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A CHARACTERIZATION OF OSTERWALDER-SCHRADER PATH SPACES BY THE ASSOCIATED SEMIGROUP

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This note is to announce a characterization of (generalized) path spaces satisfying the Osterwalder-Schrader positivity condition by the associated semigroup, on the lines of the characterization of Markov path spaces by positivity preserving semigroups (e.g. Simon [5], Klein and Landau [3]). In the semigroup characterization Osterwalder-Schrader path spaces are seen to be the natural generalization of Markov path spaces. As an application we discuss the existence of Euclidean fields given a relativistic Wightman field theory.

I. **Path spaces and semigroups.** A (generalized) *path space* $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ consists of a probability space (Q, Σ, μ); a distinguished sub-σ-algebra Σ_0; a one-parameter group $U(t)$ of measure preserving automorphisms of $L_\infty(Q, \Sigma, \mu)$ which are strongly continuous in measure; a measure preserving automorphism R of $L_\infty(Q, \Sigma, \mu)$ such that $R^2 = I$, $RU(t) = U(-t)R$, and $RE_0 = E_0R$ where E_0 is the conditional expectation with respect to Σ_0; where Σ is generated by $\bigcup_{t\in R} \Sigma_t$, $\Sigma_t = U(t)\,\Sigma_0$. By E_+ (E_-) we will denote the conditional expectation with respect to $\Sigma_+(\Sigma_-)$, the σ-algebra generated by $\bigcup_{t\geqslant 0} \Sigma_t$ $(\bigcup_{t\leqslant 0} \Sigma_t)$. The path space is said to be *Osterwalder-Schrader* if $\langle RF, F\rangle \geqslant 0$ for every $F \in L_2(Q, \Sigma_+, \mu)$. It is said to be *Markov* if $RE_0 = E_0$ and $E_+E_- = E_+E_0E_-$.

Every Markov path space is Osterwalder-Schrader [4]. In the case of a Markov path space $P(t) = E_0U(t)E_0$ gives a positivity preserving semigroup on $L_2(Q, \Sigma_0, \mu)$ [5], [3]. Given an Osterwalder-Schrader path space there exists [4] a Hilbert space $\mathcal{H}$ and a contraction $V: L_2(Q, \Sigma_+, \mu) \longrightarrow \mathcal{H}$ such that V has dense range and $P(t)V(F) = V(U(t)F)$ for $F \in L_2(Q, \Sigma_+, \mu)$ and $t \geqslant 0$ defines a strongly continuous selfadjoint contraction semigroup on $\mathcal{H}$. If $\Omega = V(1)$, then $\|\Omega\| = 1$ and $P(t)\Omega = \Omega$ for all $t \geqslant 0$.

For Osterwalder-Schrader path spaces we must look at another piece of structure, which is hidden in the Markov case.

LEMMA. *Let $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ be an Osterwalder-Schrader path*

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space, and let $\mathcal{H}$, V, $P(t)$, Ω *be as above. Then, if* $f \in L_\infty(Q, \Sigma_0, \mu)$, $\tilde{f}V(F) = V(fF)$ *for* $F \in L_2(Q, \Sigma_+, \mu)$ *defines a bounded operator on* $\mathcal{H}$ *with* $\|\tilde{f}\| = \|f\|_\infty$, *and* $\mathfrak{A} = \{\tilde{f} \mid f \in L_\infty(Q, \Sigma_0, \mu)\}$ *is a commutative von Neumann algebra of operators on* $\mathcal{H}$, *with* Ω *as a separating vector. Moreover, for any* $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$, $f_{t_i} = U(t_i)f_i$ *where* $f_i \in L_\infty(Q, \Sigma_0, \mu)$ *and* $i = 1, 2, \ldots, n$,

$$\int f_{t_1} f_{t_2} \cdots f_{t_n}\, d\mu = \langle \Omega, \tilde{f}_1 P(t_2 - t_1)\tilde{f}_2 \cdots P(t_n - t_{n-1})\tilde{f}_n \Omega\rangle.$$

$(\mathcal{H}, P(t), \mathfrak{A}, \Omega)$ is called the *associated semigroup structure. If* $((Q, \Sigma_0, \mu), \Sigma_0, U(t), R)$ is a Markov path space, $(L_2(Q, \Sigma_0, \mu), E_0U(t)E_0, L_\infty(Q, \Sigma_0, \mu), 1)$ is its associated semigroup structure.

Definition. A positive semigroup structure $(\mathcal{H}, P(t), \mathfrak{A}, \Omega)$ consists of a Hilbert space $\mathcal{H}$; a strongly continuous selfadjoint contraction semigroup $P(t)$ on $\mathcal{H}$; a commutative von Neumann algebra $\mathfrak{A}$ of operators on $\mathcal{H}$; a unit vector $\Omega \in \mathcal{H}$; such that $P(t)\Omega = \Omega$ for all $t \geqslant 0$; Ω is a cyclic vector for the algebra generated by $\mathfrak{A} \cup \{P(t) \mid t \geqslant 0\}$, i.e. the linear span of $\{P(t_1)f_1P(t_2) \ldots P(t_n)f_n\Omega \mid f_1, \ldots, f_n \in \mathfrak{A}, t_1, \ldots, t_n \geqslant 0\}$ is dense in $\mathcal{H}$; and for all $f_1, \ldots, f_n \in \mathfrak{A}^+ = \{f \in \mathfrak{A} \mid f \geqslant 0\}$ and $t_1, \ldots, t_n \geqslant 0$, $\langle \Omega, P(t_1)f_1P(t_2) \cdots P(t_n)f_n\Omega\rangle \geqslant 0$.

Osterwalder-Schrader path spaces are characterized by positive semigroup structures.

Theorem. *Let* $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ *be an Osterwalder-Schrader path space and* $(\mathcal{H}, P(t), \mathfrak{A}, \Omega)$ *its associated semigroup structure. Then* $(\mathcal{H}, P(t), \mathfrak{A}, \Omega)$ *forms a positive semigroup structure.*

Conversely, let $(\mathcal{H}, P(t), \mathfrak{A}, \Omega)$ *be a positive semigroup structure. Then there exists an Osterwalder-Schrader path space such that* $(\mathcal{H}, P(t), \mathfrak{A}, \Omega)$ *is its associated semigroup structure.*

Corollary. *Let* $((Q, \Sigma, \mu), \Sigma_0, U(t), R)$ *be an Osterwalder-Schrader path space, and* $(\mathcal{H}, P(t), \mathfrak{A}, \Omega)$ *its associated semigroup structure. The path space is Markov if and only if* Ω *is a cyclic vector for* $\mathfrak{A}$.

The details will appear elsewhere [2].

II. **Existence of Euclidean fields.** Our Theorem can be used to construct Euclidean fields given a relativistic Wightman field theory, in the same way Simon ([5], [6, Chapter IV]) used the similar result for Markov path spaces and positivity preserving semigroups to construct Euclidean fields. In Simon's scheme Axioms (S3) and (S4) [6, p. 120] are the basic elements in the construction of Euclidean fields. We can replace these axioms by the weaker:

Axiom 3′. The von Neumann algebra $\mathfrak{A}$ generated by the time zero fields is abelian; and the vacuum Ω is a cyclic vector for the von Neumann algebra generated by the fields at all fixed times.

AXIOM 4′. For all $F_1, \ldots, F_n \in \mathfrak{A}^+ = \{F \in \mathfrak{A} \mid F \geqslant 0\}$ and $t_1, \ldots, t_n \geqslant 0$, $\langle \Omega, e^{-t_1 H} F_1 e^{-t_2 H} F_2 \cdots e^{-t_n H} F_n \Omega \rangle \geqslant 0$.

We can then construct Euclidean fields satisfying Nelson's axioms, except for the Markov property which is replaced by the Osterwalder-Schrader positivity condition.

A detailed version of our axiom scheme will appear elsewhere [1], [2].

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