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BOUNDED HARMONIC BUT NO DIRICHLET-FINITE HARMONIC

BY YOUNG K. KWON

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ABSTRACT. The purpose of the present note is to announce that for each $n \geqq 3$ there exists a Riemannian n-manifold, which carries nonconstant bounded harmonic functions but no nonconstant Dirichlet-finite harmonic functions.

1. A C^2-function u on a Riemannian n-manifold M is harmonic on M if $\Delta u = 0$, where

$$\Delta u = \frac{-1}{g^{1/2}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(g^{1/2} g^{ij} \frac{\partial u}{\partial x^j}\right).$$

Here (g_{ij}) is the metric tensor for M, $(g^{ij}) = (g_{ij})^{-1}$, and $g = \det(g_{ij})$.

It is not known (cf. Sario-Nakai [**4**, p. 406]) whether or not for each $n \geqq 3$ there exists a Riemannian n-manifold M which carries nonconstant bounded harmonic functions but every harmonic function u on M is a constant whenever its Dirichlet integral

$$D(u) = \int_M \sum_{i,j=1}^{n} g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j}\, dV < \infty,$$

where $dV = g^{1/2}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ is the volume element. For $n = 2$ the problem was solved in the affirmative by Tôki [**5**], his example known as Tôki's example. Royden [**2**] and Sario [**3**] also obtained a similar result.

The purpose of the present note is to announce that for each $n \geqq 3$ there does exist a Riemannian n-manifold which solves the problem in the affirmative.[1] Details will be published elsewhere.

2. Fix $n \geqq 3$. Denote by M_0 the punctured Euclidean n-space $R^n - 0$ with the metric tensor

$$g_{ij}(x) = |x|^{-4}(1 + |x|^{n-2})^{4/(n-2)}\delta_{ij}, \quad 1 \leqq i,j \leqq n,$$

where $|x| = [\sum_{i=1}^n (x^i)^2]^{1/2}$ for $x = (x^1, x^2, \ldots, x^n) \in M_0$.

LEMMA. *Every positive harmonic function u on M_0 has the form*:

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[1] Professor Sario informed me that he recently obtained a similar result with Professors Wang and Hada.

$$u(x) = a/(1 + |x|^{n-2}) + b$$

for some constants a, b.

3. For each pair (m, l) of positive integers m, l, and $k = z^{m-1}(2l - 1) - 1$, set

$$H_{ml} = \{8^k x = (8^k x^1, 8^k x^2, \ldots, 8^k x^n) \in M_0 \mid |x| = 1 \text{ and } x^1 \geqq 0\},$$

$$H'_{ml} = \{8^{-k} x = (8^{-k} x^1, 8^{-k} x^2, \ldots, 8^{-k} x^n) \in M_0 \mid |x| = 1 \text{ and } x^1 \geqq 0\}.$$

Denote by M'_0 the manifold obtained from M_0 by deleting all the closed hemispheres H_{ml} and H'_{ml}.

Take two sequences $\{M'_0(l)\}_1^\infty$ and $\{M''_0(l)\}_1^\infty$ of duplicates of M'_0. For each fixed $m \geqq 1$ and subsequently fixed $j \geqq 0$ and $1 \leqq i \leqq m$, connect $M'_0(i + mj)$ with $M''_0(i + m + mj)$ for even j and $M'_0(i + mj)$ with $M''_0(i - m + mj)$ for odd j, crosswise along all the hemispheres H_{ml} and H'_{ml} $(l \geqq 1)$. The resulting Riemannian n-manifold N is an infinitely sheeted covering manifold of M_0.

THEOREM. *For the manifold N the following are true:*

$$(1) \quad \dim HB(N) = 2, \qquad (2) \quad \dim HD(N) = 1,$$

where H is the space of harmonic functions, D the space of Dirichlet-finite functions, B the space of bounded functions and HX stands for $H \cap X$.

It can be shown that every bounded harmonic function u on N takes the same value at all the points in N which lie over the same point in M_0. Therefore the function u is Dirichlet-finite only if u is a constant. Here we use the q-Lemma (cf. Rodin-Sario [**1**, p. 39]).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78712