mappings of the upper half-plane into itself) and a study of quasi-self-adjoint extensions. (If A is symmetric with defect index (m, m), $m < \infty$, a quasi-self-adjoint extension is an operator B such that $A \subset B \subset A^*$ and such that the co-dimension of the domain of A in the domain of B is equal to m.) The second appendix studies self-adjoint extensions of differential operators; it concludes with some concrete examples (such as the differential equations satisfied by the Bessel functions).

A student would find the detailed reading of the book highly profitable. If the authors need an analytic fact, they prove it. They prove, in particular, the completeness of L_2, they prove Plancherel's theorem, and they prove Bochner's theorem on the representation of positive definite functions. (The latter is used to prove the spectral theorem for self-adjoint operators; similarly, the spectral theorem for unitary operators is made to follow from the solution of the trigonometric moment problem.) The book contains many refreshing comments. Examples: the distance from a vector y to the span of the linearly independent vectors $x_1, \cdots, x_n$ is the quotient of the Gramian of $x_1, \cdots, x_n, y$ by the Gramian of $x_1, \cdots, x_n$; a Jacobi matrix represents a completely continuous operator if and only if its coefficients tend to zero; if a one-to-one mapping of a Hilbert space onto itself preserves inner products, then it is linear (and therefore unitary). The book also contains many illuminating examples worked out in complete detail; they include multiplication and differentiation on function spaces and shifts on sequence spaces. The authors' treatment of cyclic (self-adjoint or unitary) operators provides a very good introduction to multiplicity theory (which they do not treat). The style throughout is unhurried, precise, and clear.

PAUL R. HALMOS

Quadratische Formen und orthogonale Gruppen. By Martin Eichler. (Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 63.) Berlin, Göttingen, Heidelberg, Springer, 1952. 12+220 pp. 24.60 DM; bound, 27.60 DM.

This is a competent and highly original monograph on the algebraic and arithmetic theory of quadratic forms from the modern point of view of Witt's Crelle 176 paper (1937). Much of the work is done in an arbitrary classical product formula (c.p.f.) field—namely a finite algebraic extension of the rational field or a field of rational functions of one variable over a Galois field—subject only to the perpetual assumption that the characteristic is not 2. It contains the classical theory of equivalence of forms under linear transformations with coefficients in the ground field, a new arithmetical theory mainly due to the author, and a new treatment, with proofs, of a considerable part of the recent deep analytical work of Hecke and Siegel, all from a new point of view to be described soon. Obviously it is an extremely important and valuable contribution to the literature of its subject.

Like Witt, Eichler defines a *semisimple metric space* (here called simply "space") to be a structure of a finite dimensional vector space R over a field k with a commutative bilinear vector product of R, R into k, such that no nonzero vector of R is orthogonal to all elements of R. The problem of equivalence of two q. f. under linear transformations with coefficients in k is equivalent to the problem of isomorphism of the corresponding spaces. The *orthogonal group* $\mathfrak{O}$ of R is the group of all isometries—i.e. of all isomorphisms of the entire structure which leave elements of k fixed. The "orthogonal group" of analysis is a very special case. Eichler states in the Introduction that he considers the theory of q. f. as the geometry of the space R with respect to the group of operators $\mathfrak{O}$. He compares R and $\mathfrak{O}$ to the additive and multiplicative groups of a ring, and suggests that the theory of q. f. has the same relation to the theory of algebras as this orthogonal group has to the affine group.

When k is a c. p. f. field, Eichler proves that two spaces are isometric if and only if everywhere locally isometric and completely characterizes spaces by local invariants. His proof, using Witt's theory of space-types and the local-global theory of quaternion algebras, is much the same as Witt's (Crelle 176) and is technically simpler in some places; but it produces a much less unified impression because, not wanting to use the general theory of algebras, Eichler lists a set of formally described local invariants which look quite mysterious at the archimedean primes while Witt can simply list "the algebra S" as one big invariant; and for the same reason Eichler's proof finishes with a complicated construction instead of Witt's simple induction. This great theorem (proof for rational field sketched by Minkowski; general proof by Hasse, Crelle 152–153; idea of using algebras due to Artin) concludes the algebraic theory.

Eichler's framework for arithmetic is the set of lattices in a space and the group of similarity transformations (s. t.) of that space. Let $\mathfrak{o}$ be an integral domain in k, and R a space. A *lattice* $\mathfrak{J}$ is a finitely generated $\mathfrak{o}$-module contained in R and containing a k-basis for R. A *similarity transformation* Σ is a linear mapping of R onto itself such that the product of images of vectors equals a fixed constant, called the *norm* $n(\Sigma)$ of Σ, times product of original vectors. An s. t. is a *unit* if it is a mapping of $\mathfrak{J}$ onto itself and its norm is then a unit in $\mathfrak{o}$; it is an *automorphic unit* if it is an automorphism and its norm is then 1. In spaces of even dimension Eichler defines the slightly smaller group of the *proper* s. t. and in what follows "s. t." should usually be interpreted "proper s. t."

If $\mathfrak{o}$ has class number 1 then every lattice is a linear form module and there is a one-one correspondence between classes of isomorphic lattices and classes of integrally equivalent forms; but if the class number of $\mathfrak{o}$ is greater than 1 neither of these statements is so. For lattices it is no longer true that everywhere local isomorphism implies global isomorphism; also the invariant characterization of local lattices is difficult at those prime spots where 2 is not a unit, and hasn't been completely done when 2 is moreover not a prime element. For an excellent discussion of this latter problem see O'Meara, Amer. J. Math. vol. 77 (1955). Eichler says little about it but determines the structure, which turns out to be very simple, of all maximal local lattices—i.e. those which are contained in no larger lattice of the same norm in the same space.

Being interested in the ideal theory of s.t., Eichler studies not isomorphism classes but similarity classes of lattices. When $\mathfrak{o}$ is the ordinary integers these can as he points out be made to correspond to equivalence classes. Otherwise they are somewhat different and Eichler does not discuss the connection. A *local ideal* $\mathfrak{K}_\mathfrak{p}/\mathfrak{J}_\mathfrak{p}$ is the set of all s. t. which take the local lattice $\mathfrak{J}_\mathfrak{p}$ onto $\mathfrak{K}_\mathfrak{p}$. Two global lattices $\mathfrak{J}$ and $\mathfrak{K}$ are called *ideal related* if for every $\mathfrak{p}$ the $\mathfrak{p}$-adic completions $\mathfrak{J}_\mathfrak{p}$ and $\mathfrak{K}_\mathfrak{p}$ are similar; and the family, indexed by the set of all valuations, of the local ideals $\mathfrak{K}_\mathfrak{p}/\mathfrak{J}_\mathfrak{p}$ is called an *ideal.* Ideals have a theory much like the ideal theory of a division algebra. *Genera* are defined as follows: two lattices $\mathfrak{J}$ and $\mathfrak{K}$ are called *related* if there is a global s. t. Σ such that $\Sigma\mathfrak{J}$ and $\mathfrak{K}$ are everywhere locally isomorphic, and the equivalence classes of related lattices are called genera. When $\mathfrak{o}$ is the ordinary integers these genera correspond to the ordinary ones. There are the usual sort of finiteness theorems, whose proof in general requires Minkowski's theorem on linear forms (although the reduction theory is omitted from the book). For example the number of similarity classes in a genus is finite.

Just as in the older theory characterization of similarity classes by invariants is a big unsolved problem. Here Eichler makes a notable advance by deriving a new necessary condition for two related lattices to be similar. Namely, a space defines in a simple way a Clifford algebra (slightly different from the one used by Witt in Crelle 176). The s. t. correspond to certain inner automorphisms of this algebra. Using this correspondence Eichler defines for each s. t. an element of the multiplicative group of k modulo squares called its *spinor-norm.* (The formalism resembles that of spinors in quantum mechanics, though the actual structures are quite different.) Sometimes it is possible by studying local and global spinor-norms to prove that two related lattices cannot be similar. If this is not possible they are said to be in the same *spinor-genus*. In many cases—for example in all lattices of dimension more than 2 which contain a vector orthogonal to itself—the spinor-genera are the similarity classes. Eichler gives an example to prove that this is not always so.

This survey of the main features of Eichler's new branch of arithmetic has covered, though with many omissions, most of the first three of the five chapters together with the beginning of the fifth. The fourth chapter begins by studying numbers of ideals taking lattices of one similarity class into those of another, numbers of vectors of given square in a lattice, etc. Special cases give such things as known formulas for number of representations of an integer as sum of 4 squares. The rest of this chapter proves some of Hecke's recent results on theta functions (E. Hecke, *Analytische Arithmetik der positiven quadratischen Formen*, Kgl. Danske Videnskab. Selskab, Math.-fys Medd. XVII, 12). In this last part Eichler specializes to the rational field and uses forms instead of lattices. The part of the fifth chapter not already discussed studies the mass (German *Mass*) of a genus, concluding with a proof of certain formulas due to Siegel (Ann. of Math. vols. 36 and 37 (1935)). Here again the analytic work is restricted to the rational field. There are many original things in these last two chapters, but the reviewer lacks the exhaustive knowledge of analytic number theory necessary for critically comparing Eichler's methods with Hecke's and Siegel's.

The style of this book must be accepted, since it is not unusual in mathematical monographs; but the reviewer regrets that such an important book wasn't written in a more expansive way. In his opinion, it hasn't enough motivation, unifying explanation, and references clarifying the relation of the book's content to the whole literature. For example, in the theory of spaces the reviewer considers it an important mathematical fact that the "characters" have a unified description as invariants of an algebra class, and is surprised that Eichler gives no hint of this—not even a reference to Crelle 176. In many other places the reviewer got a similar impression of disunity, usually without knowing how to correct it. This is not only pedagoguery; careful striving for simplicity and unification has helped a lot in putting class field theory into its present elegant form and, as Eichler points out in his Introduction, the subject of quadratic forms needs similar improvement.

Except for hiding the footnotes in the back, Springer-Verlag continues its tradition of impeccable printing. The reviewer has never before seen upper case Fraktur I and J used in the same discussion (p. 105) and hopes he never sees it again.

G. WHAPLES

Higher transcendental functions. Vol. 3. Bateman Project Staff, Editor A. Erdélyi. New York, McGraw-Hill, 1955. 17+292 pp. $6.50.

The present publication is a continuation of the two volumes of the *Higher transcendental functions* completed under the Bateman Manuscript Project sponsored by the California Institute of Technology (cf. this Bulletin vol. 60 (1954) pp. 405–408). In the meantime also two volumes of the *Tables of integral transforms* have been published which are a helpful supplement to the other volumes. (For reviews see this Bulletin, vol. 60 (1954) pp. 491–493 and vol. 61 (1955) pp. 239–240.) The present book shows the same erudition and competency which were apparent in the previous ones. At the same time it is clear again how difficult it is to define natural boundaries and dividing lines in this field. Considering the present material this is particularly conspicuous for the chapter on number-theory and for some parts of the chapter on automorphic functions which deal with special functions very different in origin, aspect, and interconnections from those functions to which this term is usually applied. The book follows in general the same ordering principles and organization as the previous volumes; its usefulness is on the same level.

Chapter XIV deals with automorphic functions, XV with Lamé functions, XVI with Mathieu and similar functions. (In the meantime the important book of Meixner and Schäfke has been published on this subject.) Chapter XVII is an introduction to functions occurring in number-theory while Chapters XVIII and XIX deal with more scattered topics. Naturally the concept of "generating function" (Chapter XIX) is not sufficiently substantial in order to form the backbone to many rather heterogeneous subjects.

The overwhelming credit for the preparation of the book goes, of course, to the Editor, Professor Arthur Erdélyi, who was the Director of the Bateman Project Staff. Professors W. Magnus and T. M. Apostol have participated with success in the present work. The completion of the whole project by this volume is a good opportunity to stress again that this unique work will occupy for a long time to come an outstanding place in the modern literature on special functions occurring in mathematical physics. This well organized collection of important concepts and results will be a useful tool for those who deal with the application of special functions.

G. SZEGÖ