BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 2, March 1977

AUTOMORPHIC CUSP FORMS CONSTRUCTED FROM THE WEIL REPRESENTATION

BY S. RALLIS AND G. SCHIFFMANN

Communicated by J. A. Wolf, September 10, 1976

We recall the notation and results of [3].

Let $\mathbf{Q}$ be the rational numbers.

We let L be a Q integral lattice in $\mathbf{Q}^k$, i.e. $Q(\xi_1, \xi_2) \in \mathbf{Z}$ for all $\xi_1, \xi_2 \in L$. Let $L_*(Q)$ be the Q dual of L, i.e. $L_*(Q) = \{\eta \in \mathbf{R}^k \mid Q(\eta, \xi) \in \mathbf{Z}, \forall \xi \in L\}$. Then $L_*(Q)/L$ is a finite Abelian group, and we let N_L be the exponent of $L_*(Q)/L$, i.e. the smallest positive integer x so that $x \cdot \xi \in L$ for all $\xi \in L_*(Q)$. Choosing a $\mathbf{Z}$-basis X_i of L, we let $D_{Q(L)} = \det\{Q(X_i, X_j)\}$. Then the integer $D_{Q(L)}$ is independent of the choice of basis of L.

Then we define

$$\Gamma_L(Q) = \{g \in O(Q) \mid g(L) = L\}$$

and

$$\Gamma^L(Q) = \left\{\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \epsilon\right) \middle| \, a, b, c, d \in \mathbf{Z}, ad - bc = 1, \right.$$
$$\left. b \equiv 0 \bmod 2 \text{ and } c \equiv 0 \bmod 2N_L \right\}.$$

Then $\Gamma_L(Q)$ is an arithmetic subgroup of $O(Q)$ and $\Gamma^L(Q)/$(cyclic group of order 4) is an arithmetic subgroup of $\mathrm{PSl}_2(\mathbf{R})$ (contained in the Γ_ϑ theta group). Then using the corollary to Theorem 5 of [3] we have

THEOREM 1. *Let φ be a $\widetilde{K} \times K$ finite function in $\mathbf{F}_Q^+(s^2 - 2s)$ with $s > \frac{1}{2}k$. Then the sum with $(G, g) \in \widetilde{\mathrm{Sl}_2} \times O(Q)$,*

$$T_\varphi^L(G, g) = \sum_{\xi \in L} \pi_Q(G, g)^{-1}(\varphi)(\xi), \tag{1.1}$$

is absolutely convergent. Moreover, for $(\Omega, \gamma) \in \Gamma^L(Q) \times \Gamma_L(Q)$, we have the functional equation

$$T_\varphi^L(G\Omega, g\gamma) = \sigma_Q^L(\Omega, \gamma) T_\varphi^L(G, g), \tag{1.2}$$

where σ_Q^L is a unitary character on $\Gamma^L(Q) \times \Gamma_L(Q)$ taking values in S_4 (where $S_j = \{z \in \mathbf{C} \mid z^j = 1\}$ for j any positive integer). Moreover, T_φ^L is a C^∞ function on $\widetilde{\mathrm{Sl}_2} \times O(Q)$ satisfying $D * T_\varphi^L(G, g) = T_{\pi_Q(D)\varphi}^L(G, g)$ for any D in the

---

AMS (MOS) subject classifications (1970). Primary 10D20.

universal enveloping algebra of $\widetilde{\mathrm{Sl}_2} \times O(Q)$ ($*$ *represents differentiation on the left*). *In particular,* $\omega_{\mathrm{Sl}_2} * T^L_\varphi = (s^2 - 2s)T^L_\varphi$. *Finally we have the estimate*

$$|T^L_\varphi(G, g)| \leqslant M r_G^{s-1/2} \|g^{-1}\|_k^{s+k/2-2} \tag{1.3}$$

where M *is some positive constant independent of* (G, g), r_G *denotes the* A *part of* G *in the Iwasawa decomposition of* $G = K_G a(r_G) n(x_G)$, *and* $\| \;\|_k$ *denotes the Frobenius norm of a linear operator on* $\mathbf{R}^k$.

REMARK 1. The function T^L_φ is an automorphic form on $\widetilde{\mathrm{Sl}_2} \times O(Q)$ in the sense of the definitions in [1].

REMARK 2. The unitary character σ^L_Q on $\Gamma^L(Q) \times \Gamma_L(Q)$ is given as $\sigma^L_Q(\Omega, \omega) = c(\Omega)$, where the map $\Omega \rightsquigarrow c(\Omega)$ on $\Gamma^L(Q)$ is given by

$$c\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \epsilon\right) = (\operatorname{sgn} \epsilon)^k b_\delta^k \left(\frac{2\gamma}{\delta}\right)^k \left(\frac{D_{Q(L)}}{\delta}\right)$$

where $\gamma \neq 0$ with

$$b_\delta = \begin{cases} 1 & \text{if } \delta \equiv 1 \bmod 4, \\ \sqrt{-1} & \text{if } \delta \equiv 3 \bmod 4, \end{cases}$$

and $(\overline{\;\;})$ the quadratic residue symbol as given in [4].

Using Remark 2 we then construct on $P = \{z \in \mathbf{C} \mid \operatorname{Im}(z) > 0\}$, the upper half plane, a half-integral multiplier system for the discrete arithmetic group

$$\Delta_{N_L} = \left\{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{Sl}_2(\mathbf{Z}) \mid \gamma \equiv 0 \bmod 2N_L, \beta \equiv 0 \bmod 2\right\}$$

of degree s, taking values in S_4. That is: if $v_Q(G) = \{c((G, 1))\}^{-1} \psi_2(G)$ with

$$\psi_2(G) = \begin{cases} 1 & \text{if } c_G \neq 0, \\ \operatorname{sgn}(d_G) & \text{if } c_G = 0, \end{cases}$$

then

$$v_Q(G_1 G_2)(c_3 z + d_3)^s = v_Q(G_1) v_Q(G_2)(c_1 z + d_1)^s (c_2 z + d_2)^s,$$

where $G_1, G_2 \in \Delta_{N_L}$ with

$$G_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \quad \text{and} \quad G_1 G_2 = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}$$

(where $z^s = |z|^s e^{\sqrt{-1}(\arg z)s}$ with $-\pi < \arg z \leqslant \pi$).

Then using Theorem 1 and the corollary to Theorem 5 of [3], we deduce the following.

THEOREM 2. *Let φ be a function belonging to $E_Q(s^2 - 2s, s, s_1, 0)$ (with $s > \frac{1}{2}k$ and $s_1 = s - \frac{1}{2}(a - b)$) of the form on Ω_+: $\varphi(X) =$*

$$Q(X, X)^{s-1} e^{-\pi Q(X,X)} \|X_+\|^{-(s+s_1+k/2-2)} Q(X, \xi_+)^{s_1},$$

where $\xi_+ \in \mathbf{C}^a$ is a nonzero complex isotropic vector, i.e. $Q(\xi_+, \xi_+) = 0$. Then we let

$$\widetilde{T}^L_\varphi(z, g) = (\operatorname{Im} z)^{s/2}\, T^L_\varphi\left(\left(\begin{bmatrix} x & y \\ 0 & x^{-1}\end{bmatrix}, 1\right), g\right) \tag{1.4}$$

with $z = -y/x + \sqrt{-1}\,x^2 \in P$. Then we have the expansion

$$\widetilde{T}^L_\varphi(z, g) = \sum_{n\in \mathbf{Z};\, n\geq 1} n^{s-1} e^{\pi\sqrt{-1}\,zn} \varphi_n^{s_1}(g), \tag{1.5}$$

where

$$\varphi_n^{s_1}(g) = \sum_{\{M\in L \mid Q(M,M)=n\}} Q(M, g^{-1}\xi_+)^{s_1} \|(gM)_+\|^{-(s+s_1+k/2-2)}$$

Then $\mathbf{T}^L_\varphi(z, g)$ is an antiholomorphic cusp form in z for Δ_{N_L} of degree $|s|$ *form for Δ_{N_L} with multiplier v_Q of degree s, that is*

$$\widetilde{T}^L_\varphi\left(\frac{az+b}{cz+d}, g\right) = v_Q(G)\,(cz+d)^s\, \widetilde{T}^L_\varphi(z, g)$$

with $G = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] \in \Delta_{N_L}$. Moreover, we have that $\widetilde{T}^L_\varphi(z, g)$ is a cusp form in z for Δ_{N_L}, that is, $\widetilde{T}^L_\varphi$ is holomorphic at ∞ (from 1.5) *and $\widetilde{T}^L_\varphi\,(u + \sqrt{-1}\,v, g) = O(v^{-(s-1/4)})$ as $v \longrightarrow 0$ uniformly in u.*

REMARK 3. Choosing the quadratic form $Q = xy + zw$ on $\mathbf{R}^4$ and a suitable Q integral lattice $L \subseteq \mathbf{R}^4$, we obtain from the construction above automorphic forms similar to $\Omega(\tau_1, \tau_2, z)$ in [6]. We also note the construction of related automorphic forms in [2] and [5] for the case $k = 3$.

REMARK 4. From the invariance of $\widetilde{T}^L_\varphi$ in the $O(Q)$ variable relative to $\Gamma_L(Q)$, we see that $\varphi_n^{s_1}(g\gamma) = \varphi_n^{s_1}(g)$ for all $g \in O(Q)$, $\gamma \in \Gamma^L(Q)$. The interpretation of formula (1.5) for $\widetilde{T}^L_\varphi$ is simply the Fourier expansion of $\widetilde{T}^L_\varphi$ at ∞ with each Fourier coefficient $\varphi_n^{s_1}(g) n^{s-1}$ an automrophic form for $O(Q)$ relative to $\Gamma_L(Q)$.

REMARK 5. In a manner similar to the construction above (with the added assumption that $b = 2$), we start with the function $\varphi \in E_Q(s^2 + 2s, s, 0, s_2) \subseteq \mathbf{F}^-_Q(s^2 + 2s)$ given by

$$\varphi(Y) = |Q(Y, Y)|^{|s|-1} e^{\pi Q(Y,Y)} Q(X, \xi_-)^{-s_2} \quad \text{on } \Omega_-$$

where $s < -\frac{1}{2}k$ and $s_2 = |s| + \frac{1}{2}a - 1$ and $\xi_- \in \mathbf{C}^b$, nonzero complex isotropic, i.e. $Q(\xi_-, \xi_-) = 0$. Then as in Theorem 2 we let

$$\widetilde{T}_{\varphi}^{L}(z, g) = \sum_{n \in Z; n \leqslant -1} |n|^{|s|-1} e^{\pi\sqrt{-1}nz} \widetilde{\varphi}_n^{s_2}(g), \tag{1.6}$$

where

$$\widetilde{\varphi}_n^{s_2}(g) = \sum_{\{M \in L \mid Q(M,M)=n\}} Q(M, g^{-1}\xi_-)^{-s_2},$$

$z \in \overline{P}$ = lower half plane.

Then $\mathbf{T}_{\varphi}^{L}(z, g)$ is an antiholomorphic cusp form in z for Δ_{N_L} of degree $|s|$ with multiplier v_Q.

Then we can analyze the cuspidal behavior of each $\widetilde{\varphi}_n^{s_2}$ determined in Remark 5.

THEOREM 3. *Let $b = 2$ and let $\widetilde{\varphi}_n^{s_2}$ be as in Remark 5. Then for the unipotent radical H of any rational maximal parabolic subgroup of $O(Q)$ we have*

$$\int_{H/H \cap \Gamma_L(Q)} \widetilde{\varphi}_n^{s_2}(gh)dh \equiv 0$$

(with dh an H invariant measure on $H/H \cap \Gamma_L(Q)$) for all $g \in O(Q)$ and all $n \leqslant -1$.

REMARK 6. Theorem 3 implies that the family of automorphic forms $\widetilde{\varphi}_n^{s_2}$ belongs to the *space of cusp* forms (in the sense of [1]) of $L^2(O(Q)/\Gamma_L(Q))$.

The case $b = 2$ turns out to be critical in the proof of Theorem 3. The basic idea behind the proof of Theorem 3 is what we call the *Cusp Vanishing Theorem.*

THEOREM 4. *Let $\varphi \in \mathbf{F}_Q^-(s^2 + 2s)$ be a $\widetilde{K} \times K$ finite function with $b = 2$ and $s < -\frac{1}{2}k$. Then for any $X \in \Omega_-$ and for the unipotent radical H of any rational maximal parabolic subgroup of $O(Q)$, $\int_{H/H^X} \varphi(gh(X))\, d\mu_x(h) \equiv 0$ for all $g \in O(Q)$ (with $d\mu_x$ some H invariant measure on H/H^X, H^X = isotropy group of X).*

Again we note the importance of the case $b = 2$. If $b = 2$, then $O(Q)/K$ is a Hermitian symmetric space. We let $\mathfrak{F} = \mathfrak{k} + p$ be the Cartan decomposition of the Lie algebra of $O(Q)$. Then we have the direct sum $\mathfrak{F}_C = \mathfrak{k}_C \oplus p^+ \oplus p^-$, where p^- and p^+ span the holomorphic and antiholomorphic tangent vectors at the "origin" in $O(Q)/K$. Then we recall the construction of a family of holomorphic discrete series representations of $O(Q)$. We consider $K = O(a) \times O(2)$, and let χ_n: $K \longrightarrow S^1$ be the unitary character on K which is trivial on $O(a)$ and maps

$$O(2) = \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \middle| -\pi < \theta \leqslant \pi \right\}$$

to $e^{\sqrt{-1}n\theta}$ $(n \in \mathbf{Z})$. Then we form the "holomorphic" unitarily induced representation space $H(O(Q)/k, \chi_n) = \{\varphi: O(Q) \longrightarrow \mathbf{C} \mid \varphi(gk) = \varphi(g)\chi_n(k)$ for all $g \in O(Q)$, $k \in K$, $\varphi * W \equiv 0$ for all $W \in p^+$, and $\int_{O(Q)/K} |\varphi(g)|^2 \, d\sigma(g) < \infty\}$ with $* W$, convolution on the left and $d\sigma$ some $O(Q)$ invariant measure on $O(Q)/K$. Then we have

THEOREM 5. *The representation on $O(Q)$ in A_s^-* (see Remark 1 in [3]) *is equivalent to the "holomorphic" induced representation of $O(Q)$ in $H(O(Q)/K, \chi_{s_2})$ where $s_2 = |s| + \frac{1}{2}a - 1$.*

REMARK 7. The representation of $O(Q)$ in A_s^- (for $b = 2$) is thus always "square integrable". Moreover A_s^- is "integrable" if $s < 2 - k$.

COROLLARY TO THEOREM 5. *Let $s < 2 - k$. Then each $\widetilde{\varphi}_n^{s_2}$ given in Remark 5 is a "Poincaré series" on $O(Q)/\Gamma_L(Q)$. That is, there exists a K finite function $q_n \in H(O(Q)/K, \chi_{s_2})$ so that $\widetilde{\varphi}_n(g) = \Sigma_{\gamma \in \Gamma_L(Q)} q_n(\gamma g^{-1})$.*

BIBLIOGRAPHY

1. Harish-Chandra, *Automorphic forms on semisimple Lie groups*, Lecture Notes in Math., vol. 62, Springer-Verlag, Berlin and New York, 1968. MR 38 #1216.

2. S. Niwa, *Modular forms of half integral weight and the integral of certain theta-functions*, Nagoya Math. J. 56 (1975), 147–161. MR 51 #361.

3. S. Rallis and G. Schiffmann, *Discrete spectrum of the Weil representation*, Bull. Amer. Math. Soc. **83** (1977), 267–270.

4. G. Shimura, *On modular forms of half integral weight*, Ann. of Math. (2) 97 (1973), 440–481. MR 48 #10989.

5. T. Shintani, *On construction of holomorphic cusp forms of half integral weight*, Nagoya Math. J. 58 (1975), 83–136.

6. D. Zagier, *Modular forms associated to real quadratic fields* Invent. Math. **30** (1975), 1–48.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556

INSTITUTE DE RECHERCHE MATHÉMATIQUE AVANCÉE, UNIVERSITÉ LOUIS PASTEUR, 7, RUE RENÉ DESCARTES, 67084–STRASBOURG, CEDEX, FRANCE (Current address of G. Schiffmann)

Current address (Stephen Rallis): Department of Mathematics, Princeton University, Princeton, New Jersey 08540