BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 5, September 1972

TWO CONSISTENCY RESULTS IN TOPOLOGY

BY A. HAJNAL AND I. JUHÁSZ

Communicated by Mary Ellen Rudin, March 24, 1972

The present authors proved in 1966 that for any cardinal α, if a Hausdorff space X is of cardinality $> 2^{2^\alpha}$ then it contains a discrete subspace of cardinality $> \alpha$. The following result shows that this cannot be improved (even for completely regular spaces).

THEOREM 1. Con(ZF) $\rightarrow$ Con(ZFC & $2^\alpha = \alpha^+$ & 2^{α^+} *is anything reasonable & there exists a zero-dimensional* T_2 *space* X *of cardinality* $2^{\alpha^+} = 2^{2^\alpha}$ *which is hereditary* α*-separable).*

(A space is hereditary α-separable iff every subspace in it has a dense subset of cardinality $\leqq \alpha$.)

COROLLARY. *It is consistent to assume that* 2^{ω_1} *is big and there exists a zero-dimensional* T_2 *space in which the number of all open sets is any cardinal* β *with* $\omega_1 \leqq \beta < 2^{\omega_1}$ *and* $\mathrm{cf}(\beta) \neq \omega$.

This relates to a problem raised by J. de Groot.

A space X is called α-Lindelöf if every open cover of it can be reduced to a subcover of cardinality $\leqq \alpha$. X is hereditary α-Lindelöf iff every subspace of X is α-Lindelöf.

THEOREM 2. *Let* α *be a given cardinal number. Then* Con(ZF) $\rightarrow$ Con(ZFC & GCH & $\exists$ *zero-dimensional* T_2 *space* X *so that*

(i) $|X| = \alpha^+$ *and* X *is hereditary* α*-Lindelöf;*

(ii) $Y \subset X$ *and* $|Y| = \alpha^+$ *imply that the weight of the subspace* Y *is* α^{++};

(iii) $Y \subset X$ *and* $|Y| \leqq \alpha$ *imply that* Y *is closed and discrete).*

COROLLARY. *For any given* α, *it is consistent to assume that there exists a zero-dimensional* T_2 *space of weight* α^{++} *in which no subspace has the weight* α^+.

AMS 1969 *subject classifications.* Primary 5440, 5423, 0430, 0415, 0250, 0263.
Key words and phrases. Hereditarily separable, zero dimensional, cardinality, Lindelöf, weight.

MATHEMATICAL INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES, BUDAPEST, HUNGARY