Kernel functions and elliptic differential equations in mathematical physics. By S. Bergman and M. Schiffer. New York, Academic Press, 1953. 14+432 pp. $8.00.

This book consists of two parts. Part A, entitled *Boundary value problems for partial differential equations of elliptic type*, discusses a number of general problems of classical mathematical physics which can be reduced to boundary value problems either for the Laplace equation or for the equation

$$(1) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - p(x, y)u = 0, \qquad p(x, y) > 0,$$

and its three-dimensional analogue, and the theory of these problems is developed from a more or less elementary and, occasionally, heuristic point of view. In part B (*Kernel function methods in the theory of boundary value problems*) equation (1) is subjected to detailed and rigorous analysis.

Part A consists of the following chapters: I. Theory of heat conduction; II. Fluid dynamics; III. Electro- and magnetostatics; IV. Elasticity. Chapter I starts with the derivation of the equation of conduction of heat in a nonhomogeneous medium. Steady heat flows are examined further. The homogeneous case leads to Laplace's equation, and the equation div (κ grad T) $=0$, which arises in the case of variable conductivity κ, is shown to be reducible to an equation of type (1) by a simple transformation. The Green function, the Neumann function, and the corresponding function for the boundary condition $\partial T/\partial\nu - \lambda T = 0$ (denoted by $R_\lambda(P, Q)$ and called Robin's function in the book) are introduced and are given the proper physical interpretation. Uniqueness and some descriptive properties are proved for these functions; questions of existence are postponed to part B. The minimum property of the Dirichlet integral is formulated.

The second chapter, dealing with fluid dynamics, is by far the largest chapter of the book. It consists of 24 paragraphs. After deriving the equations for general (§1) and stationary irrotational compressible flows (§2), the authors turn to the special case of three-dimensional incompressible flow, i.e. again to Laplace's equation. Here Neumann's function is studied also for infinite domains, and the notion of regularity at infinity of a harmonic function is discussed. The tensor of virtual mass of a body is linked to the power series expansion at infinity of certain velocity potentials, and the Diaz-Weinstein minimum property of the virtual mass is derived. Variational formulas for Neumann's function are established and applied to the study of free-boundary problems. The next 7 paragraphs are devoted to two-dimensional steady irrotational incompressible flows. The connection to analytic functions of a complex variable is made at the outset and is used to express Green's function in terms of a mapping function. The flow around an obstacle is treated next; the discussion includes a proof of the paradox of d'Alembert and of the formulas of Blasius and Kutta-Joukowski. At this point the complex kernel function is introduced as a certain combination of second derivatives of Green's function possessing a simple variational formula. In §19 and §20 axially symmetric motions of an incompressible fluid are discussed. As in the two-dimensional case the velocity potential and the stream function are related to each other by a system of first-order partial differential equations which can be considered as a generalization of the Cauchy-Riemann equations. The theory of such systems is studied from a general point of view. Finally (§21–24) the authors study two-dimensional compressible fluid flow. The relevant equation for the potential, nonlinear in the physical plane, is shown to become linear (and reducible to (1)) in the hodograph plane. Again this equation can be split up into a system of first-order equations. Particular solutions are obtained (a) by using the theory of l-monogenic functions (as developed by L. Bers et al.) and (b) by separation of variables. The first procedure foreshadows the use of integral operators in part B for the same purpose.

In chapter III the basic equations of electro- and magnetostatics are derived. Whereas in chapter II Neumann's function was fundamental, the significant quantities of the field are now expressed in terms of Green's function. To show an application of the general theory, the authors determine the magnetic field in the metal core of a transformer. An elegant treatment, based on a variational formula for Green's function, is given for the theory of forces and moments in a conductor system. For Laplace's equation the kernel function is introduced as the difference of Neumann's and Green's functions, and a number of its properties are derived.

Chapter IV starts with a derivation of the basic equations of elasticity and a discussion of the simplest boundary value problems. The rôles of the Green, the Neumann and the Robin function are now taken over by the corresponding tensor functions. The difference of the Neumann and the Green tensor is called the kernel tensor and is shown to have properties analogous to those of the kernel functions defined in chapters II and III. Some "general solutions" of the differential equations are indicated, and the theory of thin plates is touched at.

The chapters of part B are: I. Properties of solutions; II. The kernel functions and their properties; III. Variational and comparison theory; IV. Existence theory; V. Dependence of kernels on boundary conditions and the differential equation; VI. Generalizations. It is assumed throughout that the function $p(x, y)$ in (1) is continuous and continuously differentiable in the closed domain $D+C$, and at a few places also analyticity of $p(x, y)$ is required. The boundary C of D is assumed to consist of a finite number of closed analytic curves, although it is stated that most results remain valid for smooth or even piecewise smooth boundaries.

In chapter I of part B the linear space Ω of functions $u(x, y)$ which are twice continuously differentiable in $D+C$ is considered. The expression

$$E\{u\} = \iint_D (u_x^2 + u_y^2 + pu^2)dxdy$$

is used as norm of an element $u \in \Omega$ and is shown to have the usual properties. Correspondingly, a scalar product is defined by

$$E\{u, v\} = \iint_D (u_x v_x + u_y v_y + puv)dxdy.$$

Two subspaces of Ω are considered: (a) The linear space Ω^0 of all functions $v \in \Omega$ which vanish on C; (b) the linear space Σ of all functions $u \in \Omega$ which satisfy (1). It is shown that $\Omega = \Sigma + \Omega^0$. Some maximum properties for the functions $u \in \Sigma$ are derived. The equation $\Delta u - pu = f(P)$ is considered for the boundary conditions $u=0$ and $\partial u/\partial \nu = 0$. The existence of Green's function $G(P, Q)$ and Neumann's function $N(P, Q)$ for equation (1) is proved by the method of integral equations under the assumption that Green's and Neumann's functions exist for Laplace's equation. Robin's function is introduced in the same way. It is shown that

$$\begin{aligned} E\{N(P,Q), v(P)\} &= v(Q), && v \in \Omega, \\ E\{G(P,Q), v(P)\} &= v(Q), && v \in \Omega^0, \\ E\{G(P,Q), v(P)\} &= 0, && v \in \Sigma. \end{aligned}$$

In chapter II the kernel function $K(P, Q) = N(P, Q) - G(P, Q)$ is introduced. Unlike $N(P, Q)$ and $G(P, Q)$, $K(P, Q)$ is in Σ. We have

$$E\{K(P,Q), v(P)\} = v(P), \qquad v \in \Sigma.$$

The relations

$$u(Q) = -\int_C K(P, Q) \frac{\partial u}{\partial \nu_P} ds_P = \int_C u \frac{\partial K(P, Q)}{\partial \nu_P} ds_P$$

hold for $u \in \Sigma$; they show that the knowledge of $K(P, Q)$ alone is sufficient for the solution of the first as well as of the second boundary value problem. The kernel function can also be characterized independently as follows: Of all functions $u \in \Sigma$ having the value 1 at a point $Q \in D$, the function $u(P) = K(P, Q)/K(Q, Q)$ has the least norm. The kernel function has the series expansion

$$K(P, Q) = \sum_{n=0}^{\infty} u_n(P) u_n(Q),$$

where $\{u_n(P)\}$ is an arbitrary complete orthonormal set of solutions of (1) (complete in Σ and in the sense of the above norm). Two methods for the construction of a complete system are given. For a fixed fundamental singularity $S(P, Q)$ the functions

$$g(P, Q) = G(P, Q) - S(P, Q), \qquad n(P, Q) = N(P, Q) - S(P, Q)$$

are investigated. It is shown that $l(P, Q) = g(P, Q) + n(P, Q)$ is continuously differentiable in $D + C$ with respect to both argument points. Finally, the kernel $K(P, Q)$ is developed into a series of iterated integrals of $S(P, Q)$.

Chapter III contains a study of the behavior of the fundamental functions and of the kernel function under infinitesimal and finite changes of the domain D. A number of variational formulas are given here.

The remaining three chapters are shorter. In chapter IV the existence of a solution of the first boundary value problem for equation (1) is proved by the method of orthogonal projections. The "projection" of an arbitrary function $v \in \Omega$ with the prescribed boundary values is made into Σ rather than Ω^0, as was done in the classical treatments of the existence problem. In chapter V the dependence of the solutions on the character of the boundary conditions and on the coefficient $p(x, y)$ are studied. The Robin function $R_\lambda(P, Q)$ is shown to decrease with increasing λ, so that in particular $N(P, Q) \geqq R_\lambda(P, Q) \geqq G(P, Q)$. Also, for fixed boundary values the solutions of (1) decrease with increasing $p(x, y)$. In chapter IV, besides $E\{u, v\}$ which in Σ can be written

$$E\{u, v\} = -\int_C u \frac{\partial v}{\partial \nu} ds,$$

two new metrics are introduced in Σ; they are defined, respectively, by

$$(u, v) = \int_C u(P)v(P)ds$$

and

$$[u, v] = \int_C \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} ds.$$

The Stekloff functions $\chi_m(P)$, defined as those solutions of (1) which on C are the normalized eigenfunctions of the integral equation

$$\chi(P) = \int_C N(P, Q)\chi(Q)ds_Q, \qquad P \in C,$$

are shown to be orthogonal in all three metrics considered. They yield interesting new representations for the various fundamental functions and for the kernel function. The two metrics based on (u, v) and $[u, v]$ remain also applicable when the assumption $p(x, y) > 0$ is dropped.

At the end there are a list of symbols, an extensive bibliography and two indexes.

The book is written in a constructive spirit and the style is very readable. In spite of a remark in the preface this reviewer can see no reason why at least the earlier parts of the book could not be used successfully as basis of an advanced course in mathematical physics.

PETER HENRICI