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GENERATORS FOR ALGEBRAS OF RELATIONS[1]

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Let $\mathcal{B}_n$ denote the collection of all binary relations on the set $X = \{1, 2, \ldots, n\}$. The purpose of this paper is to observe that there exists a *pair* of relations on X that *generate* all of $\mathcal{B}_n$ under the boolean operations and relational composition.

In [1] C. J. Everett and S. M. Ulam introduced the notion of an abstract projective algebra. McKinsey [2] showed that every projective algebra is isomorphic to a subalgebra of a complete atomic projective algebra and thus, in view of the representation given in [1], every projective algebra is isomorphic to a projective algebra of subsets of a direct product; that is, to an algebra of relations.

PROJECTIVE ALGEBRA. A boolean algebra $\mathcal{P}$ with unit 1 and zero 0, so that for all $x \in \mathcal{P}$, $0 \leqslant x \leqslant 1$, is said to be a projective algebra if there are defined two mappings π_1 and π_2 of $\mathcal{P}$ into $\mathcal{P}$ satisfying the following:

P_1. $\pi_i(a \vee b) = \pi_i a \vee \pi_i b$.

P_2. $\pi_1\pi_2 1 = p_0 = \pi_2\pi_1 1$ where p_0 is an atom of $\mathcal{P}$.

P_3. $\pi_i a = 0$ if and only if $a = 0$.

P_4. $\pi_i\pi_i a = \pi_i a$.

P_5. For $0 < a \leqslant \pi_1 1$, $0 < b \leqslant \pi_2 1$, there exists an element $a \,\square\, b$ such that $\pi_1(a \,\square\, b) = a$, $\pi_2(a \,\square\, b) = b$, with the property that $x \in \mathcal{P}$, $\pi_1 x = a$, $\pi_2 x = b$ implies $x \leqslant a \,\square\, b$.

P_6. $\pi_1 1 \,\square\, p_0 = \pi_1 1$; $p_0 \,\square\, \pi_2 1 = \pi_2 1$.

P_7. $0 < x, y \leqslant \pi_1 1$ implies $(x \vee y) \,\square\, \pi_2 1 = (x \,\square\, \pi_2 1) \vee (y \,\square\, \pi_2 1)$; and $0 < u, v \leqslant \pi_2 1$ implies $\pi_1 1 \,\square\, (u \vee v) = (\pi_1 1 \,\square\, u) \vee (\pi_1 1 \,\square\, v)$.

If the projective algebra $\mathcal{P}$ is a complete atomic boolean algebra, then $\mathcal{P}$ is called a *complete atomic projective algebra.* The projective algebra $\mathcal{P}$ is said to be *projectively generated* by a subset $\mathcal{A}$ if $\mathcal{P}$ can be obtained from $\mathcal{A}$ using π_1, π_2, $\square$ and the boolean operations.

Consider $\mathcal{B}_n$ and let $p_0 = (1, 1)$. We define the mappings $\pi_1, \pi_2\colon \mathcal{B}_n \longrightarrow \mathcal{B}_n$ and a product $\square\colon \mathcal{B}_n \times \mathcal{B}_n \longrightarrow \mathcal{B}_n$ as follows:

(i) $\pi_1\alpha = \alpha((X \times X)p_0)$,

(ii) $\pi_2\alpha = (p_0(X \times X))\alpha$,

(iii) $\alpha \,\square\, \beta = (\alpha(X \times X))\beta$,

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where juxtaposition denotes the *composition* of relations.

It is easy to verify axioms P_1–P_7 to establish that $\mathcal{B}_n$ with the atom p_0 and the mappings π_1, π_2 and $\square$ as defined is a projective algebra.

The verification above, as well as the calculations below are made easier by noting the following equivalent forms of (i), (ii) and (iii):

(i) $\pi_1\alpha = \text{domain}(\alpha) \times \{1\}$;

(ii) $\pi_2\alpha = \{1\} \times \text{range}(\alpha)$;

(iii) $\alpha \square \beta = \text{domain}(\alpha) \times \text{range}(\beta)$.

THEOREM 1. *The projective algebra $\mathcal{B}_n$ can be projectively generated by a pair of disjoint elements.*

PROOF. We observe first that if we generate the atoms $(1, k)$ and $(k, 1)$, $1 \leqslant k \leqslant n$, then all others are obtained by taking the $\square$-product of suitable pairs of these.

Let $\alpha_0 = \{(x, y) | x < y\}$ and $\beta_0 = \{(x, y) | y < x\}$. Now $p_0 = (1, 1) = \pi_2\pi_1\beta_0$. If we let $\alpha_1 = \alpha_0 - (p_0 \square \pi_2\alpha_0)$ and $\beta_1 = \beta_0 - (\pi_1\beta_0 \square p_0)$, we get $(\pi_1\alpha_1 - \pi_1\beta_1) = (2, 1)$ and $(\pi_2\beta_1 - \pi_2\alpha_1) = (1, 2)$. Using the recursions $\alpha_{k+1} = \alpha_k - ((k+1, 1) \square \pi_2\alpha_k)$ and $\beta_{k+1} = \beta_k - (\pi_1\beta_k \square (1, k+1))$, noting that $\alpha_k = \{(x, y) | k < x < y\}$ and $\beta_k = \{(x, y) | k < y < x\}$, we see that $(\pi_1\alpha_k - \pi_1\beta_k) = (k+1, 1)$ and $(\pi_2\beta_k - \pi_2\alpha_k) = (1, k+1)$, for all $0 \leqslant k \leqslant n-2$. Also $\pi_1\beta_{n-2} = (n, 1)$ and $\pi_2\alpha_{n-2} = (1, n)$, so that we have generated all of the atoms mentioned above.

THEOREM 2. *The algebra of relations $\mathcal{B}_n$ can be generated, with respect to the boolean operations and composition, by two relations.*

PROOF. Let $\bar{\alpha} = \alpha_0 \cup \{(1, 1)\}$ and $\bar{\beta} = \beta_0 \cup \{(1, 1)\}$. Now $\bar{\alpha} \cap \bar{\beta} = \{(1, 1)\} = p_0$, $\bar{\alpha} \cup \bar{\beta} \cup \bar{\beta}\bar{\alpha} = X \times X$, $\alpha_0 = \bar{\alpha} - p_0$ and $\beta_0 = \bar{\beta} - p_0$. Since we defined the mappings π_1, π_2 and $\square$ in terms of the composition in $\mathcal{B}_n$, Theorem 2 is an immediate consequence of Theorem 1.

REMARK. It is well-known that $\mathcal{B}_n$ cannot be generated by a pair of elements using only the boolean operations. Moreover one can show that the compositional semigroup $\mathcal{B}_n$ cannot be generated by a pair of relations.

REFERENCES

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