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DISCRETE SPECTRUM OF THE WEIL REPRESENTATION

BY S. RALLIS AND G. SCHIFFMANN

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1. Weil representation. Let Q be a nondegenerate quadratic form on $\mathbf{R}^k$. Let $O(Q)$ be the orthogonal group of Q. One owes to A. Weil [4] the construction of a certain unitary representation π_Q of the group $\widetilde{\mathrm{Sl}_2} \times O(Q)$ in $L^2(\mathbf{R}^k)$, where $\widetilde{\mathrm{Sl}_2}$ is a two fold covering of $\mathrm{Sl}_2(\mathbf{R})$, i.e. given by pairs (g, ϵ) with $g \in \mathrm{Sl}_2(R)$ and $\epsilon = \pm 1$ satisfying the group law $(g, \epsilon)(g', \epsilon') = (gg', V(g, g')\, \epsilon\epsilon')$, where V is the Kubota cocycle on $\mathrm{Sl}_2(\mathbf{R})$ (with values in $\mathbf{Z}_2$). Let $w_0 \in \widetilde{\mathrm{Sl}_2}$ be the element $\left(\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], -1\right)$. Then π_Q is given by

$$\text{(i)} \qquad \pi_Q(w_0)\varphi(X) = \delta_Q \hat{\varphi}(-M_Q(X)), \; \varphi \in L^2(\mathbf{R}^k),$$

where $M_Q \in \mathrm{Aut}(\mathbf{R}^k)$ so that $[X, M_Q(Y)] = Q(X, Y)$ for all $X, Y \in \mathbf{R}^k$ (with $[\ ,\ ]$ the usual dot product on $\mathbf{R}^k$) and $\delta_Q = |\det Q|^{-1/2} u_Q$ with u_Q a certain eighth root of unity determined explicitly in [2]. Moreover, ^ denotes the Fourier transform on $L^2(\mathbf{R}^k)$. Also we have

$$\text{(ii)} \qquad \pi_Q\left(\begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix}, 1\right)\varphi(X) = |\alpha|^{k/2} e^{\sqrt{-1}\,\pi\beta\alpha Q(X,X)}\varphi(\alpha X), \quad \text{with } \alpha > 0$$

and

$$\text{(iii)} \qquad \pi_Q(g)\varphi(X) = \varphi(g^{-1}X) \quad \text{for } g \in O(Q).$$

Then (i), (ii), and (iii) determine π_Q explicitly. The main problem is to give a spectral decomposition of π_Q.

2. Discrete spectrum of π_Q. Let $\widetilde{K}$ be the maximal compact subgroup of $\widetilde{\mathrm{Sl}_2}$ given by

$$\left\{\left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \epsilon\right) \Big| -\pi \leqslant \theta < \pi, \epsilon = \pm 1\right\}.$$

Then every unitary character of K is given by

$$k(\theta, \epsilon) \rightsquigarrow (\operatorname{sgn} \epsilon)^{2m} e^{\sqrt{-1}\, m\theta} \quad \text{with } m \in \tfrac{1}{2}\mathbf{Z}.$$

We let

$$A = \left\{ a(r) = \left(\begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix}, 1\right) \Big| r > 0 \right\}$$

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and

$$N = \left\{ n(x) = \left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, 1 \right) \mid x \in \mathbf{R} \right\}.$$

Let $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{k}$ be the infinitesimal generators of A, N, and K, respectively. Then

$$\omega_{\mathrm{Sl}_2} = -\mathfrak{k}^2 + \mathfrak{a}^2 + (\mathfrak{n} + \mathrm{Ad}(w_0)\mathfrak{n})^2$$

is the Casimir element of $\widetilde{\mathrm{Sl}_2}$. We let

$$E_+ = \mathfrak{k} + \sqrt{-1}\,(\mathfrak{n} + \mathrm{Ad}(w_0)\mathfrak{n}) \quad \text{and} \quad E_- = k - \sqrt{-1}\,(\mathfrak{n} + \mathrm{Ad}(w_0)\mathfrak{n}).$$

We assume that Q has inertia type (a, b) where $a \geqslant b \geqslant 1$ and $a + b = k \geqslant 3$. Then we choose a splitting of Q on $\mathbf{R}^k = \mathbf{R}^a \oplus \mathbf{R}^b$ so that $X = X_+ + X_-$ with $X_+ \in \mathbf{R}^a$, $X_- \in \mathbf{R}^b$ and $Q(X, X) = \|X_+\|^2 - \|X_-\|^2$ ($\|\ \|$ = usual length of vector in $\mathbf{R}^t$).

We consider $\mathbf{F}_Q(\lambda) = \{\varphi \in \mathbf{F}_Q \mid \omega_{\mathrm{Sl}_2} \cdot \varphi = \lambda\varphi\}$ where $\mathbf{F}_Q$ is the space of C^∞ vectors in $L^2(\mathbf{R}^k)$ of π_Q. Let $\Omega_+ = \{X \mid Q(X, X) > 0\}$ and $\Omega_- = \{X \mid Q(X, X) < 0\}$.

THEOREM 1. *The spaces* $\mathbf{F}_Q^+(\lambda) = \{\varphi \in \mathbf{F}_Q(\lambda) \mid \varphi|_{\Omega_-} \equiv 0\}$ *and* $\mathbf{F}_Q^-(\lambda) = \{\varphi \in \mathbf{F}_Q(\lambda) \mid \varphi|_{\Omega_+} \equiv 0\}$ *are* $\widetilde{\mathrm{Sl}_2} \times O(Q)$ *stable subspaces. Moreover,* $\mathbf{F}_Q^+(\lambda)$ *and* $\mathbf{F}_Q^-(\lambda)$ *(if nonzero) determine topologically irreducible representations of* $\widetilde{\mathrm{Sl}_2} \times O(Q)$ *which are inequivalent. Also* $\mathbf{F}_Q(\lambda)$ *is the direct sum of* $\mathbf{F}_Q^+(\lambda)$ *and* $\mathbf{F}_Q^-(\lambda)$.

We let

$$L^2(\mathrm{Whit}) = \left\{ f\colon \widetilde{\mathrm{Sl}_2} \to \mathbf{C} \mid f(gn(x)) = f(g)e^{2\pi\sqrt{-1}x} \right.$$
$$\left. \text{for all } g \in \widetilde{\mathrm{Sl}_2},\ x \in \mathbf{R} \text{ and } \int_{\widetilde{\mathrm{Sl}_2}/N} |f(g)|^2\, d\mu(g) < \infty \right\},$$

where $d\mu$ is an $\widetilde{\mathrm{Sl}_2}$ invariant measure on $\widetilde{\mathrm{Sl}_2}/N$. We consider the subspace $L^2(\mathrm{Whit})_\lambda = \{\psi \in L^2(\mathrm{Whit})_\infty \mid \omega_{\mathrm{Sl}_2} * \psi = \lambda\psi\}$. (($)_\infty$ denotes C^∞ vectors of representation.)

THEOREM 2. *The discrete spectrum of* $L^2(\mathrm{Whit})_\infty$ *is the direct sum* $\bigoplus_{s \in \widetilde{A}} L^2(\mathrm{Whit})_{s^2 - 2s}$, *where* $\widetilde{A} = \{\tfrac{1}{2}m > 0 \mid m \in \mathbf{Z}\}$. *Moreover, each* $L^2(\mathrm{Whit})_{s^2 - 2s}$ *is* $\widetilde{\mathrm{Sl}_2}$ *irreducible and corresponds to a square integrable representation of* $\widetilde{\mathrm{Sl}_2}$.

("Discrete spectrum" means the sum of all those irreducible representations of $\widetilde{\mathrm{Sl}_2}$ which occur discretely in $L^2(\mathrm{Whit})$.)

THEOREM 3. *The space* $\mathbf{F}_Q^+(\lambda) \neq 0$ *if and only if* $\lambda = s^2 - 2s$ *with* $s \in \widetilde{A} - \{1/2\}$ *and* $s \equiv 1/2 k \bmod 1$. *The representation of* $\widetilde{\mathrm{Sl}_2} \times O(Q)$ *in* $\mathbf{F}_Q^+(s^2 - 2s)$ *is equivalent to the tensor product of* $L^2(\mathrm{Whit})_{s^2-2s} \hat{\otimes} \mathsf{A}_s^+$, *where* $\mathsf{A}_s^+ = \{\varphi \in \mathbf{F}_Q \mid \mathfrak{k} \cdot \varphi = \sqrt{-1}\, s\varphi$ *and* $E_+\varphi = 0\}$. *Moreover,* A_s^+ *is an irreducible* $O(Q)$ *module.*

We note that for the case $k = 3$ an analogous tensor product as in Theorem 3 is discussed in [1].

REMARK 1. If $b = 1$, then $\mathbf{F}_Q^-(\lambda) = 0$ for all λ. And if $b > 1$, then as in Theorem 2, $\mathbf{F}_Q^-(\lambda) \neq 0$ if and only if $\lambda = s^2 - 2s$ with $s \in \widetilde{A} - \{1/2\}$ and $s \equiv 1/2 k \bmod 1$. Similarly $F_Q^-(s^2 - 2s)$ is $\widetilde{\mathrm{Sl}_2} \times O(Q)$ equivalent to the tensor product $L^2(\mathrm{Whit})^*_{s^2-2s} \hat{\otimes} \mathsf{A}_{-s}^-$, with $L^2(\mathrm{Whit})^*_{s^2-2s}$ the representation of $\widetilde{\mathrm{Sl}_2}$ in $L^2(\mathrm{Whit})_{s^2-2s}$ after conjugation by the unique outer automorphism of $\widetilde{\mathrm{Sl}_2}$, and $\mathsf{A}_{-s}^- = \{\varphi \in \mathbf{F}_Q \mid \mathfrak{k}\varphi = -\sqrt{-1}\, s\varphi,\ E_-(\varphi) = 0\}$.

Then the space A_s^+ is characterized in several ways.

THEOREM 4. A_s^+ *is* $O(Q)$ *equivalent to the representation of* $O(Q)$ *in the spaces* $\{\beta \in L^2(\Gamma_1)_\infty \mid W_\xi^+ * \beta = (s^2 - 2s + k - 1/4 k^2)\beta\}$ *where* Γ_1 *is the hyperboloid* $\{X \in \mathbf{R}^k \mid Q(X, X) = 1\}$ *and* W_ξ^+ *the Laplace Beltrami operator on* Γ_1 *determined by the separation of variables of*

$$\partial(Q) = \frac{\partial^2}{\partial t^2} + \frac{k-1}{t}\frac{\partial}{\partial t} - \frac{1}{t^2} W_\xi^+$$

(*with* $X = t \cdot \xi$, $\xi \in \Gamma_1$).

REMARK 2. We note here results on the discrete spectrum of the hyperboloid similar to Theorem 4 are obtained in [3] in a different framework.

We let K be the maximal compact subgroup of $O(Q)$. Then K is isomorphic to the product $O(a) \times O(b)$, where $O(t)$ is the standard orthogonal group in t variables. We consider the family of irreducible representations $[s_1]_a \otimes [s_2]_b$ of K, where $[x]_t$ denotes the representation of $O(t)$ on spherical harmonics of degree t. Then let $E_Q(s^2 - 2s, m, s_1, s_2)$ be the $\widetilde{K} \times K$ isotypic component in $\mathbf{F}_Q^+(s^2 - 2s)$ which transforms according to the character

$$k(\theta, \epsilon) \rightsquigarrow (\operatorname{sgn} \epsilon)^{2m} e^{\sqrt{-1}\,\theta m}$$

on $\widetilde{K}$ and according to $[s_1]_a \otimes [s_2]_b$ on K.

THEOREM 5. *The space of* $\widetilde{K} \times K$ *finite vectors in* $\mathbf{F}_Q^+(s^2 - 2s)$ *is the direct sum of the* $E_Q(s^2 - 2s, m, s_1, s_2)$, *where* $m = s + 2j$, j *a nonnegative integer and* s_1 *and* s_2 *satisfy the relation* $s_1 - s_2 = s - 1/2(a - b) + 2j$. *Moreover, each space* $E_Q(s^2 - 2s, s + 2j, s_1, s_2)$ *is spanned by elements of the form (determined on* Ω_+)

$$\psi_{s,j}(Q(X, X))Q(X, X)^{s-1}e^{-\pi Q(X,X)}\|X_-\|^{s_2}\|X_+\|^{-(s+k/2+s_2-2)},$$

$$(2.1)\qquad {}_2F_1\left(\frac{1}{2}(s+s_1+s_2)+\frac{1}{4}k-1, -j, s_2+\frac{1}{2}b, \left(\frac{\|X_-\|}{\|X_+\|}\right)^2\right)$$

$$\cdot P_{s_1}\left(\frac{X_+}{\|X_+\|}\right)P_{s_2}\left(\frac{X_-}{\|X_-\|}\right),$$

where ${}_2F_1$ *is the usual hypergeometric function,* P_{s_1} *and* P_{s_2} *are harmonic polynomials of degree* s_1 *and degree* s_2 *in* $\mathbf{R}^a$ *and* $\mathbf{R}^b$, *respectively, and* $\psi_{s,j}(u)$ *is the polynomial* $\Sigma_{\nu=0}^{\nu=j} c_\nu u^{j-\nu}$ *with*

$$c_\nu = \frac{(-1)^\nu}{2^\nu \nu!}\,\frac{\Gamma(s+j)}{\Gamma(s+j-\nu)}\,\frac{j!}{(j-\nu)!}.$$

As an important consequence of Theorem 5 we deduce growth and continuity properties of $\widetilde{K} \times K$ finite vectors in $\mathbf{F}_Q^+(s^2 - 2s)$.

COROLLARY TO THEOREM 5. *Every* $\widetilde{K} \times K$ *finite function* φ *in* $\mathbf{F}_Q^+(s^2 - 2s)$ *extends uniquely to a continuous function on* $\mathbf{R}^k - \{0\}$ *which vanishes identically on* $(\Omega_- \cup \Gamma_0) - \{0\}$. *Moreover, if* $s > \frac{1}{2}k$, *then* φ *extends uniquely to a continuous function on* $\mathbf{R}^k$ *which vanishes identically on* $\Omega_- \cup \Gamma_0$. *Also such a* φ *satisfies the Poisson Summation Formula Property, that is, for any lattice* $L \subseteq \mathbf{R}^k$ *with* $Q(L, L) \subseteq \mathbf{Z}$, *the integers,*

$$(2.2)\qquad F(X) = \sum_{\xi \in L} \varphi(X + \xi),$$

is continuous (with the summation satisfying absolute convergence) on $\mathbf{R}^k$ *and* $\Sigma_{\xi^* \in L^*}\hat{\varphi}(\xi^*)$ *is absolutely convergent* (L^* *dual lattice to* L).

We remark that similar types of statements hold for $\widetilde{K} \times K$ functions $f \in \mathbf{F}_Q^-(s^2 - 2s)$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556

INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, UNIVERSITÉ LOUIS PASTEUR, 7, RUE RENÉ DESCARTES, 67084–STRASBOURG, CEDEX, FRANCE (Current address of G. Schiffmann)

Current address (Stephen Rallis): Department of Mathematics, Princeton University, Princeton, New Jersey 08540