ON HEIGHTS IN NUMBER FIELDS[1]

BY S. SCHANUEL

Communicated by Felix Browder, May 14, 1963

Let K be a number field, of degree N over $\boldsymbol{Q}$.

Let S_∞ be the set of archimedean absolute values of K, normalized to extend the ordinary absolute value on $\boldsymbol{Q}$. For $x \in K^*$, $v \in S_\infty$, put $\|x\|_v = |x|_v^{N_v}$, where N_v is the local degree $[K_v : \boldsymbol{Q}_v]$, so $N_v = 1$ or 2. For $X = (X_1, \cdots, X_m) \in K^m$, put $\|X\|_v = \sup \|X_i\|_v$, $H_\infty(X) = \prod_{v \in S_\infty} \|X\|_v$; and let $[X]$ denote the fractional ideal generated by $X_1, \cdots, X_m$. Then the height of X is $\boldsymbol{N}[X]^{-1} H_\infty(X)$.

The *class* of a point in projective space $P^{m-1}(K)$ is the class (modulo principal ideals) of the fractional ideal generated by homogeneous coordinates for X.

THEOREM 1. *The number of points in $P^{m-1}(K)$ of a given class, with height at most B, is*

$$\frac{\kappa_m}{\zeta_K(m)} B^m + O(B^{m-1/N}),$$

where

$$\kappa_m = \left(\frac{2^{r_1} (2\pi)^{r_2}}{\sqrt{d}} \right)^m \frac{R}{w} m^r;$$

except that for $m = 2$, $N = 1$, the error term is to be replaced by $O(B \log B)$.

The notation is standard (cf. [1]). For a discussion of the setting of the problem, see [2, Chapter III]. The burden of the proof is carried by Theorem 2.

An S_∞-divisor, or simply *divisor* on K is a pair $\mathfrak{d} = (\mathfrak{a}, B)$, where $\mathfrak{a}$ is a nonzero fractional ideal and B is a positive real number. The *norm* of $\mathfrak{d}$ is $\|\mathfrak{d}\| = \boldsymbol{N}\mathfrak{a}^{-1} B$. Map $K^m - 0^m$ to the group of divisors by $\mathfrak{d}_X = ([X], H_\infty(X))$. For $m = 1$, this is a homomorphism, with kernel U (the group of units of K) and image the *principal divisors.*

Let U act on K^m by componentwise multiplication. Then associated to any divisor $\mathfrak{d}$ is an *S_∞-parallelotope* $L_m(\mathfrak{d}) \subset (K^m - 0^m)/U$; it is the set of all orbits UX for which $\mathfrak{d}_X \leqq \mathfrak{d}$, that is: $[X] \subset \mathfrak{a}$, $H_\infty(X) \leqq B$. Similarly, the *restricted S_∞-parallelotope* $L'_m(\mathfrak{d})$ is the set of all UX for which $[X] = \mathfrak{a}$, $H_\infty(X) \leqq B$. Let λ_m, λ'_m be the cardinalities of L_m, L'_m.

[1] Details and related results will appear in a forthcoming paper.

REMARK 1. $\lambda_m(\mathfrak{d})$, $\lambda'_m(\mathfrak{d})$, $\|\mathfrak{d}\|$ depend only on the class of $\mathfrak{d}$ modulo principal divisors.

REMARK 2. $\lambda_m(\mathfrak{d}) = \lambda'_m(\mathfrak{d}) = 0$ for $\|\mathfrak{d}\| < 1$.

THEOREM 2. $\lambda_m(\mathfrak{d}) = \kappa_m \|\mathfrak{d}\|^m + O(\|\mathfrak{d}\|^{m-1/N})$.

One may restrict $\mathfrak{d}$ to range over divisors $(\mathfrak{a}_i, B)$, where $\mathfrak{a}_i$ are representatives for the ideal classes, by Remark 1; thus it suffices to consider divisors with $\mathfrak{a}$ fixed. For $m=1$, $L_1(\mathfrak{a}, B)$ is the set of principal ideals contained in $\mathfrak{a}$, of norm at most B. Hence Theorem 2 reduces in this case to a classical theorem due to Dedekind and Weber [3].

The reduction of Theorem 1 to Theorem 2 is based on two easy observations. First, there is a bijection from $L'_m(\mathfrak{a}, BN\mathfrak{a})$ to the set of points of $P^{m-1}(K)$ of class $\mathrm{Cl}(\mathfrak{a})$, by $UX \to K^*X$, so that the problem is to estimate λ'_m. Second, $L_m(\mathfrak{a}, B) = \bigcup L'_m(\mathfrak{a}\mathfrak{b}, B)$, the (disjoint) union extending over all integral ideals $\mathfrak{b}$. Thus

$$\lambda_m(\mathfrak{a}, B) = \sum \lambda'_m(\mathfrak{a}\mathfrak{b}, B).$$

(The sum is finite, since $\lambda'_m(\mathfrak{a}\mathfrak{b}, B) = 0$ for $N\mathfrak{b} > BN\mathfrak{a}^{-1}$, by Remark 2.) A variant of the Möbius inversion formula gives

$$\lambda'_m(\mathfrak{a}, B) = \sum \mu(\mathfrak{b}) \lambda_m(\mathfrak{a}\mathfrak{b}, B).$$

By Theorem 2, the sum on the right is

$$\sum \mu(\mathfrak{b}) \left(\kappa_m \left(\frac{\|\mathfrak{d}\|}{N\mathfrak{b}} \right)^m + O\left(\left(\frac{\|\mathfrak{d}\|}{N\mathfrak{b}} \right)^{m-1/N} \right) \right),$$

summed over all integral $\mathfrak{b}$ with $N\mathfrak{b} \leqq \|\mathfrak{d}\| = BN\mathfrak{a}^{-1}$. The first term contributes

$$\kappa_m \|\mathfrak{d}\|^m \left(\frac{1}{\zeta_K(m)} + O(\|\mathfrak{d}\|^{1-m}) \right),$$

and the second is easily estimated to yield Theorem 1.

BIBLIOGRAPHY

1. Erich Hecke, *Vorlesungen über die Theorie der algebraischen Zahlen*, Akademie-Verlag, Leipzig, 1923.

2. Serge Lang, *Diophantine geometry*, Interscience, New York, 1962.

3. Heinrich Weber, *Lehrbuch der Algebra*, Band II, Absch. 20, 21, Vieweg, Braunschweig, 1899.

COLUMBIA UNIVERSITY