SCHAUDER BASES IN SPACES OF DIFFERENTIABLE FUNCTIONS

BY STEVEN SCHONEFELD

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Banach [1, p. 238] states that Schauder bases are known for the spaces $C^k(I)$ but it is not known if $C^1(I\times I)$ has a Schauder basis. In this note we construct a Schauder basis for $C^1(I\times I)$.

1. **Definitions and notation.** We say that $\{x_n;\alpha_n\}$ (or simply $\{x_n\}$) is a Schauder basis for a Banach space X if for each $x\in X$ there exist unique scalars $a_i=\alpha_i(x)$ such that $x=\sum_{i=1}^{\infty} a_ix_i$ (i.e. the sequence of partial sums $\{\sum_{i=1}^{n} a_ix_i\}$ converges to x in norm).

It is well known [4] that each α_n is a continuous linear functional on X. Also, a total[1] set $\{x_n\}$ is a Schauder basis for X if and only if there exists a constant M such that

$$\left\|\sum_{i=1}^{p} a_ix_i\right\| \leqq M\left\|\sum_{i=1}^{p+q} a_ix_i\right\| \tag{1}$$

for any sequence $\{a_i\}$ of scalars and any natural numbers p, q. In the sequel we simply say "basis" for "Schauder basis".

We will denote by I the closed interval $[0, 1]$, by $C(I)$ the Banach space of real-valued continuous functions f defined on I with norm $\|f\|_\infty=\sup_{x\in I}|f(x)|$. $C^k(I)$ is the Banach space of real-valued f having k continuous derivatives with norm $\|f\|_k=\|f\|_\infty+\|f'\|_\infty+\cdots+\|f^{(k)}\|_\infty$. Finally, $C^1(I\times I)$ is the Banach space of real-valued functions $h=h(x, y)$ defined on $I\times I$ with continuous first partial derivatives. The norm for $C^1(I\times I)$ is given by

$$\|h\| = \sup_{(x,y)\in I\times I}|h(x,y)| + \sup_{(x,y)\in I\times I}\left|\frac{\partial}{\partial x}h(x,y)\right| + \sup_{(x,y)\in I\times I}\left|\frac{\partial}{\partial y}h(x,y)\right|.$$

2. **Construction of bases for $C^k(I)$.** Let $\{\phi_n;\mu_n\}$ be any basis for $C(I)$ and let

$$\begin{aligned} f_1(x) &= 1, & \alpha_1(f) &= f(0),\\ f_n(x) &= \int_0^x \phi_{n-1}(t)dt, & \alpha_n(f) &= \mu_{n-1}(f'), \end{aligned} \tag{2}$$

[1] total = finite linear combinations are dense in X.

for

$$f \in C^1(I), \qquad x \in I, \qquad n = 2, 3, \cdots,$$

then $\{f_n; \alpha_n\}$ is a basis for $C^1(I)$.

One can obtain a basis for $C^k(I)$ by repeating the above process k times. The resulting basis for $C^k(I)$ is

$$(3) \qquad \begin{aligned} f_1(x) &= 1, & \alpha_1(f) &= f(0), \\ &\cdots\cdots\cdots\cdot & &\cdots\cdots\cdots\cdot \\ f_k(x) &= \frac{1}{(k-1)!} x^{k-1}, & \alpha_k(f) &= f^{(k-1)}(0), \\ f_n(x) &= \int_0^x \int_0^{t_{k-1}} \cdots \int_0^{t_2} \int_0^{t_1} \phi_{n-k}(t)\,dt\,dt_1 \cdots dt_{k-1}, \\ \alpha_n(f) &= \mu_{n-k}(f^{(k)}) \end{aligned}$$

for

$$f \in C^k(I), \quad x \in I, \quad \text{and} \quad n = k+1, k+2, \cdots.$$

Conversely, let $\{g_n; \beta_n\}$ be a basis for $C^k(I)$ which satisfies

$$g_i(x) = \frac{1}{(i-1)!} x^{i-1}, \qquad \beta_i(f) = f^{(i-1)}(0)$$

for $i = 1, 2, \cdots, k$, then $\{g_n^{(k)}; n = k+1, n+2, \cdots\}$ is a basis for $C(I)$

3. **A basis for $C^1(I \times I)$.** Ciesielski [3, p. 320] has shown that if $\{\phi_n\}$ is the Franklin basis for $C(I)$, then the resulting indefinite integral basis (2) is a basis for both $C^1(I)$ and $C(I)$. A theorem of Kreĭn, Milman and Rutman [6] assures us that "many" bases will do this.

LEMMA 1. *The polynomials in x and y are dense in $C^1(I \times I)$.*

The proof is an immediate consequence of Lemma 2 [2, p. 109].

Next, let $\{\phi_n; \mu_n\}$, $\{\psi_n; \nu_n\}$ be bases for $C(I)$ for which the indefinite integral bases $\{f_n; \alpha_n\}$, $\{g_n; \beta_n\}$ are also bases for $C(I)$.

Note. The expansion $\sum_{i=1}^{\infty} \alpha_i(f) f_i$ for a differentiable function f is the same if we consider f as an element of $C^1(I)$ or $C(I)$. Let

$$(4) \qquad \begin{aligned} S_n f &= \sum_{i=1}^{n} \alpha_i(f) f_i, & \Phi_n \phi &= \sum_{i=1}^{n} \mu_i(\phi) \phi_i, \\ T_n f &= \sum_{i=1}^{n} \beta_i(f) g_i, & \Psi_n \phi &= \sum_{i=1}^{n} \nu_i(\phi) \psi_i, \end{aligned}$$

for $f \in C^1(I)$, $\phi \in C(I)$, and $n = 1, 2, \cdots$.

There exist constants L_1, L_2, M_1, M_2, as in (1) such that

$$\|S_nf\|_\infty \leqq L_1\|f\|_\infty, \qquad \|\Phi_n\phi\|_\infty \leqq L_2\|\phi\|_\infty,$$
$$\|T_nf\|_\infty \leqq M_1\|f\|_\infty, \qquad \|\Psi_n\phi\|_\infty \leqq M_2\|\phi\|_\infty.$$

For $h(x, y) = f(x)g(y)$, we define

$$\begin{aligned} S_nh &= \left[\sum_{i=1}^{n} \alpha_i(f)f_i\right]g, & \Phi_nh &= \left[\sum_{i=1}^{n} \mu_i(f)\phi_i\right]g, \\ T_nh &= f\left[\sum_{i=1}^{n} \beta_i(g)g_i\right], & \Psi_nh &= f\left[\sum_{i=1}^{n} \nu_i(g)\psi_i\right]. \end{aligned} \tag{5}$$

We extend the operators (5) to all polynomials h by linearity. (We feel justified in confusing the operators in (4) with those in (5) since there will be no ambiguity if one keeps track of the function to which the operators are being applied.)

Thus, for a polynomial $h(x, y)$ equations (2) give us

$$\frac{\partial}{\partial x} S_nh = \Phi_{n-1}\frac{\partial}{\partial x} h$$

and

$$\frac{\partial}{\partial y} S_nh = S_n\frac{\partial}{\partial y} h$$

for $n = 1, 2, \cdots$ with $\Phi_0 h = 0$. Therefore, we get

$$\begin{aligned} \|S_nh\| &= \sup_{(x,y)} |S_nh(x, y)| + \sup_{(x,y)}\left|\frac{\partial}{\partial x} S_nh(x, y)\right| \\ &\quad + \sup_{(x,y)}\left|\frac{\partial}{\partial y} S_nh(x, y)\right| \\ &= \sup_y\left[\sup_x |S_nh(x, y)|\right] + \sup_y\left[\sup_x\left|\Phi_{n-1}\frac{\partial}{\partial x} h(x, y)\right|\right] \\ &\quad + \sup_y\left[\sup_x\left|S_n\frac{\partial}{\partial y} h(x, y)\right|\right] \\ &\leqq L_1\left[\sup_{(x,y)} |h(x, y)|\right] + L_2\left[\sup_{(x,y)}\left|\frac{\partial}{\partial x} h(x, y)\right|\right] \\ &\quad + L_1\left[\sup_{(x,y)}\left|\frac{\partial}{\partial y} h(x, y)\right|\right] \\ &\leqq L\|h\| \end{aligned}$$

where $L=\max[L_1, L_2]$.

Similar calculations give

$$\|T_n h\| \leqq M\|h\|.$$

This permits us to extend S_n and T_n to all of $C^1(I\times I)$ with

$$\|S_n h\| \leqq L\|h\| \quad \text{and} \quad \|T_n h\| \leqq M\|h\| \tag{6}$$

for any $h\in C^1(I\times I)$.

We enumerate $N\times N$ in the following way.

$$\begin{aligned} &\{(1, 1), (1, 2), (2, 1), (2, 2), \cdots, (n, n), (1, n+1), \\ &(2, n+1), \cdots, (n, n+1), (n+1, 1)(n+1, 2), \cdots, \\ &\qquad\qquad\qquad\qquad\qquad\qquad (n+1, n+1), \cdots\}. \end{aligned} \tag{7}$$

We let $h_p(x, y)=f_i(x)g_j(y)$, where (i, j) is the pth element in the enumeration (7).

THEOREM 1. *The functions* $\{h_p\}$ *form a basis for* $C^1(I\times I)$.

PROOF. First we show that $\{h_p\}$ is total in $C^1(I\times I)$. In view of Lemma 1, we need only show that we can approximate a function of the form $h(x, y)=f(x)g(y)$. We have

$$\begin{aligned} \|h - S_m T_n h\| &\leqq \|h - S_m h\| + \|S_m h - S_m T_n h\| \\ &\leqq \|h - S_m h\| + L\|h - T_n h\|. \end{aligned} \tag{8}$$

The right-hand side of (8) converges to zero as $(m, n)\rightarrow(\infty, \infty)$ since

$$\begin{aligned} \|h - S_m h\| &\leqq \|f - S_m f\|_\infty \|g\|_\infty \\ &\quad + \|f' - \Phi_{m-1} f'\|_\infty \|g\|_\infty + \|f - S_m f\|_\infty \left\| \frac{\partial}{\partial y} g \right\|_\infty \\ &= o(1) \end{aligned}$$

as $m\rightarrow\infty$ and a similar calculation shows that $\|h-T_n h\|=o(1)$ as $n\rightarrow\infty$. Next, let $h=\sum_{i=1}^{p+q} a_i h_i$. We show that

$$\left\| \sum_{i=1}^{p} a_i h \right\| < 3LM\|h\|, \tag{9}$$

where L and M are the constants in (6).

Case 1. $p=n^2+m$ with $1\leqq m\leqq n$, then

$$\sum_{i=1}^{p} a_i h_i = S_n T_n h + S_m(T_{n+1} - T_n)h.$$

Case 2. $p=n^2+n+m$ with $1\leqq m\leqq n+1$ then

$$\sum_{i=1}^{p} a_i h_i = S_n T_{n+1} h + (S_{n+1} - S_n) T_m h.$$

In either case, the triangle inequality gives (9).

4. **Comments.** The above construction will generalize to give a basis for $C^1(I\times I\times \cdots \times I)$ with little difficulty. In order to use this method to get a basis for $C^k(I\times I)$ one needs the basis $\{f_n; \alpha_n\}$, defined in (3), to be a basis for $C(I)$. However, this is impossible for $k>1$ since the functional α_2 can not be extended to a continuous functional on $C(I)$. Finally, we notice that the basis $\{h^p\}$ for $C^1(I\times I)$ is also a basis for $C(I\times I)$ (see [5] and [7]).

References

1. S. Banach, *Théorie des opérations linéaires*, Monografie Mat., PWN, Warsaw, 1932; reprint, Chelsea, New York, 1955.
2. P. L. Butzer, *On two-dimensional Bernstein polynomials*, Canad. J. Math **5** (1953), 107–113.
3. Z. Ciesielski, *Properties of the orthonormal Franklin system.* II, Studia Math. **27** (1966), 289–323.
4. M. M. Day, *Normed linear spaces*, Academic Press, New York, 1962.
5. B. R. Gelbaum and J. Gil de Lamadrid, *Bases of tensor products of Banach spaces*, Pacific J. Math **11** (1961), 1281–1286.
6. M. Kreĭn, D. Milman and M. Rutman, *A note on bases in Banach spaces*, Comm. Inst. Sci. Math. Mec. Univ. Kharkoff [Zapiski Inst. Mat. Mec.] (4) **16** (1940), 106–110.
7. Z. Semadeni, *Product Schauder bases and approximation with nodes in spaces of continuous functions*, Bull. Acad. Polon. Sci. **11** (1963), 387–391.

Purdue University, Lafayette, Indiana 47907