already have formed an opinion about the former question.

D.A. I am beginning to infer that when the London Mathematical Society decided to publish this work, they didn't seek your opinion?

E.W. I do wish you wouldn't ask me about matters which are confidential.

D.A. I go on to infer that they must have preferred some other opinion; perhaps someone better qualified by being closer to the subject or more sympathetic to it?

EXPERT WITNESS. This conjecture follows from the former one.

DEVIL'S ADVOCATE. Let us try another expert witness; they come two a penny.

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Foundations of quantum physics, by C. Piron, Mathematical Physics Monograph Series, no. 19, W. A. Benjamin, Inc., Reading, Massachusetts, 1976, xii + 123 pp., \$17.50 (cloth) and \$8.50 (paper).

Ever since the physicists' discovery that a logically coherent and physically acceptable treatment of atomic and subatomic systems has to be based on principles that are profoundly different from those of classical physics, the problem of understanding and clarifying these principles has engaged the attention of many mathematicians, theoretical physicists, and philosophers. That such discussions continue to go on, and often reveal new aspects fifty years after the original discoveries of the physicists, indicates the remarkable nature of these new ideas as well as the extent of their departure from classical lines of thought.

To trace the origin and development of these ideas is a formidable task; in the framework of the present review it is an impossible one. Suffice it to say that the tremendous difficulties in explaining the mass of spectroscopic data based on a logically sound conception of the atom led the physicists through a succession of approximations culminating in the currently held quantum theoretic view of the atomic world. Bohr's orbit rules, his complementarity principle, the mechanics of Heisenberg and Schrödinger, the Dirac equation of the relativistic electron, Pauli's exclusion principle, and so on, represent various stages in this continuing evolution. The end is even now not in sight because there are still serious difficulties in building a coherent picture of the elementary particles and their interactions.

Throughout this period, the logical foundations of quantum theory have been subjected to very close scrutiny. The aim of most of the investigations has been to gain an understanding of the strange and remarkable manner in which the physical observables and states of a microscopic system are interconnected, and of the startlingly nonclassical nature of the mathematical objects used to represent them. In classical mechanics, when we want to study a finite system of particles, we introduce the phase space P of the system which is a $2N$-dimensional C^∞ manifold; the states of the system are represented by points of P, its observables by real functions on P, and its dynamical evolution by a suitable one-parameter group of diffeomorphisms of P; if F is an observable and σ a state, represented by the function f and the point $x \in P, f(x)$ is the value of F when the system is in the state σ. In statistical mechanics we work with mixed states which are represented by probability measures on the σ-algebra of Borel subsets of P; observables are still functions (Borel measurable) on P; if the observable F and the state σ are represented by the function f and the probability distribution ξ, the probability distribution of F in the state σ is the probability measure $E \mapsto \xi(f^{-1}(E))$ on the σ-algebra $\mathfrak{B}(\mathbf{R})$ of Borel subsets of the real line $\mathbf{R}$. In quantum mechanics things are completely different. Corresponding to the given microscopic system which is the object of study, there is a complex Hilbert space (usually separable) $\mathfrak{H}$; the observables of the system are represented by (not necessarily bounded) selfadjoint operators on $\mathfrak{H}$, while its states are represented by the one-dimensional subspaces of $\mathfrak{H}$ (= rays), i.e., by the points of the projective space $\mathfrak{P}(\mathfrak{H})$ associated with $\mathfrak{H}$. In quantum statistical mechanics where mixed states are required, the states are represented by von Neumann operators on $\mathfrak{H}$, these being defined as the bounded selfadjoint operators which are $\geqslant 0$, are of trace class, and have trace 1. If the observable a and the state σ are represented respectively by the selfadjoint operator A and the one-dimensional subspace S, then quantum mechanics allows us only to calculate the probability distribution of a when the system is in the state σ; this is the probability measure $E \mapsto \langle P_E^A \varphi, \varphi \rangle$ on the σ-algebra $\mathfrak{B}(\mathbf{R})$, where φ is a unit vector in S and P^A is the spectral measure of A. Note that this is also describable as the probability measure $E \mapsto \mathrm{tr}(P_E^A P_S)$ where P_S is the orthogonal projection $\mathfrak{H} \to S$; if σ is a mixed state represented by the von Neumann operator D, the probability measure of a in σ is given by $E \mapsto \mathrm{tr}(P_E^A D)$. The dynamics of the system is represented by a one-parameter group $t \mapsto U_t$ $(-\infty < t < \infty)$ of unitary operators of $\mathfrak{H}$; the action of this group on $\mathfrak{P}(\mathfrak{H})$ then describes the evolution of the pure states of the system with time. If the system has some symmetries governed by some symmetry group G, this is put in display by specifying a homomorphism of G into the group of all automorphisms of $\mathfrak{P}(\mathfrak{H})$; in many cases, this homomorphism comes from a unitary representation of G in $\mathfrak{H}$.

The foregoing descriptions of observables, states, and their probability distributions are further supplemented by certain rules of interpretation which tell us something about the nature of the measurement process in quantum mechanics. As a rule, measurements disturb the state in an unpredictable fashion, so that the principle of causality is applicable to miscroscopic systems only in the periods where no measurement is made. If a is a physical observable with possible values $\lambda_1, \lambda_2, \ldots, \ldots$ it is represented by a selfadjoint operator A having discrete spectrum and eigenvalues $\lambda_1, \lambda_2, \ldots$; if $S(\lambda_n)$ is the eigensubspace of A corresponding to λ_n and the state of the system is represented by the one-dimensional subspace S, then the measurement of a will yield the values $\lambda_1, \lambda_2, \ldots$ with respective probabilities $p_1, p_2, \ldots$ where $p_n = \langle P_{S(\lambda_n)}\varphi, \varphi\rangle$, φ being a unit vector in S, and $P_{S(\lambda_n)}$ the orthogonal projection $\mathfrak{H} \to S(\lambda_n)$. The effect of measurement is to force the state into a state contained in one of the $S(\lambda_n)$. In particular, if $S \subset S(\lambda_n)$, the measurement of a is certain to yield the value λ_n; we say then that a has a sharply defined value, namely λ_n, in S. Furthermore, if b is another observable with values $\mu_1, \mu_2, \ldots$ represented by a selfadjoint operator B, there may exist no state in which both a and b have sharply defined values; in order that $\mathfrak{H}$ admit an orthonormal basis of such states it is necessary and sufficient that A and B commute. The restriction to operators with discrete spectra is made only for simplicity of exposition and there is no difficulty in including more general observables.

There is no need to emphasize that this is an astonishing set of rules constituting a severe departure from classical physics. One of the most striking consequences of these rules is that it is in general impossible to "prepare" a system to be in a state in which all the observables have sharply defined values. The Heisenberg uncertainty principle, for instance, sets quantitative limits to the precision with which the position and momentum of a microscopic particle may be simultaneously measured. As another consequence may be mentioned the fact that an object such as the electron or the photon has both particle and wave properties, i.e., that it has states in which corpuscular properties are exhibited and also states in which wave phenomena arise. The elegant and simple mathematical manner in which quantum theory can explain this complementarity is one of its great accomplishments.

Pioneered by Bohr and developed further by Heisenberg, Schrödinger, Dirac, Pauli and others, these principles and their consequences have succeeded in explaining and predicting a truly impressive range of phenomena. However, these rules and interpretations were so revolutionary that they provoked fierce debate and controversy ever since they emerged out of the chaos and excitement of the early years of this century. The resulting scrutiny of the foundations has been very beneficial to the theory, leading to the creation of a large body of mathematical, physical, and philosophical literature, attempting to show, at least in broad outline, the inevitability of the fundamental rules and their interpretations.

From the mathematicians' point of view, the central contributions in this area came from John von Neumann, and were set forth in his famous book *Mathematische Grundlagen der Quantenmechanik,* and in a number of papers, of which the one with Birkhoff (Ann. of Math. **37** (1936)) is most relevant for us. His main ideas may be briefly summarized as follows. Let $\mathfrak{S}$ be the physical system to be investigated. There is then associated to $\mathfrak{S}$ a partially ordered set $\mathfrak{L} = \mathfrak{L}(\mathfrak{S})$ whose members correspond to the *experimentally verifiable* statements concerning $\mathfrak{S}$; the ordering in $\mathfrak{L}$ corresponds to the relation of implication. $\mathfrak{L}$ also admits an orthogonal complementation corresponding to negation. If $\mathfrak{S}$ is classical, $\mathfrak{L}$ would be a Boolean σ-algebra. However, if $\mathfrak{S}$ is a quantum mechanical system, $\mathfrak{L}$ would be highly *nondistributive and would resemble a projective geometry.* Observations of physical observables are then processes that single out specific Boolean σ-algebras contained in $\mathfrak{L}$; but, in general, these various σ-algebras cannot all be contained in a single Boolean σ-algebra, and the lattice-theoretic structure of $\mathfrak{L}$ is expressed essentially in the manner in which the various Boolean σ-algebras contained in $\mathfrak{L}$ interlock with each other. In particular, the presence of Boolean σ-algebras not contained in a single one corresponds to the phenomenon of complementarity of $\mathfrak{S}$. $\mathfrak{L}$ is called the *logic* of $\mathfrak{S}$.

This axiomatic point of view led von Neumann to the problem of characterizing, within the category of logics (= partially ordered sets with suitable orthocomplementations), the subcategory of projective geometries associated with Hilbert spaces; it is convenient to refer to the latter as *standard logics.* The work of von Neumann showed the existence of many logics other than the standard ones (see his Collected Works); however, none of these others is as simple as the standard ones.

Interest in the axiomatic viewpoint revived after the Second World War, especially the late fifties and early sixties. In addition to the papers of Irving Segal, there were the lectures and articles of George Mackey, especially his book (Benjamin), and the publication, by Gleason, of his theorem determining all the states of a standard logic. My own interest in the subject dates back to this time; and I worked out, in two volumes (Van Nostrand), the main features of the axiomatic approach, including one of the notable contributions to the axiomatic approach that came out at that time, namely, the characterization, due to Piron, of the standard logics within the category of all logics. The entire subject is still very active; among its most interesting recent developments are the investigations of Kochen, and the theorem of Lodkin determining all the states of the logic of projections in an arbitrary von Neumann algebra (A. A. Lodkin, *Functional analysis and its applications* (Russian), **8**, 4 (1974), 54–58).

This brings us to the book under review in which Professor Piron has presented his view of the axiomatic foundations of quantum physics. It is a brief exposition (123 pp.). It begins with a quick summary of the main features of classical mechanical systems emphasizing the fact that their logics are Boolean algebras. It then goes on to discuss a suitable cateogry of partially ordered sets suitable for representing the logics of quantum mechanical systems. It contains, in this general context, descriptions of the notions of states and observables. Its highlight is a proof of Piron's own theorem singling out the standard logics by suitable lattice-theoretic properties. The book concludes with a discussion of particles, both classical and quantum mechanical, obeying the Galilean principle of relativity.

So far as the essential content of this book is concerned, there is nothing in it that is not already covered (with much more detail) in the books of Mackey and the reviewer. It would have been very useful if Piron had gone into some of the aspects not treated fully in these earlier works. It is also unfortunate that Piron does not make any effort to discuss the phenomenological basis of his axioms. While an "explanation" of the axioms is nowadays regarded as unnecessary in a mathematical treatise, it is still an important part in any exposition of the mathematical nature of physical theories; in the present situation where many departures from long accepted patterns of thought are demanded of the reader, the physical basis of the axioms is, at least in this reviewer's opinion, a crucial part of the treatment. Piron's exposition therefore takes on a dogmatic character. For instance, he assumes that the logic of a system is a complete atomic lattice. In fact, he *proves* that it is a complete lattice and *postulates* its atomicity. As far as the atomicity is concerned, no arguments are given as to why this is a reasonable assumption. It excludes, for example, logics arising out of the projections in a von Neumann algebra which is not of type I. Already at the classical level, this assumption forces Piron to consider only probability measures defined on all subsets of the phase space; as such measures are atomic, this would make the theory too weak to be of any use in statistical mechanics, and so Piron makes use of artificial concepts and definitions to introduce nonatomic measures. As for the fact that for Piron the logic of a system is always a complete lattice (Theorem 2.1), it is based on a definition of the infimum of a set of experimental statements that this reviewer did not find convincing at all. This attitude of Piron is in sharp contrast with the widely held view (going back to the Birkhoff-von Neumann paper) that there is no convincing phenomenological explanation behind the requirement that the logic be a lattice (not to mention a complete one) and that it is important to find one. Indeed, when we consider two elements of a logic that cannot be imbedded in a single Boolean algebra, they are nonclassically related, and are already illustrative of the phenomenon of complementarity that is so characteristic of microscopic systems; and so, any phenomenological understanding of their infimum must come to grips with this aspect of quantum theory. It is the opinion of the reviewer that this question cannot be studied by working with the logic of a system in isolation; it appears necessary to introduce the states also, and develop the theory along lines in which both the logic and the space of its states play equally fundamental roles. The phenomenon of complementarity and the problems connected with the existence of a lattice structure on a logic then appear to emerge more clearly out of the manner in which the observables and states are interconnected. The expositions of Mackey and Zierler are of this type. To dismiss one of the crucial aspects of the subject in such a perfunctory manner as Piron has done is, at the least, very misleading. I would also like to point out that Piron makes no reference to the work of Zierler on the characterization of standard logics, although Zierler's work was done more or less simultaneously with Piron's and independently of it. There are many such instances of a lack of proper care in giving references to the work of others scattered throughout this book, making this exposition somewhat distorted. The reader who wants to be informed in depth on the various aspects of the subject and the extensive literature on these questions would do well to consult the volume entitled *The logico-algebraic approach to quantum mechanics*, edited by C. A. Hooker (D. Reidel Publishing Company).

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Non-archimedean fields and asymptotic expansions, by A. H. Lightstone and A. Robinson, North-Holland Mathematical Library, vol. 13, North-Holland/American Elsevier, Amsterdam and New York, 1975, 204 + x pp., \$24.95.

The aim of this short book is to show that nonarchimedean fields and nonstandard analysis form an excellent setting for the study of asymptotic expansions. The authors have been quite successful in achieving this goal. One wishes there were more, but the terminal illness of Abraham Robinson, who wrote the first draft, prevented further collaboration on Harald Lightstone's final manuscript. Since both authors are now deceased, it will be up to others to further their ideas.

An asymptotic expansion for a function f with respect to an "asymptotic sequence" of functions ϕ_i is a formal series $\sum_{i=0}^{\infty} a_i\phi_i$ such that while the sequence of partial sums $S_n(x) = \sum_{i=0}^{n} a_i\phi_i(x)$ may diverge at a given x, there may yet exist an n_x (in practice, small) such that $S_{n_x}(x)$ is a satisfactory approximation to $f(x)$. For the most part, the book deals with real, rather than complex, valued functions. A sequence of real-valued functions $\phi_0, \phi_1, \ldots$ is called an asymptotic sequence if there is a neighborhood of $+\infty$ in the real line R on which each ϕ_i is defined and nonvanishing and for each n in the natural numbers $N = \{0, 1, 2, \ldots\}$ we have $\phi_{n+1} = o(\phi_n)$, i.e., $\lim_{x\to\infty} \phi_{n+1}(x)/\phi_n(x) = 0$. Given an asymptotic sequence $\{\phi_i\}$, a sequence of real numbers $\{a_i\}$, and a real-valued function f defined on some neighborhood $(t, +\infty)$ of $+\infty$ in R, the formal sum $\sum a_i\phi_i$ is called an asymptotic expansion for f, and we write $f \sim \sum a_i\phi_i$, if for each $n \in N$, $f - \sum_0^n a_i\phi_i = o(\phi_n)$. One may think of the nth error as being a higher order of infinitesimal than the last term adjoined to the series.