$ of the compact group $\mathcal{G}$ and let $c_n(T, \mathcal{G})$ denote the number of completely symmetric n-linear operators on $\mathfrak{N}$ which are covariant with respect to T. A generating function for the coefficients is*

$$\sum_{n=0}^{\infty} c_n(T, \mathcal{G})z^n = \int_{\mathcal{G}} \det(I - zT(g))^{-1}\overline{\chi}(g)\, d\mu(g)$$

where $\chi(g) = \operatorname{Tr} T(g)$ and $\mu(g)$ is the normalized invariant measure on $\mathcal{G}$. (In the case of a finite group, μ assigns weight $1/|\mathcal{G}|$ to each element $g \in \mathcal{G}$, where $|\mathcal{G}|$ is the order of $\mathcal{G}$.) Alternatively, the number of n-linear covariant mappings may be computed from the formula

$$\sum_{n=0}^{\infty} c_n(T, \mathcal{G})z^n = \int_{\mathcal{G}} \sum_{n=0}^{\infty} \chi_{(n)}(g)\overline{\chi}(g)z^n\, d\mu(g)$$

where

$$\chi_{(n)}(g) = \sum_{\sum l i_l = n} \frac{\chi^{i_1}(g)\chi^{i_2}(g^2) \cdots \chi^{i_n}(g^n)}{1^{i_1}i_1! 2^{i_2}i_2! \cdots n^{i_n}i_n!}.$$

For example, the character $\chi_{(2)}$ is given by

$$\chi_{(2)}(g) = \tfrac{1}{2}(\chi^2(g) + \chi(g^2)).$$

This theorem is very closely related to the Molien function which is the generating function for the invariants of a given representation (see Jaric and Birman [39]). The proof of Theorem 14 is given in [86], [87].

EXAMPLE. Let the group in question be D_3, the symmetry group of the equilateral triangle. This group is generated by the elements $g = (1\ 2\ 3)$ and $h = (1\ 2)$. It has three irreducible representations:
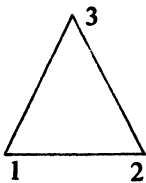
two one-dimensional and one two-dimensional arising from the natural action of D_3 in the plane. The conjugacy classes are $\{e\}$, $\{g, g^2\}$, and $\{h, gh, g^2h\}$ and the character table is

D_3	$\{e\}$	$\{g, g^2\}$	$\{h, gh, g^2h\}$
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	1	-1
$\chi^{(3)}$	2	1	0

Let T be the representation $T = T^{(1)} \oplus T^{(2)} \oplus T^{(3)}$ and let us compute the number of covariant quadratic mappings. The characters are

	$\{e\}$	$\{g, g^2\}$	$\{h, gh, g^2h\}$
χ	4	1	0
$\chi_{(2)}$	10	1	2
$\chi_{(2)}\bar{\chi}$	40	1	0

Accordingly, the number of maps is

$$c_2(T, D_3) = \frac{1}{6} \sum_{\sigma \in D_3} \chi_{(2)}(\sigma)\bar{\chi}(\sigma) = \frac{1}{6}(1 \cdot 40 + 2 \cdot 1 + 3 \cdot 0) = 7.$$

There are thus 7 independent mappings for this representation, and therefore 7 scalar quantities must be tabulated to completely determine the bifurcation. Let us calculate covariant mappings. We represent the vector space $\mathfrak{N}$ as linear polynomials in four variables x, y, z, and $\bar{z}$ (x and y are real). These variables transform as follows under the group action

$$\begin{aligned} gx &= x & hx &= x \\ gy &= y & hy &= -y \\ gz &= e^{2\pi i/3}z & hz &= \bar{z} \\ g\bar{z} &= e^{-2\pi i/3}\bar{z} & h\bar{z} &= z. \end{aligned}$$

(We are using complex conjugate variables z and $\bar{z}$ for the two-dimensional action.) The mapping $F = (F_1, F_2, F_3, F_4)$ must also transform as x, y, z, and $\bar{z}$. In particular, F_1 is an invariant and so is a function of the invariants $x, y^2, |x|^2, z^3, \bar{z}^3$:

$$F_1 = A(x, y^2, |z|^2, z^3, \bar{z}^3).$$

F_2 must transform like y, so

$$F_2 = yB(x, y^2, |z|^2, z^3, \bar{z}^3).$$

The functions z and $\bar{z}^2$ both transform like z, so

$$F_3 = zC(x, y, |z|^2, z^3, \bar{z}^3) + \bar{z}^2 D(x, y, |z|^2, z^3, \bar{z}^3).$$

Now applying the group operation h, we get

$$hF_3 = F_4 = F_3h,$$

$$F_4(x, y, z, \bar{z}) = F_3(x, -y, \bar{z}, z)$$

$$= \bar{z}C(x, -y, |z|^2, \bar{z}^3,) + z^2 D(x, -y, |z|^2, \bar{z}^3, z^2).$$

Now we can write down the lowest order terms, beginning with linear, then quadratic functions of the four variables:

$$F_1 = \lambda_1 x + ax^2 + by^2 + c|z|^2 + \cdots$$

$$F_2 = (\lambda_2 + dx) + \cdots$$

$$F_3 = \lambda_3 z + ezy + fxz + g\bar{z}^2 + \cdots$$

$$F_4 = \lambda_3 \bar{z} - e\bar{z}y + fx\bar{z} + gz^2 + \cdots.$$

The seven scalars $a, \ldots, g$ index the seven independent quadratic covariant mappings. In practice, seven variables is too many to deal with, and I only give this example as an illustration of the techniques involved. We turn to some computations in actual physical situations.

4. *The Bénard problem.* The equations modeling convection are the Boussinesq equations

$$\Delta u_k + \delta_{k3}R\theta - \frac{\partial P}{\partial x_k} = \frac{1}{P_r}\left[u_j \frac{\partial u_k}{\partial x_j} + \frac{\partial u_k}{\partial t} \right],$$

$$\Delta\theta + u_3 = u_j \frac{\partial\theta}{\partial x_j} + \frac{\partial\theta}{\partial t},$$

$$\frac{\partial u_j}{\partial x_j} = 0,$$

$$-\infty < x_1, x_2 < \infty, \qquad 0 < x_3 < 1$$

where P_r is the Prandtl number, R is the Rayleigh number, and u_1, u_2, u_3, θ, P are the components of velocity, temperature, and pressure. R is proportional to the temperature drop across the layer, and convection sets in when R exceeds a certain critical value R_c ($R_c = 1708$ if both the top and bottom surfaces are rigid). For a more extensive discussion see D. Joseph's book [41]. Bifurcating solutions of the nonlinear problem were first constructed formally by Malkus and Veronis [57]; the first rigorous proofs of the existence of bifurcating solutions were given by Iudovic [36] and Rabinowitz [69]. Iudovic proved the existence of bifurcating solutions by a topological degree argument. Earlier, however, Velte [99] had proved the existence of branching solutions of the Taylor problem by a topological degree argument.

The infinite plane layer model described here has the advantage that it simplifies the mathematical analysis and gives good quantitative predictions for the onset of convection and of the value of the critical wave number. It has the disadvantage, however, that it allows a high degree of nonuniqueness in the bifurcating solutions. The question then arises as to what mechanisms govern the selection of a particular pattern.

Busse [7], motivated by an earlier work of Malkus, showed that there was a functional associated with the bifurcation problem and argued that this functional would be maximized by the physically correct solutions. Busse's functional is in fact the invariant tensor corresponding to the reduced bifurcation equations, so his method is equivalent to a linearized stability analysis of the bifurcating solutions. Group theoretic ideas are already implicit in Busse's work. In [84], [85] the problem of pattern selection is discussed entirely from a group theoretic point of view.

We begin not with the Boussinesq equations on an infinite plane, but with any general system $G(\lambda, u)$ which is covariant with respect to the group of rigid motions in the plane. For simplicity we consider the case where u is a scalar, though the situation is somewhat more complicated in the case of the Bénard problem. When cellular solutions appear the symmetry is broken from the full group of rigid motions to a crystallographic subgroup. There is so far no mathematical proof that symmetry must be broken in this fashion;

we simply make the *Ansatz* that we look for bifurcating doubly periodic solutions. These are characterized by the hypothesis that the solutions be invariant under some subgroup of the full translation group. Let Λ be a lattice of vectors in the plane generated as the sum of all integer combinations $\{n\omega_1 + m\omega_2\}$ for two fixed linearly independent vectors, ω_1 and ω_2. Let $H(\Lambda)$ denote the subgroup of translations $H(\Lambda) = \{T_\omega : \omega \in \Lambda\}$ where $(T_\omega u)(x) = u(x + \omega)$. We consider the subclass of all functions which are invariant under $H(\Lambda)$. From the translational invariance of G, such subclasses are invariant under G; so we may restrict ourselves to bifurcation in subspaces of doubly periodic solutions with a given lattice structure. If ψ is Λ-periodic then $T_r\psi$ (where r is a rotation) is Λ-periodic if and only if r^{-1} leaves Λ invariant. We denote by $\mathcal{D}(\Lambda)$ the largest subgroup of $O(2)$ which leaves Λ invariant; such a subgroup is called the *holohedry*. When we restrict ourselves to the bifurcation of Λ-periodic functions the kernel $\mathcal{K}$ of $G_u(\lambda_c, 0)$ is spanned by the finite set of wave functions

$$\mathcal{K}(\Lambda) = \{e^{i\langle r\omega,\, x\rangle}\}_{r\in\mathcal{D}(\Lambda)}.$$

We denote by $\mathcal{S}(\Lambda)$ the subgroup of the group of rigid motions which includes all translations plus the elements of $\mathcal{D}(\Lambda)$. $\mathcal{S}(\Lambda)$ is the symmetry group of the equation $G(\lambda, u) = 0$ restricted to Λ-periodic functions. In the case of the Boussinesq equations the linearized operator G_u, restricted to the subspace of doubly periodic functions, is a Fredholm operator (Fife [18]).

Let us derive the branching equations for bifurcation of solutions in the hexagonal lattice. The kernel is spanned by the six wave functions

$$\{e^{i\langle\omega_j,\, x\rangle}, j = 1, \dots, 6\}$$

where the vectors ω_j are as pictured in Figure 9. Denote these wave functions by ψ_k.
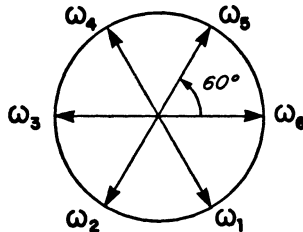
FIGURE 9

PROPOSITION 15. *Let T_σ be the representation of the group of rigid motions given by $(T_\sigma u)(x) = u(\sigma^{-1}x)$. The action T_σ on the basis functions $\{\psi_1, \dots, \psi_6\}$ is as follows:*

(i) *If $\sigma = a$ is a translation, $T_a\psi_j = e^{i\langle\omega_j,\, a\rangle}\psi_j$.*

(ii) *If $\sigma = r$ is a rotation or reflection in $\mathcal{D}(\Lambda)$ then T_r acts as a permutation on the ψ_j. Specifically, let $r(j)$ be the permutation of the vertices ω_j induced by the rotation r. Then $T_r\psi_j = \psi_{r(j)}$.*

(iii) *$T_r\psi_j = \bar{\psi}_j$ whenever $r\omega_j = -\omega_j$.*

This proposition is easily proved by direct computation. To prove (ii) note that $T_r e^{i\langle\omega,\, x\rangle} = e^{i\langle\omega,\, r^{-1}x\rangle} = e^{i\langle r\omega,\, x\rangle}$ since r is an orthogonal transformation; thus T_r moves the wave vector ω to $r\omega$.

If instead we write the bifurcation equations in the form

$$F_i(\lambda, z_1, \ldots, z_j) = 0$$

then the variables z_j should transform as the ψ_j. Let $\alpha = (1\ 2\ 3\ 4\ 5\ 6)$ and $\beta = (2\ 6)(3\ 5)$; these two permutations generate the symmetry group of the hexagon. From $\alpha F = F\alpha$ we have

$$F_2(z_1, \ldots, z_6) = F_1(z_2, z_3, \ldots, z_6, z_1)$$

and so forth, so it is only necessary to determine the first component F_1; the remaining components are obtained by cyclically permuting the variables. From $T_a F = F T_a$ we get

$$e^{i\langle \omega_1, \mathbf{a}\rangle} F_1(z_1, \ldots, z_6) = F_1\big(e^{i\langle \omega_1, \mathbf{a}\rangle} z_1, \ldots, e^{i\langle \omega_6, \mathbf{a}\rangle} z_6\big).$$

We break F_1 down into linear, quadratic, cubic terms, etc. If F_1 contains the linear term z_j then $e^{i\langle \omega_1, a\rangle} z_j = e^{i\langle \omega_j, \mathbf{a}\rangle} z_j$; therefore F_1 contains no linear terms except z_1 and $F_1 = a z_1$, $F_2 = a z_2$, ... Thus the linear term must be a scalar multiple of the identity; and according to Schur's lemma the representation T_σ restricted to the kernel $\mathfrak{N}(\Lambda)$ is irreducible. The quadratic terms $z_j z_k$ will be covariant only if

$$e^{i\langle \omega_1, \mathbf{a}\rangle} z_j z_k = \big(e^{i\langle \omega_j, \mathbf{a}\rangle} z_j\big)\big(e^{i\langle \omega_k, \mathbf{a}\rangle} z_k\big),$$

hence only if $\omega_1 = \omega_j + \omega_k$. The only vectors satisfying this condition are $\omega_j = \omega_2$ and $\omega_k = \omega_6$, so the only covariant quadratic term is $b z_2 z_6$. Then $F_2 = b z_3 z_1$, $F_3 = b z_4 z_2$, ... In the cubic case we are similarly led to the condition $\omega_1 = \omega_j + \omega_k + \omega_l$. There are three possible solutions to this equation, leading to the three terms $z_1^2 z_4$, $z_1 z_2 z_5$, and $z_1 z_3 z_6$. The symmetry condition $\beta F = F\beta$, however, implies that F_1 is symmetric in z_2, z_6 and z_3, z_5. Therefore there are really only two covariant cubic terms, namely

$$c z_1^2 z_4 + d(z_1 z_2 z_5 + z_1 z_3 z_6).$$

In the other lattices (when the basis vectors ω_1 and ω_2 make an acute angle other than $\pi/6$ between each other) there are no quadratic terms but again two cubic terms.

The structure of the bifuration equations can in each case be determined by group-theoretic arguments, and the bifurcation and stability of all possible solutions can be considered for each of the basic lattices. The reduced bifurcation equations (lowest-order terms) can be obtained from the generating functions

$$\Lambda_6 \colon F_1(\lambda, z_1, \ldots, z_6) = \lambda z_1 + a z_1^2 z_4 + b z_1(z_2 z_5 + z_3 z_6),$$

$$\Lambda_2, \Lambda_4 \colon F_1(\lambda, z_1, \ldots, z_6) = \lambda z_1 + a z_1^2 z_3 + b z_1 z_2 z_4.$$

This is the case when the quadratic term in the Λ_6 case vanishes. (Λ_6 denotes the hexagonal, Λ_4 the square, and Λ_2 the rhombic lattices.) If it does not vanish the generating function is

$$\Lambda_6 \colon F_1(\lambda, z_1, \ldots, z_6) = \lambda z_1 + c z_2 z_6.$$

The other components of the bifurcation equations are obtained by cyclic permutation of the variables $z_1, \ldots, z_6$. The coefficients a, b, c depend on

the lattice (in fact on the angle θ between ω_1 and ω_2) and on the original physical parameters of the problem.

The stability of the bifurcating solutions can be determined in terms of the parameters a and b. (By stability here we mean the formal linearized stability of a solution relative to disturbances within the same lattice class. Certainly this is a necessary condition for stability; although a rigorous mathematical treatment of stability on an unbounded domain is so far nonexistent.) In order to obtain any results on pattern selection one must evaluate the dependence of the coefficients a and b on the lattice and on the external parameters of the problem. For the Boussinesq equations there are two physical parameters to the problem, the Rayleigh number and the Prandtl number. For simplicity let us first restrict ourselves to bifurcation of solutions in Λ_6, that is, those which are invariant under translations in the hexagonal lattice. We analyze the linearized stability of the solutions with respect to disturbances in the same invariant subspace, and it can be shown that the pattern selection diagram in Figure 10 holds. Rolls and hexagons bifurcate supercritically provided $a + 2b < 0$, but they are not both simultaneously stable; their stabilities depend on the relative sizes of the parameters a and b.
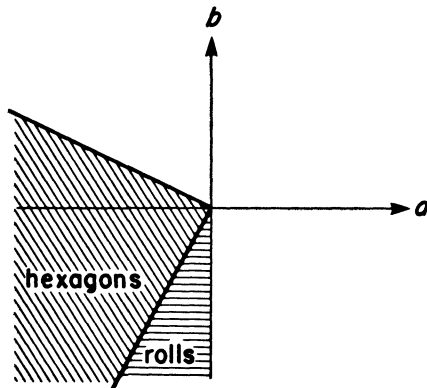
FIGURE 10

The stability results described above can all be made mathematically rigorous if we restrict ourselves to solutions (and disturbances) in the hexagonal lattice Λ_6. This, of course, does not resolve the issue for the Bénard problem, or other problems where Euclidean invariance is broken, for there is no *a priori* justification for the restriction to translational invariance under the hexagonal lattice. Nevertheless the analysis demonstrates precisely the concept of selection mechanisms for pattern formation and their relationship to group representation theory. The two parameters a and b in the bifurcation equations signify the existence of two independent covariant mappings of third order for the given representation.

In order to give a complete resolution of the Bénard problem, or more generally the breaking of Euclidean symmetry, one would have to derive selection laws which uniquely select one stable pattern from all possible patterns. This means, in particular, that the stability of patterns in one lattice must be compared to those in another. Partial results of this type can be obtained from purely group-theoretic arguments [85].

THEOREM 16. *There exists a function $q(\alpha)$ such that*

(1) $q(\alpha) = A_0 + A_2 \cos 2\alpha + A_4 \cos 4\alpha + \cdots$,

(2) $a = 3q(0)$, $b = 6q(\theta)$

where θ is the angle between the basic vectors of the lattice and the coefficients $A_0, A_2, \ldots$ depend only on the basic physical parameters of the problem.

In the case of the Bénard problem, Busse considered a simplified system of equations obtained by letting the Prandtl number go to infinity in the Boussinesq equations. As a consequence, for the simplified system, one can show that $q(\alpha) = A + B \cos 2\alpha$, and then the pattern selection diagram in Figure 11 is valid.

If the quadratic term does not vanish in the hexagonal case (which happens if the material properties such as viscosity, specific heat, or thermal diffusivity
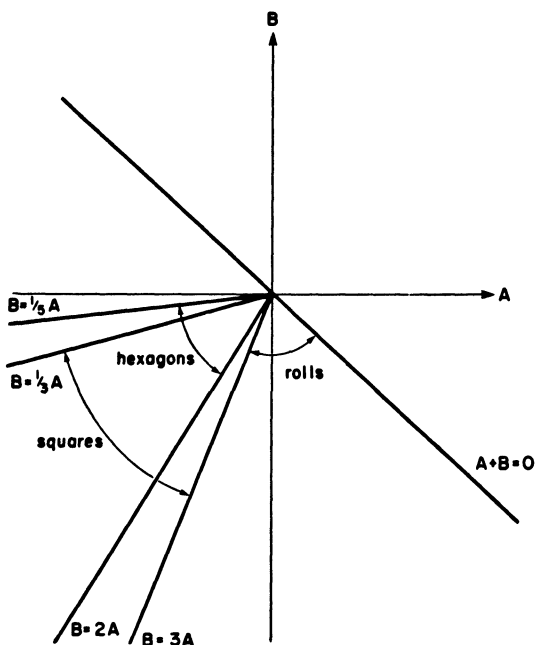
FIGURE 11.

In the sector between the lines $B = 3A$ and $A + B = 0$, rolls are the only possible doubly period solutions which can be stable. That is, if for a given set of external physical conditions the pair (A, B) lies in this sector, then rolls are the only solutions which can be stable when tested against all disturbances in the three lattice classes. In the sector $2A < B < \frac{1}{3}A < 0$ hexagons may be stable while rolls are unstable. Squares, however, are also stable in this region, as are rectangles in the rectangular lattices with angle $\theta > \pi/3$. In general, θ-rectangles may be stable in the sector

$$A(1 - 2 \cos 2\theta) < B < -\left(\frac{1 + 2 \cos 2\theta}{3}\right)A.$$

vary, when the density-temperature relation is nonlinear, or when surface tension drives the convection) then one obtains the branching of subcritical solutions in the hexagonal lattice. These solutions are invariant under the entire symmetry group of the hexagon and are unstable on both sides of criticality. The eigenvalues of the linearized equations about the bifurcating solution are, up to a scalar multiple at lowest order, $-1, 3, 2, 2, 0, 0$. The two zero eigenvalues arise because of the presence of the two-parameter translation group. The fact that one eigenvalue is negative means there is one unstable mode on the subcritical branch. These results can be obtained entirely by group theoretic arguments. Since the subcritical branch has one unstable mode it can "bend back" and regain stability, as sketched in Figure 4. There is, of course, no general proof that this branch *must* bend back, as in the diagram.

It may be useful to compare the foregoing analysis with the earlier work on bifurcating solutions of the Bénard problem by Iudovic [36] and Rabinowitz [69]. These authors construct doubly-periodic solutions by symmetrizing within a given, fixed lattice class but without regard to the stability of the solutions. Within a given lattice they considered the subspace of functions which are completely invariant under the symmetry group of the lattice, on which the branch point is simple. But at a simple eigenvalue, as we have seen, supercritical solutions are stable and subcritical solutions are unstable, so when the analysis is carried out in this fashion one is led to the erroneous conclusion that any solution which bifurcates supercritically is stable. Altogether different conclusions are obtained, and a mechanism for pattern selection is possible, if the symmetrization argument is avoided.

Busse, in his work, does not *a priori* make the symmetrization argument, but, due to other assumptions, he is led to the conclusion that only rolls can be stable. I believe that a valid theoretical explanation of the mechanisms for pattern selection does not yet exist.

I want to emphasize that I do not claim the Bénard problem to be resolved by the above considerations, though I do think the analysis sheds some light on the complexity of the problem. In any event, the group-theoretic approach to a bifurcation problem is very nicely illustrated by the Bénard problem.

5. *Bifurcation in the presence of the rotation group.* Now let us consider the derivation of the bifurcation equations when the symmetry group in question is $O(3)$, the group of rotations of the sphere in $\mathbf{R}^3$. Busse [8] was the first person to consider a general bifurcation problem in the presence of $O(3)$. Such bifurcation problems arise, for example, in the onset of convection in a spherical mass or the buckling of a perfectly uniform spherical shell. In these cases, complete rotational symmetry is broken by the bifurcating solutions. The irreducible representations of $SO(3)$ are denoted by D^l for $l = 0, 1, 2, \ldots$ and are of dimension $2l + 1$. They arise when one considers the transformation properties of the spherical harmonics $Y_m^l(\theta, l) = P_{l,m}(\cos \theta)e^{iml}$, $-l \leqslant m \leqslant l$. Let the infinitesimal generators of $SO(3)$ be L_1, L_2, and L_3; these satisfy the commutation relations

$$\left[L_i, L_j \right] = \varepsilon_{ijk} L_k$$

where ε_{ijk} is the completely antisymmetric tensor. Putting $J_{\pm} = \pm L_2 + iL_1$,

$J_3 = -iL_3$ we obtain for J_1, J_2, and J_3 the commutation relations

$$[J_+, J_-] = 2J_3, \qquad [J_3, J_\pm] = \pm J_\pm. \tag{6}$$

LEMMA 17. *Let V be a real vector space which transforms irreducibly under the rotation group according to the representation D^l. Then there exists a basis $\{f_m\}$ for the complexification of V such that*

$$J_3 f_m = m f_m, \tag{7a}$$

$$J_\pm f_m = \beta_{\pm m} f_{m \pm 1}, \tag{7b}$$

$-l \leqslant m \leqslant l$ *and* $\beta_m = \sqrt{(l - m)(l + m + 1)}$. *In addition, the f_m can be normalized so that*

$$\bar{f}_m = (-1)^m f_{-m}. \tag{8}$$

The relations (7a), (7b) can be derived entirely from the commutation relations (6) by standard methods [64]. We shall work with representations of $O(3)$ such that $D^l(R) = -D^l(R)$ for a reflection of R (see [64]).

Let $\mathfrak{N}$ be the kernel of the linearized operator $L_0 = G_u(\lambda_c, 0)$. Identify $\mathfrak{N}$ with linear polynomials in the variables $z_{-l}, \ldots, z_l$ which transform under the Lie algebra according to (7). The algebra $K[z_{-l}, \ldots, z_l]$ of polynomials in the variables $z_{-l}, \ldots, z_l$ is isomorphic to the algebra of symmetric tensors over $\mathfrak{N}$. Extend the operators J_3, $J_\pm$ to be *derivations* on the algebra K:

$$J(\alpha f + \beta y) = \alpha Jf + \beta Jg, \qquad J(fg) = (Jf)g + f(Jg)$$

where α, β are scalars and f, g are polynomials in K. Let the bifurcation equations be

$$F_m(\lambda, z_{-l}, \ldots, z_l) = 0.$$

These will be covariant with respect to D^l provided the F_m transform as the z_m; that is,

$$J_3 F_m = m F_m, \qquad J_\pm F_m = \beta_{\pm m} F_{m \pm 1}$$

where J_3, J_+, and J_- act as derivations on F_m. For example, the quadratic polynomials F_m are obtained as follows. The action of J_3 on $z_j z_k$ is

$$J_3(z_j z_k) = (J_3 z_j) z_k + z_j(J_3 z_k) = (j + k) z_j z_k$$

so $J_3 z_j z_k = m z_j z_k$ if and only if $j + k = m$. Therefore

$$F_m = \sum_{m_1 + m_2 = m} a_{m_1 m_2 m} z_{m_1} z_{m_2}.$$

In particular, when l is even,

$$F_l = a_0 z_l z_0 + a_1 z_{l-1} z_1 + \cdots + a_{l/2}(z_{l/2})^2.$$

Furthermore, $J_+ F_l = \beta_l F_l = 0$ and this condition gives us a set of linear equations for the coefficients $a_0, \ldots, a_{l/2}$. In the case $l = 2$,

$$F_2 = a z_2 z_0 + b z_1^2,$$

$$J_+ F_2 = a\beta_0 z_2 z_1 + 2b\beta_1 z_1 z_2 = (a\beta_0 + 2b\beta_1) z_1 z_2 = 0,$$

so $a\beta_0 + 2b\beta_1 = 0$. The last equation determines the coefficients a and b, hence F_2, up to a scalar multiple. Once F_l is known we get F_{l-1} from

$$J_- F_l = \beta_{-l} F_{l-1}$$

and so forth. In this way we construct all the F_m's.

This procedure extends immediately to higher-order terms. For example, to get third-order terms we write

$$F_l = \sum_{i+j+k=l} a_{ijk} z_i z_j z_k$$

and apply $J_+ F_l = 0$ to get a linear system of equations for the a_{ijk}. For $l = 1$ there is only one solution but for $l = 3$ there are two independent solutions [86].

For even l the quadratic terms of the covariant mapping are given by

$$F_m = \sum_{m_1+m_2=m} (-1)^m \begin{pmatrix} l & l & l \\ m_1 & m_2 & -m \end{pmatrix} z_{m_1} z_{m_2}$$

where

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

are the Wigner $3j$ symbols for the rotation group. For odd l the covariant quadratic mappings are antisymmetric, and one must go to third-order terms in the bifurcation equations. For even l the quadratic terms possess a gradient structure, as follows. Consider the homogeneous polynomial of degree 3,

$$p(z_{-l}, \ldots, z_l) = \frac{1}{3} \sum_{-l}^{l} F_m \bar{z}_m,$$

restricted to the real subspace of $\mathfrak{N}$ for which $\bar{z}_m = (-1)^m z_{-m}$. There we have

$$p(z_{-l}, \ldots, z_l) = \frac{1}{3} \sum_{-l}^{l} (-1)^m F_m z_{-m}$$

$$= \frac{1}{3} \sum_{m_1,m_2,m_3=-l}^{l} \begin{pmatrix} l & l & l \\ m_1 & m_2 & m_3 \end{pmatrix} z_{m_1} z_{m_2} z_{m_3}.$$

For even l the 3_j symbols are completely symmetric in m_1, m_2, m_3 and therefore

$$\frac{\partial p}{\partial z_m} = F_m(z_{-l}, \ldots, z_l).$$

Therefore the reduced bifurcation equations (linear plus quadratic terms) take the form

$$\sigma z_m + \frac{\partial p}{\partial z_m} = 0,$$

which are the Euler-Lagrange equations for the variational problem $\text{Max}_{|z|=1} p$ where $|z|^2 = \sum_{-l}^{l} (-1)^m z_m z_{-m}$. The function p is the third-order invariant for D^l; that is, $p(D^l z) = p(z)$ and the Euclidean norm $|z|^2$ is the second-order invariant.

Leon Green and I have succeeded in casting this variational problem in a slightly different way. For even l we have the Clebsch-Gordon series

$$D^{l/2} \otimes D^{l/2} = D^l \oplus D^{l-1} \oplus \cdots \oplus D^0 \tag{9}$$

and the associated representation

$$U_g A = D^{l/2}(g) A D^{l/2}(g^{-1})$$

on $(l+1) \times (l+1)$ matrices A. This representation is unitary relative to the inner product

$$\langle A, B \rangle = \tfrac{1}{2} \operatorname{tr} AB^* \tag{10}$$

where B^* is the Hermitian conjugate of B. The third-order invariant (there is only one since $D^l \otimes D^l \otimes D^l$ contains D^0 only once) is

$$p(A) = \tfrac{1}{3} \operatorname{tr} A^2 A^*.$$

The highest weight space, the one that transforms like D^l in (8) consists of Hermitian symmetric matrices, so we may rephrase our variational problem as Min $\frac{1}{3} \operatorname{tr} A^3$ subject to

$$\tfrac{1}{2} \operatorname{tr} A^2 = 1 \quad \text{and} \quad \operatorname{tr} AB_j = 0$$

where the B_j are symmetric matrices which lie in the lower weight invariant subspaces. In particular, $\operatorname{tr} AI = \operatorname{tr} A = 0$. For $l = 2$ (9) reads $D^1 \otimes D^1 = D^2 \oplus D^1 \oplus D^0$ but the matrices transforming as D^1 are antisymmetric; so we have only the constraints $\operatorname{tr} A = 0$, $\operatorname{tr} A^2 = 2$, and the Euler-Lagrange equations are $A^2 = \lambda A + \gamma I$ where A and I are 3×3 matrices. (The gradient of the functional $\frac{1}{3} \operatorname{tr} A^3$ is the mapping $A \to A^2$.) These equations can be completely solved as follows.

Taking the trace we get $\gamma = \frac{2}{3}$. When $l = 2$, A is a 3×3 symmetric matrix, and we can choose a rotation g such that $D^1(g) A D^1(g^{-1})$ is diagonal, since $D^1(g)$ ranges over all orthogonal matrices as g ranges over $O(3)$. So, assuming A is diagonal, we arrive at the equations

$$\mu_i^2 = \lambda \mu_i + \tfrac{2}{3}, \qquad i = 1, 2, 3$$

where μ_i are the eigenvalues of A. The constraints are

$$\mu_1^2 + \mu_2^2 + \mu_3^3 = 2, \qquad \mu_1 + \mu_2 + \mu_3 = 0.$$

There are two sets of solutions to these equations, namely,

$$\lambda = -\frac{1}{\sqrt{3}}, \qquad A = \begin{bmatrix} \dfrac{1}{\sqrt{3}} & 0 & 0 \\[2mm] 0 & \dfrac{1}{\sqrt{3}} & 0 \\[2mm] 0 & 0 & -\dfrac{2}{\sqrt{3}} \end{bmatrix}$$

and

$$\lambda = \frac{1}{\sqrt{3}}, \qquad A = \begin{bmatrix} \dfrac{1}{\sqrt{3}} & 0 & 0 \\ 0 & -\dfrac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \dfrac{2}{\sqrt{3}} \end{bmatrix}.$$

The order of the eigenvalues of the diagonal of A is immaterial, for any permutation of the diagonal entries of A produces a point on the same orbit. Indeed, any such permutation is accomplished by the operation PAP^{-1}, where P is a permutation matrix, and such a P is an element for $O(3)$. One of the orbits above gives the maximum of the functional $\frac{1}{3} \operatorname{tr} A^3$ on the sphere $\frac{1}{2} \operatorname{tr} A^2 = 1$; the other orbit gives the minimum. The isotropy subgroup in each case is $O(2)$ (rotations which leave

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

invariant), so each extremal is axisymmetric. The case $l = 2$ was first treated by Busse by another method. The above method, while quite straightforward in the case $l = 2$, becomes extremely complicated already in the case $l = 4$ and so does not seem to be a practical approach to the resolution of the bifurcation equations in the general case.

Busse has found a series of special solutions of the reduced bifurcation equations for all even l. These solutions belong to one of two special classes:

$$z_0 \neq 0, \qquad z_n, z_{2n} \neq 0, \qquad \tfrac{1}{3}l < n \le \tfrac{1}{2}l, \qquad z_m = 0 \text{ otherwise;} \quad (11a)$$

$$z_0 \neq 0, \qquad z_n \neq 0 \text{ for a single } n > l/2, \qquad z_m = 0 \text{ otherwise.} \quad (11b)$$

The axisymmetric solutions never give a minimum to the variational problem except in the case $l = 2$.

The stability analysis of the bifurcating solutions yields the following result [77], [86], [87].

THEOREM 18. *Suppose the reduced bifurcation equations have a gradient structure and that a solution v_0 is obtained as a maximum of the corresponding variational problem. Then one eigenvalue of the Jacobian $Q_\xi(\sigma_0, v_0)$ is negative and the rest are nonnegative.*

Accordingly, from Corollary 5, it follows that the corresponding bifurcating solution has one subcritical unstable mode (the branch is transcritical, that is, it appears as in Figure 4). Thus, for l even the bifurcating solutions are unstable on both sides of the branch point. The solution branch with one unstable mode, however, may "bend back and regain stability", as indicated in Figure 4. There is no general theorem to this effect, but it is a common feature of nonlinear problems. In such a situation there exists the possibility of a discontinuous jump from the symmetric (rotationally-invariant) state to the nontrivial branch possessing a smaller symmetry group. Such a situation

in an elastic structure would take the form of a sharp buckling below the critical parameter value (at subcritical loads).

6. *Bifurcation of waves.* We have seen how group-theoretic methods can be applied to analyze the bifurcation of stationary solutions at a multiple eigenvalue when the multiplicity is due to symmetry. The same methods are applicable when eigenvalues cross the imaginary axis but are multiple due to spatial symmetries in the problem. In this case we should expect to see symmetry breaking in space-time, which would manifest itself as the onset of wave-like phenomena. The relevant invariance group would be the group $G \times \mathbf{R}^+$ where G is the group of spatial symmetries and $\mathbf{R}^+$ is the group of time translations $t \to t + \delta$ (but not $t \to -t!$). This is the natural generalization of the Hopf bifurcation theorem to group-invariant problems.

The analysis of bifurcating waves at the onset of instability has at this point not yet been fully developed, although there has been some initial work on the subject. Turing [96] discusses the possibility of traveling wave solutions to a particular system of reaction and diffusion equations on a circle. Such solutions have been obtained in a model chemical reaction analyzed by Auchmuty and Nicholis [3]. Both rotating and standing wave solutions have been obtained by Erneux and Herschkowitz-Kaufman [27].

J. T. Stuart formally obtained traveling wave solutions (Tollmien-Schlicting waves) in his analysis of the onset of instability in plane Poiseuille flow [90]. Auchmuty [1] has obtained rigorous results on the bifurcation of waves in circular geometries.

To illustrate the use of group-theoretic methods in the analysis of bifurcating waves let me take the simple problem treated by the aforementioned authors and derive the structure of the bifurcation equations. Consider a bifurcation problem which is invariant under space and time translations and spatial reflections. The group operations are $T_\gamma x = x + \gamma$, $Rx = -x$, and $S_\delta t = t + \delta$. If the null functions have wave number k and frequency ω then the null space for the linear operator is four dimensional and is spanned by $\psi_1 = e^{i(kx + \omega t)}$, $\psi_2 = e^{-i(kx + \omega t)}$, $\psi_3 = e^{+i(kx - \omega t)}$, $\psi_4 = e^{-i(kx - \omega t)}$. Relative to $\psi_1, \ldots, \psi_4$ the group operations have the matrix representations

$$
T_\gamma = \begin{bmatrix} e^{ik\gamma} & & & 0 \\ & e^{-ik\gamma} & & \\ & & e^{ik\gamma} & \\ 0 & & & e^{-ik\gamma} \end{bmatrix}, \qquad
S_\delta = \begin{bmatrix} e^{i\omega\delta} & & & \\ & e^{-i\omega\delta} & & \\ & & e^{-i\omega\delta} & \\ & & & e^{i\omega\delta} \end{bmatrix}
$$

$$
R = \begin{bmatrix} & & & 1 \\ & 0 & 1 & \\ & 1 & 0 & \\ 1 & & & \end{bmatrix}.
$$

The problem can be reduced by the Lyapounov-Schmidt method to a system of bifurcation equations, and these must take the form

$$
F_i(\lambda, \omega, z_1, \ldots, z_4) = 0, \qquad i = 1, 2, 3, 4.
$$

Since we are looking for real solutions we want $z_2 = \bar{z}_1$, $z_4 = \bar{z}_3$ and $F_2 = \bar{F}_1$,

$F_4 = \bar{F}_3$. The symmetry $RF = FR$ leads to

$$F_4(\lambda, \omega, z_1, \ldots, z_4) = F_1(\lambda, \omega, z_4, z_3, z_2, z_1),$$

$$F_3(\lambda, \omega, z_1, \ldots, z_4) = F_2(\lambda, \omega, z_4, z_3, z_2, z_1).$$

Therefore it is enough to find F_1, for the other components can be found by applying R or taking complex conjugates.

Suppose F_1 contains a term $z_1^a z_2^b z_3^c z_4^d$. The symmetry $T_\gamma F = FT_\gamma$ implies

$$e^{ik\gamma}\left(a_1^a z_2^b z_3^c z_4^d\right) = \left(e^{ik\gamma}z_1\right)^a \cdots = e^{ik\gamma(a-b+c-d)}\left(z_1^a z_2^b z_3^c z_4^d\right)$$

hence $a - b + c - d = 1$. Similarly the symmetry $S_\delta F = FS_\delta$ leads to $a - b - c + d = 1$. Combining these two equations we get $a = b + 1$, $c = d$ so F_1 has the general term $z_1(z_1 z_2)^b(z_3 z_4)^d$, or $z|z|^{2b}|w|^{2d}$ where $z = z_1$ and $w = z_3$. The most general F_1 is therefore

$$F_1(\lambda, \omega, z, \bar{z}, \omega, \bar{\omega}) = zg(\lambda, \omega, |z|^2, |\omega|^2).$$

Then F_4 must be of the form

$$F_4(\lambda, \omega, z, \bar{z}, \omega, \bar{\omega}) = \omega g(\lambda, \omega, |\omega|^2, |z|^2).$$

The other two equations are complex conjugates of these so nothing new is obtained, and the bifurcation equations are

$$zg(\lambda, \omega, |z|^2, |\omega|^2) = 0, \qquad wg(\lambda, \omega, |w|^2, |z|^2) = 0$$

and the elements of the null space are of the form $z\psi_1 + \bar{z}\psi_2 + w\psi_3 + \bar{w}\psi_4 = 2\operatorname{Re}[z\psi_1 + w\psi_3]$. There are two basic solution types:

(i) $|z| = |w|$ and $g(\lambda, \omega, |z|^2, |z|^2) = 0$;

(ii) $w = 0$ and $g(\lambda, \omega, |z|^2, 0) = 0$.

The case $z = 0$ is equivalent to $w = 0$ under the reflection R.

If $|z| = |w|$ we take $z > 0$ by a suitable translation in x and then $w = ze^{i\gamma}$. Then the solution to first order is

$$z\left(\psi_1 + \psi_2 + e^{i\gamma}\psi_3 + e^{-i\gamma}\psi_4\right) = 2z(\cos(kk + \omega t) + \cos(kx - \omega t + \gamma)).$$

By suitable translations in x and t this is equivalent to $4z \cos kx \cos \omega t$; these waves appear as standing waves. In case (ii) with $w = 0$ we get $\psi_1 + \psi_2 = 2\cos(kx + \omega t)$, which are traveling waves. The possibility of two distinct modes of bifurcating waves was observed by Erneus and Herschkowitz-Kaufman and has also been discussed by Cowan and Ermentrout [12].

A stability theory has not yet been worked out for such bifurcating waves, but a stability analysis may produce a mechanism which will select one or the other mode of propagation, depending on the sizes of other parameters in the problem.

Symmetry breaking in natural phenomena. In this section I wish to discuss a variety of scientific disciplines in which spontaneous symmetry-breaking manifests itself as a primary feature of the problem.

1. *Morphogenesis*. A striking example of symmetry breaking in a biological system is the breakdown of rotational symmetry in the *Fucus* seaweed egg [53]. At a critical stage in the development of the egg a transition is made from a spherically symmetric membrane potential distribution to a polarized state with an axial symmetry, and a net transcellular current leaving one pole
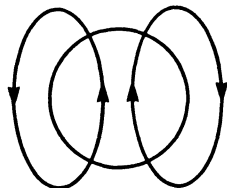
FIGURE 12

Self-electrophoresis currents in the *Fucus* egg

and entering the opposite. This phenomenon is termed "self-electrophoresis". According to Larter and Ortoleva [53], who are conducting a theoretical investigation of this effect, "The net transcellular potential gradient is believed to be essential in the development of the asymmetry which leads to dramatically different rhizoid and thallus cells after the first division of the egg". Larter and Ortoleva base their theory on models which allow for the full electrochemical nature of cellular phenomena. For slow processes these equations reduce to Poisson's equation and a system of rate-reaction equations:

$$\Delta V = \frac{-4\pi \mathscr{F}}{\varepsilon} \sum_{j=1}^{n} z_j C_j,$$

$$\frac{\partial C}{\partial t} i = \text{div}\left\{ \sum_{j=1}^{n} \mathscr{D}_{ij} \text{ grad } C_j + M_{ij} C_j \text{ grad } V \right\} + G_i(C)$$

where $\mathscr{F}$ is Faraday's constant, ε the dielectric constant, C_i are species concentrations, z_i are valences, $\mathscr{D}_{ij}$ is a matrix of diffusion coefficients, M_{ij} a matrix of mobilities, and $G_i(C)$ are chemical reaction rate terms.

The symmetry breakdown in the Fucus egg is of the form rotational invariance to axial invariance. That is, prior to self-electrophoresis the solutions are invariant under the entire rotation group $O(3)$, while the bifurcating solutions are invariant only under a subgroup of rotations about a fixed axis. The solutions thus appear in two-dimensional orbits with one-dimensional isotropy subgroup. This, however, is by no means the only symmetry breakdown which can occur in rotationally-invariant systems; and more complicated symmetry breakdowns are relevant in geophysical problems.

2. *Dynamo problem*. One problem in which nonaxisymmetric solutions are of interest is the *Dynamo problem*: By what mechanism is the earth's magnetic field sustained? The current consensus among geophysicists is that the magnetic field is sustained by electric currents flowing in the electrically conducting earth's core which interact with the convective fluid motions of that core. It is known (Cowlings theorem) that purely axisymmetric magnetic fields cannot satisfy the dynamo equation. The dynamo process in generators depends on a multiply-connected distribution of electrical conductivity, which permits a simple rotational motion. The uniform conductivity of a homogeneous, spherical dynamo may require a more complex velocity field for dynamo action. Thus the question of nonaxisymmetric flow patterns appearing at the onset of convection becomes extremely interesting for the Dynamo problem. The equations of dynamo theory are the convection

equations plus an equation governing the magnetic field:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} + 2\vec{\Omega} \times \vec{u} = -\nabla p - \beta \vec{g}\theta + \nu\Delta\vec{u} + \frac{1}{\mu}(\nabla \times \vec{B}) \times \vec{B}, \text{ (12a)}$$

$$\text{div } \vec{u} = 0, \tag{12b}$$

$$\frac{\partial \theta}{\partial t} + (\vec{u} \cdot \nabla)\theta = -u \cdot \nabla T_0 + \kappa\Delta\theta, \tag{12c}$$

$$\frac{\partial \vec{B}}{\partial t} = \text{curl}(\vec{u} \times \vec{B}) + \eta\Delta\vec{B} \tag{12d}$$

where T_0 is the base temperature profile in the absence of convection and $2\vec{\Omega} \times \vec{u}$ is the Coriolis term.

When no magnetic field is present the quadratic term $(\nabla \times \vec{B}) \times \vec{B}$ in the first equation vanishes, and we have a pure convection problem. When the convective velocity $\vec{u}$ reaches a magnitude and configuration which can sustain a growing magnetic field in (12d) then a bifurcation may take place and a nontrivial magnetic field may be sustained. Equation (12d) is linear so for a given $\vec{u}$ one should expect an exponentially growing magnetic field $\vec{B}$; but the nonlinear coupling between $\vec{B}$ and $\vec{u}$ via (12a) may be expected to prevent unlimited growth, so that a stable equilibrium is attained. Since (12d) is homogeneous in $\vec{B}$ some external magnetic field (for example, the Sun's) is required to "seed" the dynamo process. For further discussion and a survey of the literature, see the articles by Busse [9] and Roberts [72].

3. *Wave propagation in neural networks.* Bifurcation phenomena in simple mathematical models of excitatory inhibitory neuro-networks have been discussed recently by Cowan and Ermentrout [12]. Neural networks are aggregates of nerve cells which interact with other neurons in the network in either an excitatory or inhibitory way, and so it is plausible to expect these networks to exhibit such nonlinear collective phenomena as bifurcation, threshold effects, and hysteresis. Cowan and Ermentrout model these net-works by a system of equations

$$\mu Y = -Y + S(KY + P) \tag{13}$$

where Y is a two-component vector, S is a nonlinear vector-valued function, K is a linear convolution operator, and P is the external stimulus. Equation (13) may be studied in one, two, or three dimensions.

Cowan and Ermentrout seek to model the patterns of activity of the central nervous system by showing how organized space-time neuronal activity patterns can arise through the mechanisms of bifurcation from an initially uniform resting state. They investigate the structure of the bifurcation point when two pairs of complex conjugate eigenvalues cross the imaginary axis simultaneously. In that case one gets secondary bifurcation as some of the parameters in the problem are varied.

Cowan and Ermentrout have also treated hallucinatory phenomena from the standpoint of symmetry breaking bifurcations [13]. Recent experiments on mescaline induced hallucinations have led to the conclusion that most simple hallucinations could be classified into one of four categories: "(a) grating, lattice, honeycomb or chessboard; (b) cobweb; (c) funnel, tunnel, cone or

vessel; (d) spiral". Cowan and Ermentrout base their analysis on the conten-
tion that "simple formed hallucinations arise from an instability of the resting
state leading to concomitant spatial patterns of activity in the cortex. This
instability arises from a combination of enhanced excitatory modulation and
decreased inhibition. [We] demonstrate that such spatial patterns are a
property of neural nets with strong lateral interactions acting to provide a
dominant negative feedback. [We] formalize these postulates into a simple
mathematical model and then use bifurcation theory to demonstrate the
existence of the relevant spatial patterns".

The relevant spatial patterns are none other than those crystallographic
patterns which have already made their appearance in the Bénard problem,
with one additional factor. Experimental observations have established that in
primates there is a conformal transformation from the retinal field, which is
circular, to the cortical field, which has Cartesian (rectangular) symmetry.
This implies that the transformation from retinal polar coordinates to cortical
rectangular coordinates must be essentially logarithmic in nature. Such a
logarithmic transformation would take a tunnel pattern consisting of con-
centric circles of activity to a pattern of rolls parallel to the y axis. Similarly,
spirals are transforms of rolls with some other direction. Thus the patterns
observed in hallucinatory phenomena are images under the log transforma-
tions of the cellular patterns familiar in the analysis of the Bénard problem:
hexagons, squares, rectangles, and rolls. Cowan and Ermentrout then assume
that, as some parameter λ increases, the strength of the excitation increases
until, beyond some critical value λ_0, the rest state becomes unstable and gives
way to the stationary patterns of spatial activity. Thus, according to their
theory, the drug-induced hallucinatory patterns are precisely those which one
would see when Euclidean invariance is broken.

4. *Phase transitions in statistical mechanics*. The notion of symmetry break-
ing is fundamental to phase transitions, yet much harder to treat mathemati-
cally. Until the renormalization theories developed in recent years, the
primary approach to phase transitions was, in one way or another, a mean-
field approximation coupled with a bifurcation analysis of the mean-field
equations. The simplest mean-field theories for critical phenomena were the
scalar equations of state, such as the Van der Waals equation for a gas of the
Curie-Weiss model for a ferromagnet. In more elaborate theories the state of
the ensemble is described, for example, by a single particle density function,
and an integral equation is derived for this function by some kind of closure
hypothesis for the hierarchy of higher-order (multiple particle) correlation
functions. This line of analysis was first pursued by Kirkwood and Monroe,
[48] and more recently by Raveché and Stuart [70], and Rice and his
coworkers. (See [50], [101].) Nevertheless, these approximations are still
mean-field theories, and depend, for their validity, on the assumption that
fluctuations are negligible; the major difficulty is that in many cases, large
fluctuations become important precisely at the critical point. In fact, at a
critical point the fluctuations very often diverge to infinity, making the
mean-field approximation invalid, and it is this fact which accounts for the
deviation of the critical exponents from the "classical exponents" predicted
by bifurcation (mean-field) models. All this notwithstanding, the bifurcation

models do have some areas of validity, and are generally successful in predicting the symmetry changes actually observed. Landau's [6], [56] theory of second-order phase transitions is a phenominological description of phase transitions which is essentially a theory of "symmetry-breaking bifurcations".

This point of view has been expressed very concisely by Tareva and Trapezina [92], [93]: "The generalized mean-field approximation usually brings us to the formulation of the broken-symmetry problem in terms of the bifurcation on a nonlinear integral equation solution for the Bogolyubov quasi-average. Particularly in a number of papers the liquid-solid phase transition is considered as a bifurcation of the solution of the equation of Hammerstein type

$$\Phi(r_1) - \mu \int K(r_1, r_2) f(\Phi(r_2), r_2) \, dr_2 = 0".$$

The phase transitions of the ensemble are described in terms of bifurcations of this integral equation. See also the review article by Kozak [50].

In the area of nonequilibrium thermodynamics the operation of the laser can be described by a mean-field theory which is amenable to a bifurcation analysis. In the Dicke-Haken-Lax model of the laser it is possible to describe the many body photon field by a mean-field theory as N (the number of degrees of freedom) tends to infinity. Thus it is possible in this case to solve a nonlinear quantum mechanical model, far from equilibrium, by reducing the problem to a system of ordinary differential equations for the expectation values of the extensive variables. (See Haken [23], and Hepp and Lieb [26].) The onset of laser action in these theories is then described by the bifurcation of time-periodic solutions from the equilibrium solution, that is, so-called Hopf bifurcation.

5. *Symmetry breaking in elementary particle physics.* The notion of "spontaneous symmetry breaking" plays a fundamental role in quantum field theory. It is not possible for me to discuss this in depth here, but let me briefly describe the work of Michel and Radicati [61], [62], [63] in symmetry breaking in elementary particle theory. They study the adjoint action of $SU(3)$ on the vector space Q of traceless Hermitian matrices A. The inner production on Q is $(A, B) = \frac{1}{2} \operatorname{tr} AB^*$, and the group action is $A \to UAU^*$ where $U \in SU(3)$. This action leaves the inner product invariant. There are two third-order invariants of this action, namely

$$\{A, B, C\} = \frac{1}{2}\sqrt{3} \ \operatorname{tr}(AB + BA)C, \qquad \{A, B, C\} = \frac{-i}{2} \operatorname{tr}[A, B]C$$

where $\{ \ , \ , \ \}$ is completely symmetric and $[\ , \ , \ ]$ is antisymmetric. These two third-order invariants induce two algebras on Q via the following device. The form $C \to \{A, B, C\}$ is a linear functional on Q, so may be represented in the form $\{A, B, C\} = (A \vee B, C)$ where $A \vee B$ is an element of Q depending on A and B. It is clear that $A \vee B$ is bilinear and so defines a multiplication from $Q \times Q$ to Q. It is easily seen that $A \vee B$ must take the form

$$A \vee B = \frac{\sqrt{3}}{2}(AB + BA) - \frac{1}{\sqrt{3}} \operatorname{tr} AB.$$

The operation $\vee$ is commutative but not associative. The other algebra is

induced by the antisymmetric third-order invariant. This induces a skew product $\wedge$:

$$A \wedge B = \frac{-i}{2}[A, B], \qquad [A, B, C] = (A \wedge B, C).$$

The algebra Q with product $\wedge$ is a Lie algebra.

The important physical quantities in Gell-Mann's "Eight fold way", namely the electric charge, hypercharge, etc., correspond to solutions of the equation

$$q \vee q + \eta(q)q = 0 \tag{14}$$

where $\eta(q)$ is a real number. A solution q of this equation is an *essential idempotent* of the $\vee$ algebra. These essential idempotents are the directions of symmetry breaking for the adjoint representation: that is, they are the critical points of the invariant functional $\{A, A, A\}$ on the unit sphere $(A, A) = 1$.

Equation (14) is mathematically identical to the reduced bifurcation equations which we have already encountered, and in fact are the $SU(3)$ equivalent of the equations which arise in the analysis of bifurcation from rotationally-invariant states. A general algebraic theory of such equations and their solutions would be extremely useful.

I regard the problem of developing algebraic algorithms for finding all solutions of (14), their stability, and their symmetry, for a general irreducible representation of a compact group, as the most important problem to be resolved in the theory of symmetry breaking.

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SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455