A CHARACTERIZATION OF DEDEKIND STRUCTURES*

MORGAN WARD

If Σ is a Dedekind structure,† then for any two elements A and B of Σ, the quotient structures $[A, B]/A$ and $B/(A, B)$ are isomorphic. (Dedekind [**2**], Ore [**3**].) I prove here a converse result.

THEOREM. *Let Σ be a structure in which for every pair of elements A and B, the quotient structures $[A, B]/A$ and $B/(A, B)$ are isomorphic. Then if either the ascending or descending chain condition holds in Σ, the structure is Dedekindian.*

This result is comparatively trivial if *both* the ascending and descending chain conditions hold. That some sort of chain condition is necessary may be seen by a simple example. Consider a structure Σ with an all element O_0 and a unit element E_0 built up out of three ordered structures Σ_1, Σ_2, Σ_3 meeting only at O_0 and E_0, so that if $S_u \in \Sigma_u$, then

$$(S_u, S_v) = E_0, \qquad [S_u, S_v] = O_0$$

for $u, v = 1, 2, 3$, $u \neq v$. Then if each Σ_i is a series of the type of the real numbers in the closed interval 0, 1, the quotient structures of any pair $[S_u, S_v]/S_u$, $S_v/(S_u, S_v)$ are obviously isomorphic. But Σ is clearly non-Dedekindian.

The theorem is of some interest in view of the generalizations Ore has given of his decomposition theorems in Ore [**4**].

It suffices to prove the result under the hypothesis that the descending chain axiom holds in Σ (Ore [**3**, p. 410]). We formulate this axiom as follows:

(β) *If for any two elements A and B of Σ,*

$$A \supset X_1 \supset X_2 \supset X_3 \supset \cdots \supset B$$

for an infinity of X_i in Σ, all the X_i are equal from a certain point on.

Our proof rests upon several lemmas which we collect here.

LEMMA 1. (Dedekind [**2**].) *Σ is a Dedekind structure if and only if Σ contains no substructure Σ_0 of order five which is non-Dedekindian.*

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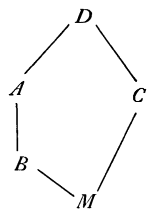
† We use the notation and terminology of Ore's fundamental paper, Ore [**3**], with the following two exceptions. (i) We write $A \supset B$, $B \subset A$ for Ore's $A \geqq B$, $B \leqq A$. (ii) If A is prime over B (Ore [**3**, p. 411]), we shall say "A covers B" or "B is covered by A" (Birkhoff [**1**]) and write $A > B$ or $B < A$.

The type of substructure in question is well known; its diagram is given in the figure. Since we utilize such substructures frequently in our proof, we shall introduce the notation $\{D, A, B, C, M\}$ for Σ_0, writing the all element D and unit element M in the first and last



Σ_0

places in the symbol while the elements A and B where $A \supset B$ occupy the second and third places.

LEMMA 2. (Ore [3].) *If (β) holds in the structure Σ, then every set of elements of Σ which divide a fixed element A contains at least one minimal element dividing no other element of the set.*

LEMMA 3. *If (β) holds in the structure Σ, then for any two distinct elements A and C of Σ such that C divides A, there exists an element B such that C divides B and B covers A.*

For we need only pick a minimal element in the subset of all elements X such that $C \supset X \supset A$, $X \neq A$.

The following lemma is obvious:

LEMMA 4. *Let Σ be a structure in which*

$$(\epsilon) \qquad [A, B]/A \cong B/(A, B)$$

for every A, B of Σ. Then $[A, B]$ covers A if and only if B covers (A, B).

LEMMA 5. *Let Σ be a structure in which (ϵ) holds. Then if A covers B and M is any other element of Σ, either $[M, A]$ equals $[M, B]$ or $[M, A]$ covers $[M, B]$.*

For clearly $[M, A] \supset [M, B]$. Since $A \supset (A, [M, B]) \supset B$ and $A > B$, either $(A, [M, B]) = A$ or $(A, [M, B]) = B$. If $(A, [M, B]) = A$, then $[M, B] \supset A \supset [M, A]$, so that $[M, B] = [M, A]$. If $(A, [M, B]) = B$, then $A > (A, [M, B])$. Hence by Lemma 4, $[A, [M, B]] > [M, B]$. But since $A \supset B$,

$$[A, [M, B]] = [M, A].$$

Our final lemma is the dual of Lemma 5.

LEMMA 6. *Let Σ be a structure in which* (ϵ) *holds. Then if A covers B and M is any other element of Σ, either (M, A) equals (M, B) or (M, A) covers (M, B).*

We shall prove our theorem indirectly. Assume that conditions (β) and (ϵ) hold in the structure Σ, but that Σ is non-Dedekindian. Then by Lemma 1, Σ contains a non-Dedekindian substructure

$$\Sigma_0 = \{D, A, B, C, M\}$$

of order five.*

We may assume that A covers B. For by Lemma 3, there exists an element N of Σ such that $A \supset N$, $N > B$. Thus

$$[A, C] \supset [N, C] \supset [B, C], \qquad (A, C) \supset (N, C) \supset (B, C);$$

that is, $[N, C] = D$, $(N, C) = M$. Hence $\{D, N, B, C, M\}$ is a non-Dedekindian substructure where $N > B$.

We assume henceforth that A covers B. Since $[A, C] = D$, $(A, C) = M$, and $[B, C] = D$, $(B, C) = M$, $D/C \cong A/M$, and $D/C \cong B/M$ by (ϵ). Hence $A/M \cong B/M$. But B lies in A/M and $A > B$. Since A corresponds to B under the isomorphism, *there exists an element in B/M covered by B.* Denote it by B_1. Then

$$B > B_1 \supset M. \tag{1}$$

Since $B \supset B_1 \supset M$, $(B, C) \supset (B_1, C) \supset (M, C)$ or $(B_1, C) = M$. Consider next the union $D_1 = [B_1, C]$. Since $B > B_1$, by Lemma 5 either $[B, C] = [B_1, C]$ or $[B, C] > [B_1, C]$; that is, either $D = D_1$ or $D > D_1$.

If $D = D_1$, then on writing A_1 for B, we obtain a non-Dedekindian substructure $\{D_1, A_1, B_1, C, M\}$ in which $A_1 > B_1$.

Now assume that $D > D_1$. Clearly $[A, D_1] = [B, D_1] = D$. Consider the crosscut (B, D_1). Since $B > B_1$, by Lemma 6, either $(B, D_1) = (B_1, D_1)$ or $(B, D_1) > (B_1, D_1)$. That is, since $B \supset (B, D_1)$ and $D_1 \supset B_1$, either $(B, D_1) = B_1$ or $(B, D_1) = B$. *We must have* $(B, D_1) = B_1$. For if $(B, D_1) = B$, then $D_1 \supset B$. Since $D_1 \supset C$, we would have $D_1 \supset [B, C]$, $D_1 = D$, contrary to the assumption $D > D_1$.

Consider next the crosscut $A_1 = (A, D_1)$. Since $A > B$, by Lemma 5 either $(A, D_1) = (B, D_1)$ or $(A, D_1) > (B, D_1)$; that is, either $A_1 = B_1$ or $A_1 > B_1$. *We must have* $A_1 > B_1$. For if $A_1 = B_1$, then $\{D, A, B, D_1, B_1\}$ is a non-Dedekindian substructure. But since $[A, D_1] = D$ and $(A, D_1) = B_1$, by (ϵ) $A/B_1 \cong D/D_1$. This isomorphism is impossible, for $A \supset B > B_1$ while $D > D_1$.

Finally, since $A \supset A_1 \supset C$ and $B \supset B_1 \supset C$, $(A_1, C) = (B_1, C) = M$

* The reader will find a structure diagram helpful in following the argument.

while $[A_1, C]=[B_1, C]=D_1$. Thus $\{D_1, A_1, B_1, C, M\}$ is a non-Dedekindian substructure of Σ in which $A_1 > B_1$.

We now replace Σ_0 in either case by $\Sigma_1=\{D_1, A_1, B_1, C, M\}$ and obtain a non-Dedekindian substructure $\Sigma_2=\{D_2, A_2, B_2, C, M\}$ where $A_2 > B_2$ and

$$B_1 > B_2 \supset M. \tag{2}$$

On repeating this reasoning, and combining (1), (2), $\cdots$ we obtain a chain

$$B > B_1 > B_2 > B_3 > \cdots \supset M$$

of indefinite length in which all B_i are distinct, contradicting (β).

References

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4. ———, *On the theorem of Jordon-Hölder*, Transactions of this Society, vol. 41 (1937), pp. 266–273.

California Institute of Technology