SOME STRUCTURAL PROPERTIES OF HAUSDORFF MATRICES

BY B. E. RHOADES

Communicated by Einar Hille, October 10, 1958

1. **Definitions.** Let $A=(a_{nk})$ denote an infinite matrix. A is called *conservative* if A has finite norm, $a_k=\lim_{n\to\infty} a_{nk}$ exists for each k, and $\lim_{n\to\infty}\sum_k a_{nk}$ exists. A is called *multiplicative* if A is conservative and $a_k=0$ for each k.

s denotes the space of sequences, m the subspace of bounded sequences, and c the subspace of convergent sequences. E_1 is the field of complex numbers and E_∞ the set of sequences, each of which possesses only a finite number of nonzero terms.

Let x be a fixed sequence. Then $c\oplus x=\{y+x \mid y\in c\}$.

Let $H=(h_{nk})$ denote a Hausdorff matrix generated by a sequence μ. I shall use (H, μ) to denote the convergence domain of H, H_μ to denote the matrix, and $H\sim\mu$ to denote the method.

A matrix $A=(a_{nk})$ is said to be of property P, displacement m (written $c^m A$ is of property P) if, for all $k\geqq m$, a_{nk} possesses property P.

A *corridor* matrix is a matrix with the property that there exists a positive integer r such that $a_{nk}=0$ for all n and k with $|n-k|>r$. The smallest such r denotes the width of A.

2. **Introduction.** Let H denote the set of Hausdorff matrices with finite norm. H coincides with the set of conservative Hausdorff matrices as a result of [**1**, page 256, lines 8–12].

Hille [**2**] denotes the set of all multiplicative Hausdorff matrices by M, and observes that it forms a commutative Banach algebra which is also an integral domain. Hence the concepts of unit, prime, divisibility, associate, multiple, and factor can be defined in M. Hille and Tamarkin [**3**, p. 576; **4**, p. 907] observed that every moment function $\mu(z)$ of the form $\mu(z)=(z-a)/(z+b)$, $\Re(a)>0$, $\Re(b)>0$, is prime in M; i.e., $H\sim\mu$ is not equivalent to convergence, but includes only methods that are equivalent to convergence. Hille mentioned this fact in [**2**, p. 422], and again raised the open question as to whether all primes in M are of this form.

From Hille's definition of a prime moment function, a regular Hausdorff matrix H with the property that $(H, \mu)=c\oplus x$ for some unbounded sequence x would have a moment function $\mu(z)$ which would be a prime element of M. The results stated in this paper show that it is impossible to construct a Hausdorff matrix $H\in$H with $(H, \mu)=c\oplus x$, for some unbounded x, by the technique of Zeller [5].

3. **Results.** The proofs of the following theorems stem from the definition of a Hausdorff matrix, and use the fact that knowledge of the values of any two of the terms $h_{n,k}$, $h_{n+1,k}$, $h_{n+1,k+1}$ gives information about the third. For example, if for some n and k $h_{n,k}=h_{n+1,k}=0$, then $h_{n+1,k+1}=0$. See [1, p. 255, line 14].

THEOREM 1. *Suppose that for some integer k, $\{h_{nk}\}\in E_\infty$. Then c^kH is a corridor matrix of width $N-k$.*

THEOREM 2. *Let $h_{nk}=\alpha_n$ for $k=0$, $=0$ for $k>0$, $\alpha_n\in E_1$. Then H is a Hausdorff matrix$\leftrightarrow\alpha_n=c$, c a constant.*

THEOREM 3. *Let H be triangular with $h_{n,0}=c$, $c\in E_1$, for all n. Then H is Hausdorff$\leftrightarrow h_{n,k}=0$ for all $k>0$.*

THEOREM 4. *There does not exist a Hausdorff corridor matrix of width $k>1$ with the elements on the kth diagonal all the same number c.*

THEOREM 5. *Let H be a Hausdorff matrix with its $(k+1)$th column an element of E_∞. Then, for $k\neq 0$, either the kth column is an element of E_∞ or it has no two succeeding elements the same number.*

THEOREM 6. *Suppose that H is a Hausdorff corridor matrix of width $r>1$. Then $H\notin\mathrm{H}$. For $r=1$, $\mu_0\neq 0$, H is a diagonal matrix and hence is equivalent to convergence.*

THEOREM 7. *Let c^mH be a corridor matrix of width $r>1$. Then $H\notin\mathrm{H}$.*

THEOREM 8. *If c^mH is a diagonal matrix, $m>0$, then $\mu_k=\mu_{k+1}$ for $k\geqq m$.*

THEOREM 9. *Let c^mH be a diagonal matrix, $m>0$. Then c^mH cannot have $\{h_{nk}\}\in E_\infty$ for $0\leqq k\leqq m-1$.*

THEOREM 10. *Let c^mH be a diagonal matrix, $m>0$. Then $h_{n,m-1}\neq 0$ for $n>m-1$.*

THEOREM 11. *Let c^mH be a null matrix, $m>0$; i.e., $h_{nk}=0$ for all $n\geqq k\geqq m$. Then $\mu_{m-1}\neq 0$.*

THEOREM 12. *If c^mH is a diagonal matrix, $m>1$, then $H\notin\mathrm{H}$.*

4. **Conclusion.** Combining the results of the above set of theorems we see that H is composed of two types of matrices. For $H\in\mathrm{H}$, either c^mH is a diagonal matrix for $m=0, 1$, or H has no column an element of E_∞.

Let $D=\{H_\mu \mid \mu=\{\mu_0 c, c, \cdots\}, \mu_0, c\in E_1\}$. Then D is a Banach algebra.

In constructing a regular matrix A with $c_A=c\oplus x$ for some unbounded x, Zeller [**5**] uses a nondiagonal matrix with columns which are elements of E_∞. However, such a construction is not possible for any $H\in\mathrm{H}$, hence, a fortiori, not possible for any regular H.

It is still, as far as I know, an open question whether there exists a Hausdorff matrix $H\in\mathrm{H}$ with $(H, \mu)=c\oplus x$ for some unbounded x.

References

1. G. H. Hardy, *Divergent series*, Oxford, 1949.

2. E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publication, vol. 31, 1948.

3. E. Hille and J. D. Tamarkin, *Questions of relative inclusion in the domain of Hausdorff means*, Proc. Nat. Acad. Sci. U.S.A. vol. 19 (1933) pp. 573–577.

4. ———, *On moment functions*, Proc. Nat. Acad. Sci. U.S.A. vol. 19 (1933) pp. 902–908.

5. K. Zeller, *Merkwürdigkeiten bei Matrixverfahren; Einfolgenverfahren*, Arch. Math. vol. 4 (1953) pp. 1–5.

Lafayette College