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ASPECTS OF VALUE DISTRIBUTION THEORY IN SEVERAL COMPLEX VARIABLES

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During the last fifty years value distribution in one complex variable has been established as one of the most beautiful branches of complex analysis. In several variables, value distribution was slow to grow up. Only a few people were concerned and many obstacles had to be overcome. However, recently, the theory has gained wide recognition. The outlook for the future is bright and promises a theory even broader in scope than its one-dimensional counterpart.

1. The classical theory. At first let us look at some basic results in one variable. Realize the Riemann sphere $\mathbf{P}_1$ as a sphere of diameter 1 in $\mathbf{R}^3$. The chordal distance between points w and a in $\mathbf{P}_1$ is denoted by $\|w, a\|$. Then $0 \leqslant \|w, a\| \leqslant 1$. The Riemann sphere carries a rotation invariant volume element Ω giving the sphere total volume 1. As on each complex manifold, the exterior derivative $d = \partial + \bar{\partial}$ twists to

$$d^c = (i/4\pi)(\bar{\partial} - \partial).$$

On $\mathbf{P}_1 - \{a\}$, the volume element Ω is computed by

$$\Omega = -dd^c \log\|a, w\|^2. \tag{1.1}$$

If $r > 0$, let $\mathbf{C}[r]$ be the closed disc, $\mathbf{C}(r)$ be the open disc and $\mathbf{C}\langle r\rangle$ be the circle, all of radius r and with center 0. Let $f\colon \mathbf{C} \to \mathbf{P}_1$ be a nonconstant holomorphic map, i.e., a nonconstant meromorphic function. The *spherical image* of f is defined by

$$A_f(r) = \int_{\mathbf{C}(r)} f^*(\Omega) > 0.$$

For $0 < s < r$, the *Ahlfors-Shimizu characteristic* of f is defined by

$$(T_f(r, s)) = \int_s^r A_f(t)\, dt/t.$$

Then $T_f(r, s) \to \infty$ for $r \to \infty$. On $\mathbf{C}\langle r\rangle$, a rotation invariant line element σ exists which gives the circle $\mathbf{C}\langle r\rangle$ length 1. For $r > 0$, the *compensation function* of f for $a \in \mathbf{P}_1$ is defined by

$$m_f(r, a) = \int_{\mathbf{C}\langle r\rangle} \log \frac{1}{\|f, a\|}\, \sigma \geqslant 0.$$

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Let $\theta_{fa}(z)$ be the *a-multiplicity* of f at $z \in \mathbf{C}$. For $r > 0$, the *counting function*

$$n_f(r, a) = \sum_{z \in \mathbf{C}[r]} \theta_{fa}(z)$$

is the number of a-points in $\mathbf{C}[r]$. For $0 < s < r$, the *valence function* of f to $a \in \mathbf{P}_1$ is defined by

$$N_f(r, s; a) = \int_s^r n_f(t; a) \frac{dt}{t}.$$

The *First Main Theorem* asserts

$$T_f(r, s) = N_f(r, s; a) + m_f(r, a) - m_f(s, a)$$

for $0 < s < r$. The *Mean Value Theorem* states

$$T_f(r, s) = \int_{\mathbf{P}_1} N_f(r, s; a)\Omega(a).$$

So the growth measure T_f of f is the average of the growth measure of the fibers of f. The First Main Theorem implies

$$0 \leqslant \delta_f(a) = \liminf_{r\to\infty} \frac{m_f(r; a)}{T_f(r, s)} = 1 - \limsup_{r\to\infty} \frac{N_f(r, s; a)}{T_f(r, s)} \leqslant 1.$$

Here $\delta_f(a)$ is called the *Nevanlinna defect* of f at a. The Mean Value Theorem implies $\int_{\mathbf{P}_1} \delta_f(a)\Omega(a) = 0$. Hence $\delta_f(a) = 0$ for almost all $a \in \mathbf{P}_1$. Therefore almost all fibers grow as quickly as f itself. A much sharper result is *Nevanlinna's Defect Relation*

$$\sum_{a \in \mathbf{P}_1} \delta_f(a) \leqslant 2.$$

Therefore $\delta_f(a) > 0$ for at most countably many points $a \in \mathbf{P}_1$. Also $\delta_f(a) = 1$ can occur only twice. If $f^{-1}(a) = \varnothing$, then $\delta_f(a) = 1$. Hence the theorem of Picard that f omits at most two values is obtained.

2. The general First Main Theorem in several variables. Let M and N be connected, complex manifolds of dimensions m and n, respectively. Assume that a family $\mathfrak{E} = \{E_a\}_{a\in A}$ of analytic subsets E_a of N is given. Each E_a has pure dimension $n - s < n$. Let f: $M \to N$ be a holomorphic map. Define $F_a = f^{-1}(E_a)$ for all $a \in A$. Value distribution studies the magnitude of the inverse family $\mathfrak{E}_f = \{F_a\}_{a\in A}$. Do the First Main Theorem, the Mean Value Theorem, and the Defect Relation hold? Certainly some additional assumptions have to be made. We shall sketch a basic procedure underlying practically all derivations of a First Main Theorem.

Inspired by (1.1), we require the existence of a form Ω of bidegree (s, s) and class C^∞ on N and the existence of a form Λ_a of class C^∞ and of bidegree $(s - 1, s - 1)$ on $N - E_a$ for each $a \in A$, such that Λ_a has residue 1 on E_a and such that

$$dd^c\Lambda_a = \Omega \tag{2.1}$$

on $N - E_a$. The following examples show that these assumptions can be satisfied in many cases.

EXAMPLE 1. Let $\mathfrak{E}$ be the family of complex projective planes of dimension

$n - s$ in the n-dimensional complex projective space $\mathbf{P}_n$. Let ω be the exterior form of the Fubini-Study Kaehler metric on $\mathbf{P}_n$. Take $\Omega = \omega^s \geqslant 0$. Then Chern [**5**] and Levine [**17**] construct $\Lambda_a \geqslant 0$ explicitly.

EXAMPLE 2. Let $\mathfrak{E}$ be the point family, i.e., $N = A$ and $E_a = \{a\}$ for all $a \in N$. Assume A is a compact Kaehler manifold. Let Ω be the Kaehler volume form of A normalized such that $\int_A \Omega = 1$. Hodge theory provides $\Lambda_a \geqslant 0$. See Wu [**32**], Hirschfelder [**14**], and Stoll [**24**].

Let X and Y be complex spaces. Let $h: X \to Y$ be a holomorphic map. Then h is said to be *projective* if for every $z \in X$ open neighborhoods U of z and V of $h(z)$ exist such that there is a complex space W and a biholomorphic map $\alpha: U \to V \times W$ such that $h|U = P \circ \alpha$ where $P: V \times W \to V$ is the projection. Assume that h is proper, projective and has pure fiber dimension q. Then the fiber integration operator h_* is defined and associates to each form χ of bidegree (k, l) and class C^∞ on X a form $h_*\chi$ of bidegree $(k - q, l - q)$ and of class C^∞ on Y provided $k \geqslant q$ and $l \geqslant q$. See Tung [**29**].

EXAMPLE 3. The family $\mathfrak{E} = \{E_a\}_{a \in A}$ is said to be *admissible* if the following conditions are satisfied.

(i) The index set A is a connected, complex Kaehler manifold.

(ii) The incidence set $S = \{(x, a) \in M \times A | x \in E_a\}$ is analytic.

(iii) The projections $\tau: S \to M$ and $\pi: S \to a$ are surjective and τ is proper and projective.

Determine $\Omega(A)$ and $\Lambda_a(A)$ for the point family on A as in Example 2. Then $\Omega = \tau_* \pi^* \Omega(A) \geqslant 0$ and $\Lambda_a = \tau_* \pi^*(\Lambda_a(A)) \geqslant 0$ satisfy assumption (2.1). Also $\tau: \pi^{-1}(a) \to E_a$ is biholomorphic. See Tung [**29**]. Also observe that the point family in Example 2 is admissible.

We return to the general situation and assume (2.1). Also suppose that F_a is either empty or has pure dimension $m - s$ if $a \in A$. Then (2.1) pulls back to $dd^c f^*(\Lambda_a) = f^*(\Omega)$ on $M - F_a$. Let φ be a form of class C^∞ and of bidegree $(m - s, m - s)$ on M. Assume φ has compact support. The singularity of $f^*(\Lambda_a)$ on F_a is so weak that Stokes' Theorem applies:

$$\int_M f^*(\Lambda_a) \wedge dd^c\varphi = -\int_M df^*(\Lambda_a) \wedge d^c\varphi.$$

For degree reasons $df^*(\Lambda_a) \wedge d^c\varphi = d\varphi \wedge d^c f^*(\Lambda_a)$. A second application of Stokes' Theorem leaves a residue

$$\begin{aligned}\int_M f^*(\Lambda_a) \wedge dd^c\varphi &= -\int_M d(\varphi \wedge d^c f^*(\Lambda_a)) + \int_M \varphi \wedge dd^c f^*(\Lambda_a)\\ &= -\int_{F_a} \theta_{fa}\varphi + \int_M \varphi \wedge f^*(\Omega).\end{aligned}$$

Here θ_{fa} is an integral-valued nonnegative multiplicity function. The *unintegrated First Main Theorem* is obtained:

$$\int_M f^*(\Omega) \wedge \varphi = \int_{F_a} \theta_{fa}\varphi + \int_M f^*(\Lambda_a) \wedge dd^c\varphi. \tag{2.2}$$

The integral on the left does not depend on a. So (2.2) can be viewed as a preservation principle. Also (2.2) can be considered as an identity between currents. Let us consider two applications of (2.2).

1. APPLICATION. Assume $M = N$ is compact and f is the identity. Then $\theta_{fa} = 1$ on E_a. Suppose $d\varphi = 0$. Then $\int_N \Omega \wedge \varphi = \int_{E_a} \varphi$. Therefore Ω is the Poincaré dual of E_a for each $a \in A$.

2. APPLICATION. Take $N = \mathbf{C}$ and $s = 1$. Then f is a holomorphic function. Choose $\Lambda_a(z) = -\log|z - a|^2$. Then $dd^c\Lambda_a \equiv 0$. Hence $\Omega \equiv 0$. Therefore

$$\int_{F_a} \theta_f^a \varphi = \int_M \log|f - a|^2 \wedge dd^c\varphi.$$

Hence the current of integration over the zeroes of f is the current $dd^c[\log|f|^2]$, a theorem due to Lelong [**16**].

Now assume $\Omega \geqslant 0$ and $\Lambda_a \geqslant 0$, which can be satisfied in many cases, for instance in the examples. Then (2.2) shall be used to estimate the volume of F_a. Still the requirements that φ ought to be the volume element on each F_a and that φ has compact support are incompatible. We shall refine the method. Assume a form $\chi \geqslant 0$ of bidegree $(m - s, m - s)$ and class C^∞ and with $d\chi = 0$ is given on M. If M is Kaehlerian, the $(m - s)$th power of the Kaehler form will do. Let G and g be relatively compact, open subsets of M with smooth boundaries $\Gamma = \partial G$ and $\gamma = \partial g$. Assume $\bar{g} \subset G$. Let $\psi\colon M \to \mathbf{R}$ be a continuous function such that $\psi|(M - G) = 0$ and $\psi|\bar{g} = R \geqslant 0$ are constant. Assume $0 \leqslant \psi \leqslant R$ on M. Also assume that $\psi|(\bar{G} - g)$ is of class C^∞. Then (G, ψ) is called a *condensor*. The *characteristic* $T_f(G)$, the *valence function* $N_f(G, a)$, the *outer compensation function* $m_f(\Gamma, a)$, the *inner compensation function* $m_f(\gamma, a)$ are defined by

$$T_f(G) = \int_G \psi f^*(\Omega) \wedge \chi \geqslant 0,$$

$$N_f(G, a) = \int_{F_a} \theta_{fa} \psi \chi \geqslant 0,$$

$$m_f(\Gamma, a) = \int_\Gamma f^*(\Lambda_a)(-d^c\psi) \wedge \chi \geqslant 0,$$

$$m_f(\gamma, a) = \int_\gamma f^*(\Lambda_a)(-d^c\psi) \wedge \chi \geqslant 0,$$

where the integrands are nonnegative. The deficit is defined by

$$D_f(G, a) = \int_G f^*(\Lambda_a) \wedge (-dd^c\psi) \wedge \chi.$$

In general the sign of the integrand of $D_f(G, a)$ is not fixed. If the proof of (2.2) is repeated with due regard to boundary integrals, the *First Main Theorem* is obtained:

$$T_f(G) = N_f(G, a) + m_f(\Gamma, a) - m_f(\gamma, a) - D_f(G, a). \tag{2.3}$$

Assume that the index set A is a compact Kaehler manifold with volume elements $\Omega(A)$ such that $\int_A \Omega(A) = 1$. Then $I(h) = \int_A h\Omega(A)$ is the average of h on A. Define $m_f(\Gamma) = I(m_f(\Gamma, a))$ and $m_f(\gamma) = I(m_f(\gamma, a))$. Also define $D_f(G) = I(D_f(G, a))$. Then the *Mean Value Theorem*

$$T_f(G) = I(N_f(G, a)) \tag{2.4}$$

is equivalent to

$$D_f(G) = m_f(\Gamma) - m_f(\gamma) \tag{2.5}$$

by (2.3). The Mean Value Theorem may not hold. In Example 1, the identity (2.5) can be proven directly by an exchange of integrals. See [22] and [23]. In Example 3, we proceed as follows: Let $(Q, \tilde{\tau}, \tilde{f})$ be the relative product of f and τ. Define $\hat{f} = \pi \circ \tilde{f}$. Then $\hat{f}$ maps $\hat{f}^{-1}(a)$ biholomorphically onto F_a. Also $\theta_{fa}(x) = 1$ for all $x \in F_a$ and almost all $a \in A$. Moreover,

$$\begin{tikzcd}
 & A & \\
Q \arrow[ur, "\hat{f}"] \arrow[rr, "\tilde{f}"] \arrow[d, "\tilde{\tau}"'] & & S \arrow[ul, "\pi"'] \arrow[d, "\tau"] \\
M \arrow[rr, "f"] & & N
\end{tikzcd}$$

$$f^*(\Omega) = f^*\tau_*\pi^*(\Omega(A)) = \tilde{\tau}_*\tilde{f}^*\pi^*(\Omega(A)) = \tilde{\tau}_*\hat{f}^*(\Omega(A)).$$

Hence,

$$\begin{aligned} T_f(G) &= \int_M f^*(\Omega) \wedge \psi\chi = \int_M \tilde{\tau}_*\hat{f}^*(\Omega(A)) \wedge \psi\chi \\ &= \int_Q \hat{f}^*(\Omega(A)) \wedge \tilde{\tau}^*(\psi\chi) = \int_A \hat{f}_*\tilde{\tau}^*(\psi\chi)\Omega(A) \\ &= \int_A \left(\int_{F_a} \psi\chi \right) \Omega(A) = I(N_f(G, a)). \end{aligned}$$

So the Mean Value Theorem holds for admissible families.

Assume (2.4). Define $B = \{a \in A \mid F_a \neq \emptyset\}$. Then $0 \leqslant b_f = \int_B \Omega(A) \leqslant 1$ is the probability that $f(M)$ intersects E_a. Define

$$D_f(G, a)^+ = \int_G \mathrm{Max}(0, f^*(\Lambda_a) \wedge dd^c\psi \wedge \chi)$$

and $D_f(G)^+ = I(D_f(G, a)^+)$. If $a \in A - B$, then $N_f(G, a) = 0$. Hence,

$$N_f(G, a) \leqslant T_f(G) + m_f(\gamma, a) + D_f(G, a)^+ \tag{2.6}$$

implies

$$T_f(G) = \int_{B_f} N_f(G, a)\Omega(A) \leqslant b_f T_f(G) + m_f(\gamma) + D_f(G)^+.$$

If $T_f(G) > 0$, then

$$(1 - b_f) \leqslant \left(m_f(\gamma) + D_f(G)^+\right)/T_f(G). \tag{2.7}$$

Keep g fixed. Let G exhaust M with convenient $\psi = \psi_G$. If $T_f(G)$ grows more quickly than $m_f(\gamma) + D_f(G)^+$, we obtain $b_f = 1$. Then $f(M) \cap E_a \neq \emptyset$ for almost all $a \in A$ which is called a *Casorati-Weierstrass Theorem*.

At this point, more accurate choices have to be made which lead to specific situations. Especially, (G, g, ψ) has to be chosen. Some of these choices shall be mentioned here.

1. CHOICE. Take $s = 1$. The closed C^∞-form χ has bidegree $(m-1, m-1)$. Assume $\chi > 0$. Then choose $\psi = \psi_G$ and $R = R(G) > 0$ such that $dd^c\psi \wedge \chi = 0$ on $G - \bar{g}$, such that $\psi|\Gamma = 0$ and $\psi|\gamma = R(G)$ with $\int_\Gamma(-d^c\psi_G) \wedge \chi = 1$. This Dirichlet problem has one and only one solution. Then $D_f(G, a) = 0 = D_f(G, a)^+$ and $D_f(G) = 0 = D_f(G)^+$. The First Main Theorem is freed from the deficit. A Casorati-Weierstrass Theorem and a Defect Relation can be obtained for the hyperplane family. See Weyl **[30]** if $m = 1$ and **[21]** if $m > 1$.

In the other cases, a nonnegative function $\tau: M \to \mathbf{R}$ of class C^∞ is used. If $K \subseteq M$ and $r \geqslant 0$, define

$$K[r] = \{z \in K | \tau(z) \leqslant r^2\}, \tag{2.8}$$

$$K(r) = \{z \in K | \tau(z) < r^2\}, \tag{2.9}$$

$$K\langle r\rangle = \{z \in K | \tau(z) = r^2\}. \tag{2.10}$$

Then τ is said to be an *exhaustion* if $M[r]$ is compact for all $r \geqslant 0$. We call $M[r]$ the closed *pseudoball*, $M(r)$ the open *pseudoball*, and $M\langle r\rangle$ the *pseudosphere*. Let u be an increasing function. Take $0 \leq r_0 < r$ and define $G = M(r)$ and $g = M(r_0)$ and $\psi_r = u(r^2) - u \circ \tau$ on $\bar{G} - g$. Now, other choices can be made.

2. CHOICE. Take an exhaustion τ with $dd^c\tau \leqslant 0$ outside $M[r_0] \neq \varnothing$. Then τ is called *pseudoconcave*. Take $u(x) \equiv x$. Then $D_f(G, a) \leqslant 0$. Hence $D_f(G) \leqslant D_f(G)^+ = 0$. Also $m_f(\gamma)$ is constant. If $T_f(G) \to \infty$ for $r \to \infty$, a Casorati-Weierstrass Theorem holds. See Chern **[5]**, Bott and Chern **[2]**, and Cowen **[6]**.

3. CHOICE. Take an exhaustion τ with $dd^c\tau \geqslant 0$. Then τ and M are called *pseudoconvex*. Take $u(x) \equiv x$. Then $D_f(G, a) = D_f(G, a)^+ \geqslant 0$ and $D_f(G) = D_f(G)^+ \geqslant 0$. A Casorati-Weierstrass Theorem follows if $D_f(M(r))/T_f(M(r)) \to 0$ for $r \to \infty$. In many cases $D_f(M(r))$ can be computed. Observe that M is a Stein manifold if and only if a pseudoconvex exhaustion τ exists with $dd^c\tau > 0$ on M. Stein manifolds are the most important noncompact complex manifolds and value distribution applies to them.

4. CHOICE. Take an exhaustion τ such that $dd^c\log \tau \geqslant 0$ and $(dd^c\log \tau)^m \equiv 0$. Also $(dd^c\tau)^m \not\equiv 0$ and $M[0]$ of measure zero are required. Then τ is called a *parabolic* exhaustion and M is called *parabolic*. Take $u = -[(2m-2)x^m]^{-1}$. A Casorati-Weierstrass Theorem and a defect relation hold. For more details see below.

3. Some historical remarks. In 1938, H. Kneser **[15]** established the First Main Theorem for meromorphic functions on $\mathbf{C}^m$. About this time, Ahlfors **[1]**, H. Weyl and J. Weyl **[30]** proved the First Main Theorem and the Defect Relation for holomorphic maps from Riemann surfaces into the complex projective space $\mathbf{P}_n$ and for the hyperplane family. In 1970, Wu **[33]** renovated this theory. In 1953–1954, these theories were united and expanded to meromorphic maps of complex manifolds into $\mathbf{P}_n$ for the hyperplane family; see **[21]**. The First Main Theorem and the Defect Relation were proved in **[21]** under Choice 1. In 1960, Levine **[17]** and Chern **[5]** established the First Main Theorem for holomorphic maps of $\mathbf{C}^m$ into $\mathbf{P}_n$ for the family of p-planes in $\mathbf{P}_n$. Here the deficit term first appears. In 1967–1969, this theory was extended to holomorphic maps of pseudoconvex manifolds into $\mathbf{P}_n$; see [22] and [23]. Up to date, no Defect Relation for the p-plane family has been found and this remains one of the most difficult open problems in this subject matter.

Around 1968, the First Main Theorem for admissible families was proved. Wu [32] treated the point family only. Hirschfelder [13] required that the parameter space A is homogeneous. In [24], the general case was established. In 1973 Tung [29] extended this theory to complex spaces and meromorphic maps. In 1973, Ahlfors estimates were established in [26] for meromorphic maps of pseudoconvex spaces into $\mathbf{P}_n$ for the hyperplane family. The results were applied to the Bezout Problem. In 1974, Murray [19] completed these investigations by establishing a Defect Relation.

In 1965, Bott and Chern [2] created a new theory of value distribution for holomorphic sections in a holomorphic vector bundle. Let M and N be complex spaces of pure dimensions m and n respectively. Let f: $M \rightarrow N$ be a holomorphic map. Let E be a holomorphic vector bundle of fiber dimension p over N. For each holomorphic section s of E, let $Z(s)$ be the zero set of s. Assume, a finite-dimensional vector space V of holomorphic sections of E over N is given, which generates each fiber of E over $\mathbf{C}$. Then $\mathfrak{E} = \{Z(s)\}_{s \in V - \{0\}}$ is the family considered on N. Bott and Chern proved the First Main Theorem and the Casorati-Weierstrass Theorem on pseudoconcave manifolds if $M = N$ and $p = n$ and if f is the identity. In 1970, an indication was given in [25] how to extend the Bott-Chern theory to Stein manifolds if $1 \leqslant p \leqslant n$. In 1971, Cowen [6] introduced Schubert zeroes of vector bundles and proved the First Main Theorem and a Casorati-Weierstrass Theorem on pseudoconcave manifolds for Schubert zeroes. In all this $M = N$ and f is the identity. The general case f: $M \rightarrow N$ is investigated in [28] on pseudoconvex manifolds.

During the last six years, Phillip Griffiths and his school considered the case of a line bundle ($p = 1$) and introduced a wealth of new ideas into value distribution. One of the main accomplishments is a new proof of the Defect Relation. The previous proofs by Ahlfors, Weyl, Stoll, and later ones by Murray and Wong use so-called associated maps. In 1972, Carlson and Griffiths [4] constructed a singular volume form ξ on N such that Ric $\xi \geqslant 0$ and (Ric $\xi)^n \geqslant \xi$. This form yields the Defect Relation. In 1973, Griffiths and King [12] extended this theory to holomorphic maps f: $M \rightarrow N$, where M is affine algebraic and N is projective algebraic. As above, a holomorphic line bundle L over N is given and $\mathfrak{E}$ is the family of zeroes of holomorphic sections of L. An important assumption is Rank $f = \dim N \leqslant \dim M$. In 1974–1975 this theory of Carlson, Griffiths and King was made intrinsic and extended to parabolic manifolds M; see [27]. An outline of this extended theory follows. For meromorphic maps from parabolic manifolds into projective space Wong [31] proves a Defect Relation without dimension restrictions using associated maps.

4. Value distribution on parabolic spaces. Here, we will consider the Carlson-Griffiths-King theory [4], [12] as extended in [27]. This section is restricted to codimension 1. Higher codimensions are treated in the next section. Let M and N be connected, complex manifolds of dimensions m and

n respectively. Assume N is compact, but M is not compact. On N, a holomorphic line bundle L is given. Let $f: M \to N$ be a holomorphic map. This situation will prevail through the remainder of this report.

Let κ be a hermitian metric along the fibers of L. Let $c(L, \kappa)$ be the associated Chern form. If $U \neq \emptyset$ is open in M and if $s: U \to L$ is a holomorphic section without zeroes, then

$$c(L, \kappa) = -dd^c \log |s|_\kappa^2$$

on U. Assume L is nonnegative, that is, $c(L, \kappa) \geqslant 0$ for a choice of κ and choose κ so.

Since N is compact, the vector space V of all holomorphic sections of L over N has finite dimension $k + 1$. Assume $k \geqslant 0$. If $0 \neq \mathfrak{a} \in V$, define $\mathbf{P}(\mathfrak{a}) = \mathbf{C}\mathfrak{a}$. If $A \subseteq V$, define $\mathbf{P}(A) = \{\mathbf{P}(\mathfrak{a}) | 0 \neq \mathfrak{a} \in A\}$. Then $\mathbf{P}(V)$ is the complex projective space associated to V and $\mathbf{P}: V - \{0\} \to \mathbf{P}(V)$ is holomorphic. If $a \in \mathbf{P}(V)$, then $a = \mathbf{P}(\mathfrak{a})$ for some $0 \neq \mathfrak{a} \in V$ and $E_L[a] = Z(\mathfrak{a})$ does not depend on the choice of $\mathfrak{a}$ but on a only. The set $E_L[a]$ is empty or pure $(n-1)$-dimensional. Also a is called a *projective section* and $E_L[a]$ its zero set. Then $\mathfrak{E} = \{E_L[a]\}_{a \in \mathbf{P}(V)}$ is the value distribution family to be considered on N.

Take a hermitian metric l on V. If $x \in N$ and $a \in \mathbf{P}(V)$, choose $\mathfrak{a} \in \mathbf{P}^{-1}(a)$ with $|\mathfrak{a}|_l = 1$. Then $0 \leqslant \|a, x\|_\kappa = |\mathfrak{a}(x)|_\kappa$ does not depend on the choice of $\mathfrak{a}$. Then κ is said to be *distinguished* if $\|a, x\|_\kappa \leqslant 1$ for all $a \in \mathbf{P}(V)$ and $x \in N$. If κ is not already distinguished a constant $\lambda > 0$ exists such that $\lambda\kappa$ is distinguished. Observe $c(L, \lambda\kappa) = c(L, \kappa) \geqslant 0$. Hence, w.l.o.g. κ can be assumed to be distinguished. We have

$$c(L, \kappa) = -dd^c \log \|a, x\|_\kappa^2 \tag{4.1}$$

on $N - E_L[a]$. The base point set

$$E_L[\infty] = \bigcap_{a \in \mathbf{P}(V)} E_L[a] \tag{4.2}$$

is analytic and $N_\infty = N - E_L[\infty]$ is open and dense. For $x \in N$, the evaluation map $\eta_x: V \to L_x$ is defined by $\eta_x(\mathfrak{a}) = \mathfrak{a}(x)$ for all $\mathfrak{a} \in V$. Let S_x be the kernel of η_x and let $(S_x)^\perp$ be the orthogonal complement of S_x by l. If and only if $x \in N_\infty$, the map η_x is surjective and $\eta_x: (S_x)^\perp \to L_x$ is an isomorphism. In this case the restriction of l to $(S_x)^\perp$ carries over to a hermitian metric l_x on L_x.This defines a hermitian metric l along the fibers of $L|N_\infty$. Then $c(L, l) \geqslant 0$ on N_∞. Also $0 \leqslant \|a, x\|_l \leqslant 1$ if $a \in \mathbf{P}(V)$ and $x \in N_\infty$.

Also M needs a measure. Let $\tau \geqslant 0$ be a nonnegative function of class C^∞ on M. If $K \subseteq M$, define $K[r]$, $K(r)$ and $K\langle r\rangle$ as in (2.8), (2.9) and (2.10). Then τ is said to be a *parabolic exhaustion* of M if the following conditions are satisfied:

(1) For each $r \geqslant 0$, the closed *pseudoball* $M[r]$ is compact.
(2) The *center* $M[0]$ has measure zero.
(3) On $M - M[0]$, assume $\omega = dd^c \log \tau \geqslant 0$. Then $v = dd^c\tau \geqslant 0$ on M.
(4) Assume $v^m \not\equiv 0$ but $\omega^m \equiv 0$.

Then $M(r) \neq \emptyset$ if $r > 0$. Also $M[0] \neq \emptyset$. Let $\mathfrak{R}_\tau$ be the set of all $r > 0$ such that $(d\tau)(x) \neq 0$ for all $x \in M\langle r\rangle$. If $r \in \mathfrak{R}_\tau$, then $M\langle r\rangle$ is the oriented boundary manifold of $M(r)$. For $q \geqslant 0$ define

(4.3) $$\sigma_q = d^c \log \tau \wedge \omega^q \quad \text{and} \quad \sigma = \sigma_{m-1}.$$

Let $j_r\colon M\langle r\rangle \to M - M[0]$ be the inclusion. If $r \in \mathfrak{R}_\tau$, then $j_r^*(\sigma) \geqslant 0$ and the *pseudosphere* $M\langle r\rangle$ has constant volume

(4.4) $$\varsigma = \int_{M\langle r\rangle} \sigma > 0.$$

For all $r > 0$, the volume of the pseudoball is given by

$$\int_{M[r]} \upsilon^m = \int_{M(r)} \upsilon^m = \varsigma r^{2m}.$$

Now, M together with τ is called a *parabolic manifold*, also denoted by (M, τ). They are many examples of parabolic manifolds. For instance:

1. EXAMPLE. $(\mathbf{C}^m, \tau)$ is parabolic with $\tau(\mathfrak{z}) = |\mathfrak{z}|^2$.

2. EXAMPLE. Let M and $\tilde{M}$ be connected, noncompact, complex manifolds of dimension m. Let $\beta\colon \tilde{M} \to M$ be a proper, surjective, holomorphic map. Let τ be a parabolic exhaustion of M. Then $\tau \circ \beta$ is a parabolic exhaustion of $\tilde{M}$.

3. EXAMPLE. Any affine algebraic manifold is parabolic.

4. EXAMPLE. Let (M_1, τ_1) and (M_2, τ_2) be parabolic manifolds. Define $M = M_1 \times M_2$. Let $\pi_j\colon M \to M_j$ be the projections for $j = 1, 2$. Then $\tau = \tau_1 \circ \pi_1 + \tau_2 \circ \pi_2$ is a parabolic exhaustion of M. Hence the product of parabolic manifolds is parabolic.

5.EXAMPLE. Let B be a compact, connected complex manifold of dimension $m - 1$. Let M be a holomorphic line bundle over B with a hermitian metric τ along the fibers of M. Assume $c(M, \tau) \leqslant 0$ on M and $c(M, \tau)(x_0) < 0$ at some point $x_0 \in M$. Then (M, τ) is parabolic.

6. EXAMPLE. A noncompact Riemann surface is parabolic if and only if it belongs to the class $\mathfrak{O}_g$, that is, any subharmonic function bounded above is constant.

7. EXAMPLE. $(\mathbf{C} - \mathbf{Z}) \times \mathbf{C}^m$ is parabolic but not affine algebraic.

A nonnegative divisor ν can be defined in various ways. Since M is a manifold, ν can be identified with its multiplicity function. A nonnegative function $\nu\colon M \to \mathbf{Z}$ is said to be a *nonnegative divisor*, if ν is locally the zero multiplicity of a local holomorphic function. Then $A = \operatorname{supp} \nu$ is empty or a pure $(m - 1)$-dimensional analytic set. Also ν is locally constant on the set of simple points of A. For $r > 0$, the *counting function* of ν is defined by

$$n_\nu(r) = r^{2-2m} \int_{A[r]} \nu \upsilon^{m-1} \geqslant 0.$$

The function increases. Hence $n_\nu(r) \to n_\nu(0) \geqslant 0$ for $r \to 0$ exists. Then

$$n_\nu(r) = \int_{A[r]} \nu \omega^{m-1} + n_\nu(0).$$

For $0 < s < r$, the *valence function* of ν is defined by

$$N_\nu(r, s) = \int_s^r n_\nu(t) \, \frac{dt}{t}.$$

Let $f\colon M \to N$ be a holomorphic map. For simplicity, assume that $f(M)$ is not contained in $E_L[a]$ if $a \in \mathbf{P}(V)$. The *spherical image* is defined by

$$A_f(r, L, \kappa) = r^{2-2m} \int_{M[r]} f^*(c(L, \kappa)) \wedge v^{m-1} \geqslant 0.$$

The function increases. The limits $A_f(r, L, \kappa) \to A_f(0, L, \kappa)$ for $r \to 0$ and $A_f(r, L, \kappa) \to A_f(\infty, L, \kappa) \leqslant \infty$ for $r \to \infty$ exist. Then

$$A_f(r, L, \kappa) = \int_{M[r]} f^*(c(L, \kappa)) \wedge \omega^{m-1} + A_f(0, L, \kappa).$$

For $0 < s < r$, the *characteristic* of f is defined by

$$T_f(r, s, L, \kappa) = \int_s^r A_f(t, L, \kappa) \frac{dt}{t} \geqslant 0.$$

If f is not constant and $c(L, \kappa) > 0$ on N, then

$$T_f(r, s, L, \kappa) \to \infty \quad \text{for } r \to \infty, \tag{4.5}$$

$$T_f(r, s, L, \kappa)/(\log r) \to A_f(\infty, L, \kappa) > 0 \quad \text{for } r \to \infty. \tag{4.6}$$

In any case, assume $A_f(\infty, L, \kappa) > 0$ which implies (4.5) and (4.6). For $a \in \mathbf{P}(V)$ and $r \in \mathfrak{R}_\tau$, the *compensation function* is defined by

$$m_f(r, a, L, \kappa) = \int_{M\langle r\rangle} \log \frac{1}{\|a, f\|_\kappa} \sigma \geqslant 0.$$

The function extends to a continuous function of r for all $r > 0$. For $a \in \mathbf{P}(V)$, the map f defines a nonnegative divisor θ_{fa} with support $F_a = f^{-1}(E_L[a])$. The counting function of θ_{fa} is denoted by $n_f(r; a, L)$ and the valence function by $N_f(r, s; a, L)$. For $0 < s < r$ the *First Main Theorem* holds:

$$T_f(r, s, L, \kappa) = N_f(r, s; a, L) + m_f(r; a, L, \kappa) - m_f(s; a, L, \kappa). \tag{4.7}$$

Assume $F_\infty = f^{-1}(E_L[\infty])$ has at most dimension $m - 2$. Let φ be the exterior form of the Fubini-Study-Kaehler metric defined by l on $\mathbf{P}(V)$. Then $\mathbf{P}(V)$ has volume 1. Unfortunately, the Mean Value Theorem does not hold. Fortunately, the statements above remain correct if κ is replaced by l, although l is defined over N_∞ only. Then the *Mean Value Theorem* holds:

$$T_f(r, s, L, l) = \int_{\mathbf{P}(V)} N_f(r, s; a, L)\varphi(a)^k.$$

Assume (4.5) for κ and l. Define the *Nevanlinna defect* of $a \in \mathbf{P}(V)$ by

$$0 \leqslant \delta_f(a, L) = \liminf_{r\to\infty} \frac{m_f(r; a, L, \kappa)}{T_f(r, s, L, \kappa)}. \tag{4.8}$$

The First Main Theorem and (4.5) imply

$$1 \geqslant \delta_f(a, L) = 1 - \limsup_{r\to\infty} \frac{N_f(r, s; a, L)}{T_f(r, s, L, \kappa)}. \tag{4.9}$$

Then $\delta_f(a, L)$ does not depend on s and κ. However, if in (4.8) κ is replaced by l, another defect $\delta_f^0(a, L)$ may be obtained. Of course if $E_L[\infty] = \varnothing$, then $\delta_f^0(a, L) = \delta_f(a, L)$. Also (4.9) holds for l. Hence $0 \leqslant \delta_f^0(a, L) \leqslant 1$. If $F_a = \varnothing$, then $\delta_f^0(a, L) = \delta_f(a, L) = 1$ since $N_f(r, s; a, L) = 0$. The Mean Value Theorem and Fatou's Lemma imply $I(\delta_f^0(a, L)) = 0$. Hence $\delta_f^0(a, L) = 0$ for almost all $a \in \mathbf{P}(V)$. Therefore $F(M) \cap E_L[a] \neq \emptyset$ for almost all $a \in \mathbf{P}(V)$. A Casorati-Weierstrass Theorem is obtained.

The results stated so far in this section extend to complex spaces and meromorphic maps. For the remainder of this section M and N have to be complex manifolds and f is holomorphic. Now, the Second Main Theorem and the Defect Relation shall be stated, which requires a number of preparations.

To each form $\Omega > 0$ of degree $2m$ and class C^∞ on M, a *Ricci form* Ric Ω of bidegree (1,1) and class C^∞ is assigned. Let $\alpha_1, \ldots, \alpha_m$ be local holomorphic coordinates on an open subset U of M. Then a positive function Ω_α of class C^∞ exists on U such that

$$\Omega | U = \Omega_\alpha i^m d\alpha_1 \wedge d\bar{\alpha}_1 \wedge \ldots \wedge d\alpha_m \wedge d\bar{\alpha}_m.$$

Then $\operatorname{Ric} \Omega | U = dd^c \log \Omega_\alpha$ on U. For $0 < s < r$, the *Ricci function* of Ω is defined by

$$\operatorname{Ric}(r, s, \Omega) = \int_s^r t^{1-2m} \int_{M[t]} \operatorname{Ric} \Omega \wedge v^{m-1} \, dt.$$

Actually, $\operatorname{Ric}(r, s, \Omega)$ can be interpreted as the characteristic function of the canonical bundle K_M of M for a certain hermitian metric along K_M. However, $\operatorname{Ric}(r, s, \Omega)$ may not have a fixed sign. Ideally, we would like to consider $\operatorname{Ric}(r, s, v^m)$ but $v^m > 0$ may not be true on all of M. Hence an alternative is needed. A function $\mathfrak{v} \geqslant 0$ of class C^∞ on M is defined by $v^m = \mathfrak{v}\Omega$. For almost all $0 < s < r$, the *Ricci function* of τ is defined by

$$\operatorname{Ric}_\tau(r, s) = \frac{1}{2} \int_{M\langle r \rangle} \log \mathfrak{v}\sigma - \frac{1}{2} \int_{M\langle s \rangle} \log \mathfrak{v}\sigma + \operatorname{Ric}(r, s, \Omega)$$

and *does not depend on the choice of* Ω. If $v^m > 0$ on M, then $\mathfrak{v} = 1$ implies $\operatorname{Ric}_\tau(r, s) = \operatorname{Ric}(r, s, v^m)$. If $\beta \colon M \to \mathbf{C}^m$ is a proper, surjective holomorphic map and if $\tau = |\beta|^2$, let μ be the *branching divisor* of β. Then

$$\operatorname{Ric}_\tau(r, s) = N_\mu(r, s). \tag{4.10}$$

If the $(m - 1)$-dimensional component of $\beta(\operatorname{supp} \mu)$ is algebraic, then

$$N_\mu(r, s) = O(\log r) \quad \text{for } r \to \infty. \tag{4.11}$$

If $M = \mathbf{C}^m$, then $\operatorname{Ric}_\tau(r, s) \equiv 0$. If (4.5) holds, the *Ricci defect* is defined by

$$R_f = \limsup_{r \to \infty} \frac{\operatorname{Ric}_\tau(r, s)}{T_f(r, s, L, \kappa)} \leqslant \infty \tag{4.12}$$

and is independent of the choice of s and κ.

Let K_M and K_N be the canonical bundles of M and N respectively. Let K_N^* be the dual of K_N. The holomorphic map $f \colon M \to N$ pulls back K_N and K_N^* to K_{Nf} and K_{Nf}^*, respectively, such that K_{Nf}^* is dual to K_{Nf}. Call $K_f = K_N \otimes K_{Nf}^*$ the *Jacobian bundle* of f. A global holomorphic section $F \not\equiv 0$ is called a *Jacobian section* of f. Let ν_F be the divisor of F. If (4.5) holds, the *ramification defect* is defined by

$$0 \leqslant \Theta_F = \liminf_{r \to \infty} \frac{N_{\nu_F}(r, s)}{T_f(r, s, L, \kappa)} \leqslant \infty. \tag{4.13}$$

A K_M-valued inner product $(\cdot,\cdot)$ exists between $K_f = K_M \otimes K_{Nf}^*$ and $K_{N,f}$. Hence F can be interpreted as an operator on certain forms. Let $\Omega^n(U)$ be the set of all holomorphic forms of bidegree $(n, 0)$ on the open subset U of N. Assume $U_f = f^{-1}(U) \neq \varnothing$. Then F acts as a homomorphism $F\colon \Omega^n(U) \to \Omega^m(U_f)$. Take $\psi \in \Omega^n(U)$. Then ψ is a holomorphic section of K_N over U which pulls back to a holomorphic section ψ_f of $K_{N,f}$ over U_f. In general, ψ_f is not the pullback $f^*(\psi)$ of forms. The inner product defines a holomorphic section $F[\psi] = (F, \psi_f)$ of K_M over U_f which is a holomorphic form of bidegree $(m, 0)$ on U_f. This action extends to forms of degree $2n$. Let $A^{2n}(U)$ be the forms of degree $2n$ and class C^∞ on U. Then F acts as a homomorphism $F\colon A^{2n}(U) \to A^{2m}(U_f)$ such that the following condition is satisfied: For each integer $p \geqslant 0$ define

$$i_p = (-1)^{p(p-1)/2} p!\,(i/2\pi)^p.$$

If $\varphi \in \Omega^n(U)$ and $\chi \in \Omega^n(U)$, then the action of F on $A^{2n}(U)$ is determined by

$$F[\,i_n\varphi \wedge \bar{\chi}\,] = i_m F[\varphi] \wedge \operatorname{conj}(F[\chi])$$

where conj denotes conjugation. If $0 < \psi \in A^{2n}(U)$, and if $Z(F)$ is the zero set of F, then $F[\psi] > 0$ on $U_f - Z(F)$ and

$$\operatorname{Ric} F[\psi] = f^*(\operatorname{Ric}\psi) \quad \text{on } U_f - Z(F).$$

Define $M^+ = \{x \in M \mid v(x) > 0\}$ and $M^+(r) = M^+ \cap M(r)$. Then F is said to be *dominated* by τ if for each $r > 0$ there exists a minimal constant $Y(r) \geqslant 1$ such that for all open sets U of N with $M^+(r) \cap U_f \neq \varnothing$ and for all forms $\psi \geqslant 0$ of bidegree (1,1) and class C^∞ on U the inequality

$$n(F[\psi^n]/v^m)^{1/n} v^m \leqslant Y(r) f^*(\psi) \wedge v^{m-1} \tag{4.14}$$

holds on $U_f \cap M^+(r)$. The function Y increases and is called the *dominator* of F. If Y exists, then $\dim M \geqslant \dim N = \operatorname{rank} f$. If this necessary condition is satisfied, Y exists under reasonable assumptions, for instance, if $v > 0$ on M. For example, if $m = n$, then a Jacobian section dominated by $Y \equiv m$ is defined by $F[\varphi] = f^*(\varphi)$. For instance, if there exists a proper, surjective holomorphic map $\beta\colon M \to \mathbf{C}^m$ such that $\tau = |\beta|^2$, then a Jacobian section F dominated by τ exists such that $Y \equiv m$. (2. Example of parabolic manifolds).

Let F be Jacobian section of f dominated by τ with dominator Y. Then the *dominator defect* is defined by

$$0 \leqslant Y_F = \limsup_{r \to \infty} \frac{\log Y(r)}{T_f(r, s, L, \kappa)} \geqslant 0 \tag{4.15}$$

and does not depend on s and κ.

Now, the *Second Main Theorem* can be formulated. For each index $j = 1, \ldots, q$, let L_j be a holomorphic line bundle on N with a hermitian metric κ_j along the fibers of L_j. Let V_j be the vector space of all global holomorphic sections of L_j and let l_j be a hermitian metric on L_j. Take $a_j \in \mathbf{P}(V_j)$. Then $a_1, \ldots, a_q$ are said to be in *general position* (or have *strictly normal crossings*) if the following conditions are satisfied for each $x \in N$: Let v_j be a holomorphic section of L_j over an open neighborhood U_j of x with

$Z(v_j) = \emptyset$. Take $0 \neq \mathfrak{a}_j \in V$ with $\mathbf{P}(\mathfrak{a}_j) = a_j$. Holomorphic functions w_j on U are defined by $\mathfrak{a}_j = w_j v_j$. A sequence of integers $1 \leqslant \mu(1) < \cdots < \mu(t) \leqslant q$ exists such that $w_{\mu(p)}(x) = 0$ for all $p = 1, \ldots, t$, but $w_j(x) \neq 0$ if $j \neq \mu_p$ for all $p = 1, \ldots, t$. Then $(dw_{\mu(1)} \wedge \ldots \wedge dw_{\mu(t)})(x) \neq 0$, i.e., $w_{\mu(1)}, \ldots, w_{\mu(t)}$ is a subset of a set of coordinate functions at x.

Let $a_1, \ldots, a_q$ be in general position. Define $L = L_1 \otimes \ldots \otimes L_q$ and $\kappa = \kappa_1 \otimes \ldots \otimes \kappa_q$. Assume $c(L, \kappa) > 0$. Take a positive integer w. A smallest integer $p \geqslant 0$ exists such that $L^p \otimes K_N^w$ is nonnegative. *Assume* $p < w$. Take $\varepsilon > 0$ and $s > 0$. Define $c_2 = (n + \varepsilon)\varsigma$ and $c_3 = 2\varepsilon\varsigma n$. Then there exist a constant $c_1 = c_1(\varepsilon) > 0$ and a measurable subset $\Delta = \Delta(\varepsilon)$ of $\mathbf{R}^+ = \{x \in \mathbf{R} | x > 0\}$ such that the *Second Main Theorem* holds:

$$N_{\nu_F}(r, s) + \left(1 - \frac{p}{w}\right)T_f(r, s, L, \kappa) \leqslant \sum_{j=1}^{q} N_f(r, s; a_j, L_j)$$
$$+ c_1 \log T_f(r, s, L, \kappa) + c_2 \log Y(r) + c_3 \log r$$

for all $r > s$ with $r \in \mathbf{R}^+ - \Delta$ where $\int_\Delta x^\varepsilon\, dx < \infty$.

For the *Defect Relation*, assume $L_1 = \cdots = L_q$. It may be helpful to recount the assumptions here. Let N be a compact, connected complex manifold of dimension $n > 0$. Let L be a positive holomorphic line bundle on N. (Hence N is projective algebraic.) Let V be the vector space of global holomorphic sections of L on N. Let l be a hermitian metric on V. Let κ be a distinguished hermitian metric along the fibers of L with $c(L, \kappa) > 0$. Let $p = \inf u/w$ with $u \geqslant 0, w > 0$ such that $L^u \otimes K_N^w$ is nonnegative. Let $a_1, \ldots, a_q$ be given in $\mathbf{P}(V)$ such that $q > p$ and such that $a_1, \ldots, a_q$ are in general position for L. Let (M, τ) be a parabolic manifold of dimension $m > 0$. Let $f\colon M \to N$ be a holomorphic map with a Jacobian section F dominated by τ. Let Y be the dominator. Then the Defect Relation holds:

$$\Theta_F + \sum_{j=1}^{q} \delta_f(a_j, L) \leqslant p + R_f + \varsigma n Y_F. \tag{4.16}$$

Of course, (4.16) is only meaningful if $R_f < \infty$ and $Y_F < \infty$. If N is the projective space and if L is the hyperplane section bundle, then $p = n + 1$. If $m = n = \operatorname{Rank} f$, then F can be taken such that $Y \equiv m$. Also, if $m \geqslant n = \operatorname{Rank} f$ and if a proper, surjective, holomorphic map $\beta\colon M \to \mathbf{C}^m$ exists such that $\tau = |\beta|^2$, then F can be chosen such that $Y \equiv m$. In both cases, $Y_F = 0$. Hence

$$\sum_{j=1}^{q} \delta_f(a_j, L) \leqslant p + R_f. \tag{4.17}$$

Observe, that (4.17) does not depend on the choice of F. Consider the covering case. Let μ be the branching divisor of β. Assume that the $(m-1)$-dimensional component of $\beta(\operatorname{supp} \mu)$ is algebraic. (If M is affine algebraic, all this can be realized.) Assume f has transcendental growth, i.e., $A_f(\infty, L, \kappa) = \infty$. Then (4.6), (4.10), (4.11) and (4.12) imply $R_f = 0$. Hence

$$\sum_{j=1}^{q} \delta_f(a_j, L) \leqslant p. \tag{4.18}$$

If $f(M) \cap E_L[a_j] = \emptyset$, then $\delta_f(a_j, L) = 1$. Since $q > p$, at least one index j

exists such that $f(M) \cap E_L[a_j] \neq \emptyset$. *The Borel Picard Theorem* is proved.

The proof of the Defect Relation and (4.16) itself depends on the choice of the Jacobian section F. Moreover F has to satisfy requirement (4.14) which implies $m \geqslant n = \operatorname{rank} f$. If $N = \mathbf{P}_n$ is the complex projective space and if L is the hyperplane section bundle, a Defect Relation can be obtained using a holomorphic form $B \not\equiv 0$ of bidegree $(m-1, 0)$. Again, additional assumptions have to be made on B. Differentiation in the direction of B is introduced and defines associated maps. No relation between m and n is required. Again, a ramification defect, a Ricci defect and a dominator defect appear. In both cases, the task of the operator F, respectively B, is to link K_N with K_M. For details to the method using the form B and associated maps see [1], [19], [21], [26], [30], [31] and [33], where [31] gives a Defect Relation quite similar to (4.16). For references to the method using Jacobian sections see [4], [12] and [27], where F appears openly in [27] only. Also, applications are given in [4], [12] and [27].

5. Higher codimensions. The case of the common zero set of several holomorphic sections of the line bundle L shall be considered. Only the First Main Theorem and a Casorati-Weierstrass Theorem have been established. Again, let N be a connected, compact, complex manifold of dimension $n > 0$. Let L be a holomorphic line bundle on N with a hermitian metric κ along the fibers of L such that $c(L, \kappa) \geqslant 0$. Let V be the vector space of all global holomorphic sections of V. Then $\dim V = k + 1 < \infty$. Assume $k \geqslant 0$. Take a hermitian metric l on V. Take $0 \leqslant p \leqslant k$. Then $\tilde{G}_p(V) = \{\mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_p | \mathfrak{a}_j \in V\}$ is analytic in $\bigwedge_{p+1} V$ and $G_p(V) = \mathbf{P}(\tilde{G}_p(V)$ is a smooth, connected compact submanifold of dimension $d(p, n) = (n-p)(p+1)$ in $\mathbf{P}(\bigwedge_{p+1} V)$ called the *Grassmann manifold* of order p.

Take $a \in G_p(V)$. Then $a = \mathbf{P}(\mathfrak{a})$ with $\mathfrak{a} = \mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_p$. For $0 \leqslant \mu \leqslant p$, define $a_\mu = \mathbf{P}(\mathfrak{a}_\mu) \in \mathbf{P}(V)$ and

$$\mathfrak{a}^\mu = \mathfrak{a}_0 \wedge \ldots \wedge \mathfrak{a}_{\mu-1} \wedge \mathfrak{a}_{\mu+1} \wedge \ldots \wedge \mathfrak{a}_p.$$

A section $\vec{\mathfrak{a}}^\mu$ of the trivial bundle $W_p = N \times \bigwedge_p V$ is defined by $\vec{\mathfrak{a}}^\mu(x) = (x, \mathfrak{a}^\mu)$. A holomorphic section $\eta_\mathfrak{a}$ of $L \otimes W_p$ over N is defined by

$$\eta_\mathfrak{a}(x) = \sum_{\mu=0}^{p} (-1)^\mu \mathfrak{a}_\mu(x) \otimes \vec{\mathfrak{a}}^\mu(x).$$

Here $\eta_\mathfrak{a}$ depends on $\mathfrak{a}$ but not on $\mathfrak{a}_0, \ldots, \mathfrak{a}_p$. Let $Z(\eta_\mathfrak{a})$ be the zero set of $\eta_\mathfrak{a}$. The hermitian metric l on V defines a hermitian metric l on $\bigwedge_p V$ and a hermitian metric l along the fibers of the trivial bundle $W_p = N \times \bigwedge_p V$. Hence a hermitian metric $\kappa \otimes l$ is defined along the fibers of $L \otimes W_p$. Take $x \in M$, then

$$0 \leqslant \|a; x\|_\kappa = ((\mathfrak{a})_l)^{-1} |\eta_\mathfrak{a}(x)|_{\kappa \otimes l}$$

is defined and depends on a but not on the choice of $\mathfrak{a} \in \mathbf{P}^{-1}(a)$. Also $E_L[a] = Z(\eta_\mathfrak{a})$ depends on a only. Obviously,

$$E_L[a] = \{x \in M | \; \|a, x\|_\kappa = 0\} = \bigcap_{\mu=0}^{p} E_L[a_\mu].$$

The Grassmann manifold $G_k(V)$ consists of one and only one point denoted by ∞. Then $E_L[\infty]$ is the base point set of L consistent with (4.2) with $N_\infty = N - E_L[\infty] \neq \emptyset$. The hermitian metric l on V defines a hermitian metric l along the fibers of $L|N_\infty$ as explained in §4. If $x \in N_\infty$, then $0 \leqslant \|a, x\|_l \leqslant 1$ and

$$\|a, x\|_\kappa = \|a, x\|_l \|\infty, x\|_\kappa.$$

Here $\|\infty, x\|_\kappa$ is called the *deviation* of κ. By multiplying κ with a positive constant, $0 \leqslant \|\infty, x\|_\kappa \leqslant 1$ can be assumed w.l.o.g. Hence κ is distinguished and $\|a, x\|_\kappa \leqslant 1$ for all $a \in G_p(V)$ and $x \in N$ by continuity. The form

$$0 \leqslant \Phi_L[a] = c(L, \kappa) + dd^c \log\|a, x\|_\kappa^2$$

is defined on $N - E_L[a]$ and is independent of κ with $\Phi_L[a]^{p+1} = 0$. On $N - E_L[a]$, the *Chern-Levine form*

$$0 \leqslant \Lambda_L[a]_\kappa = -\log\|a, x\|_\kappa^2 \sum_{\mu=0}^{p} \Phi_L[a]^\mu \wedge c(L, \kappa)^{p-\mu}$$

is defined with

$$dd^c \Lambda_L[a]_\kappa = c(L, \kappa)^{p+1}.$$

Let (M, τ) be a parabolic manifold of dimension m. Assume $0 \leqslant p \leqslant k$ and $q = m - p - 1 \geqslant 0$. Let $f: M \to N$ be a holomorphic map. For simplicity, assume that $F_a = f^{-1}(E_L[a])$ is empty or has pure dimension q for all $a \in G_p(V)$. For $r > 0$ the *spherical image* of order p is defined by

$$A_f^p(r, L, \kappa) = r^{-2q} \int_{M[r]} f^*\left(c(L, \kappa)^{p+1}\right) \wedge v^q \geqslant 0.$$

The function increases. The limits for $r \to 0$ and $r \to \infty$ exist and are denoted by $A_f(0, L, \kappa)$ and $A_f(\infty, L, \kappa)$, respectively. Then

$$A_f^p(r, L, \kappa) = \int_{M[r]} f^*\left(c(L, \kappa)^{p+1}\right) \wedge \omega^q + A_f^p(0, L, \kappa).$$

For $0 < s < r$, the *characteristic of order p* is defined by

$$T_f^p(r, s, L, \kappa) = \int_s^r A_f^p(t, L, \kappa) \frac{dt}{t} \geqslant 0.$$

The function increases and

$$T_f^p(r, s, L, \kappa)/(\log r) \to A_f^p(\infty, L, \kappa) \quad \text{for } r \to \infty.$$

For each $a \in G_p(V)$, a multiplicity function $\theta_{fa} \geqslant 0$ is defined on M with $F_a = \operatorname{supp} \theta_{fa}$ such that θ_{fa} is locally constant on the set of simple points of F_a. For $r > 0$, the *counting function* of f for $a \in G_p(V)$ is defined by

$$n_f(r; a, L) = r^{-2q} \int_{F_a[r]} \theta_{fa} v^{2q} \geqslant 0.$$

The function increases. Hence, $n_f(r; a, L) \to n_f(0; a, L)$ for $r \to 0$. If $r > 0$, then

$$n_f(r; a, L) = \int_{F_a[r]} \theta_{fa} \omega^{2q} + n_f(0; a, L).$$

For $0 < s < r$, the *valence function* of f for $a \in G_p(V)$ is defined by

$$N_f(r, s; a, L) = \int_s^r n_f(t; a, L) \frac{dt}{t}.$$

Also define $M(s, r] = M[r] - M[s]$. Then the *deficit* of f for a is defined by

$$D_f(r, s; a, L, \kappa) = \frac{1}{2} \int_{M(s,r]} f^*(\Lambda_L[a]_\kappa) \wedge \omega^{q+1} \geqslant 0.$$

The function increases with r and is semicontinuous from the right. For $r \in \mathfrak{R}_\tau$, the *compensation function* of f for $a \in G_p(V)$ is defined by

$$m_f(r; a, L, \kappa) = \frac{1}{2} \int_{M\langle r\rangle} f^*(\Lambda_L[a]_\kappa) \wedge \sigma_q \geqslant 0$$

where σ_q is defined by (4.3). The function continues to a function which is semicontinuous from the right for all $r > 0$. For $0 < s < r$ the *First Main Theorem* holds:

$$\begin{aligned} T_f^p(r, s, L, \kappa) = {} & N_f(r, s; a, L) + m_f(r; a, L, \kappa) \\ & - m_f(s; a, L, \kappa) - D_f(r, s; a, L, \kappa). \end{aligned}$$

Assume $F_\infty = f^{-1}(E_L]\infty])$ has at most dimension $q - 1$. Then K can be replaced by l in the preceding statements, although l is singular over $E_L[\infty]$. The hermitian metric l on $\bigwedge_{p+1} V$ defines a Fubini-Study-Kaehler metric on $\mathbf{P}(\bigwedge_{p+1} V)$ which restricts to a Kaehler metric on $G_p(V)$ whose exterior form is denoted by φ_p. Let $D(p, n)$ be the degree of the Grassmann manifold $G_p(V)$ in $\mathbf{P}(\bigwedge_{p+1} V)$. If h is an integrable function on $G_p(V)$, its average is defined by

$$I_p(h) = \frac{1}{D(p, n)} \int_{G_p(V)} h\varphi_p^{d(p,n)}.$$

Then $I_p(1) = 1$. Define

$$\gamma_p = \frac{1}{2} \sum_{\nu=0}^{p} \sum_{\mu=1}^{k-p} \frac{1}{\nu + \mu}.$$

Then the following Mean Value Theorems hold:

$$\begin{aligned} I_p\big(n_f(r; a, L)\big) &= A_f^p(r, L, l), \\ I_p\big(N_f(r, s; a, L)\big) &= T_f^p(r, s; L, l), \\ I_p\big(m_f(r; a, L, l)\big) &= \gamma_p A_f^{p-1}(r; L, l), \\ I_p\big(D_f(r, s; a, L, l)\big) &= \gamma_p\big(A_f^{p-1}(r, L, l) - A_f^{p-1}(s, L, l)\big). \end{aligned}$$

Assume $F_a \neq \emptyset$ for at least one $a \in G_p(V)$. Then $T_f^p(r, s; L, l) \to \infty$ for $r \to \infty$. Define $\chi(a) = 1$ if $F_a \neq \emptyset$ and $\chi(a) = 0$ if $F_a = \emptyset$. Then χ is integrable over $G_p(V)$ with $0 \leqslant b_f(p) = I_f(\chi) \leqslant 1$. Here $b_f(p)$ is the probability that $f(M) \cap E_L[a] \neq \emptyset$. The First Main Theorem and the Mean Value Theorems imply easily that

$$0 \leqslant 1 - b_f(p) \leqslant \gamma_p A_f^{p-1}(r; L, l)/T_f^p(r, s; L, l)$$

for all $0 < s < r$. Hence, if

$$A_f^{p-1}(r; L, l)/T_f^p(r, s; L, l) \to 0 \quad \text{for } r \to \infty, \tag{5.1}$$

then $b_f(p) = 1$, which means $f(M) \cap E_L[a] \neq \varnothing$ for almost all $a \in G_p(V)$, which is a *Casorati-Wierstrass* Theorem.

The results of this section hold even if M and N are complex spaces and if f is meromorphic. Also the full force of the parabolic exhaustion τ is not needed. The assumption $\omega^m \equiv 0$ can be dropped, in which case τ is called a *logarithmic pseudoconvex exhaustion*. The results of this section extend also to the case where the line bundle L is replaced by a vector bundle E and a family of Schubert zeroes is considered. A Casorati-Weierstrass Theorem can be obtained; see **[28]**. For references to §5 see **[4]**, **[12]** and **[27]**.

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