where the terms represented by $\cdots$ are of degree less than $k-i+2$ in y. Thus (2) is established for $k+1$ and hence by induction for $k \geqq n-1$. Hence $p_1^{(k)}, p_2^{(k)}, \cdots, p_n^{(k)}$ are linearly independent for $k \geqq n-1$.

We have now proved that, for every k, every linear homogeneous polynomial of degree k which is a solution of $Of=0$ has the form

$$c_1p_1^{(k)} + c_2p_2^{(k)} + \cdots + c_np_n^{(k)},$$

where the c's are arbitrary constants.

THE UNIVERSITY OF WISCONSIN

A CHARACTERIZATION OF NULL SYSTEMS IN PROJECTIVE SPACE

BY RICHARD BRAUER

1. *Introduction.* We consider the n-dimensional projective space S_n defined analytically by means of any abstract field F. The points P of S_n are given by a set of $n+1$ elements x_i of F, $P=(x_0, x_1, \cdots, x_n)$, (not all $x_i=0$), with the convention that proportional sets define the same point. The points P whose coordinates satisfy a linear homogeneous equation $u^{(0)}x_0+u^{(1)}x_1+\cdots+u^{(n)}x_n=0$, (not all $u^{(i)}=0$), form a hyperplane $\epsilon=(u^{(0)}, u^{(1)}, \cdots, u^{(n)})$. There is no difficulty in defining such notions as those of straight lines, projections, and cross ratios, and discussing the elementary properties.

Let M be a non-singular skew-symmetric bilinear form with coefficients a_{ik} in F,

$$M = \sum_{i,k=0}^{n} a_{ik}y_ix_k, \qquad a_{ik} = -a_{ki}, \qquad \det(a_{ik}) \neq 0.$$

For every point $P=(x_0, x_1, \cdots, x_n)$ the equation $M=0$ is the equation of a hyperplane ϵ in the coordinates $(y_0, y_1, \cdots, y_n)$ of a variable point of ϵ. We obtain in this manner a one-to-one correspondence between the points $P=(x_0, x_1, \cdots, x_n)$ and hyperplanes $\epsilon=(u^{(0)}, u^{(1)}, \cdots, u^{(n)})$ of S_n which is called a *null system*. The relation between corresponding values of the $u^{(i)}$ and x_i is given by

$$u^{(i)} = \sum_{k=0}^{n} a_{ik}x_k.$$

The assumption that $\det(a_{ik})$ is different from zero implies that n is odd.

As is easily seen a null system has the following properties:

(1) Every point P lies in the associated hyperplane ϵ.

(2) If P corresponds to ϵ, then the points Q in ϵ are associated with hyperplanes through P.

Conversely, *a one-to-one correspondence between points and hyperplanes of S_n with these two properties is a null system.* A geometric proof of this fact for the case $n=3$ is contained in Chapter 11, Volume 1, of the book by O. Veblen and J. W. Young on projective geometry. In this note I will give an analytic proof for any odd number of dimensions. The proof by Veblen and Young and the following one are independent of any properties of continuity. They hold for every field F. It may be remarked that the property (1) alone is not sufficient to characterize null systems.

2. *Definition of Collineation.* We assume that a one-to-one correspondence between the elements of S_n is given by

$$\text{(1a)} \qquad P = (x_0, x_1, \cdots, x_n) \rightarrow P' = (x_0', x_1', \cdots, x_n'),$$

and, furthermore, a one-to-one correspondence between the hyperplanes by

$$\text{(1b)} \qquad \begin{aligned} \epsilon &= (u^{(0)}, u^{(1)}, \cdots, u^{(n)}) \\ \rightarrow \epsilon' &= ((u')^{(0)}, (u')^{(1)}, \cdots, (u')^{(n)}). \end{aligned}$$

We call the correspondence a *collineation*, if the point P' lies in the hyperplane ϵ' when and only when P lies in ϵ.

If the mapping $\zeta \rightarrow \phi(\zeta)$ is an automorphism† of the field F, if

† An automorphism of a field F is a one-to-one mapping of the elements of a field onto themselves, $\zeta \rightarrow \zeta^* = \phi(\zeta)$, such that

$$\phi(\zeta_1 + \zeta_2) = \phi(\zeta_1) + \phi(\zeta_2), \qquad \phi(\zeta_1\zeta_2) = \phi(\zeta_1)\phi(\zeta_2)$$

for any two elements ζ_1 and ζ_2 of F. If a symbol ∞ is added to the elements of F, we postulate $\phi(\infty) = \infty$.

The zero element 0 and the unit element 1 correspond to themselves under an automorphism of any field F.

Let, for instance, F be the field of all real and complex numbers. One obtains an automorphism by associating with any ζ the conjugate complex number $\bar{\zeta}$. There are no automorphisms of the field of all real numbers except the identical automorphism $\zeta \rightarrow \zeta$.

$A=(a_{ik})$ is a non-singular matrix of degree $n+1$, $(i,\ k=0,\ 1,\ 2,\ \cdots,\ n)$, and if $A'^{-1}=(\alpha_{ik})$ is the contragredient matrix, the transformation

$$x_i' = \sum_{k=0}^{n} a_{ik}\phi(x_k), \qquad u'^{(i)} = \sum_{k=0}^{n} \alpha_{ik}\phi(u^{(k)}), \tag{2}$$

defines a collineation.

3. *Equations of the General Collineation.* It is a well known theorem that *the most general collineation for $n>1$ is given by* (2). I will begin with a sketch of a proof of this theorem which seems to be more direct than the proofs previously given.

We consider a fixed collineation. It follows from the definition that points on a straight line are always transformed into points on a straight line. Let P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4 be two sets of four points, each set lying on a straight line such that both sets have the same cross ratio λ. Then, it is possible to carry over the first set into the second one by effecting a succession of projections. We denote by R the figure consisting of all points and lines which are necessary for this construction.

We apply the given collineation to the whole figure R. If P_ν is transformed into P_ν' and Q_ν into Q_ν', the image R' of R is a figure consisting of points and lines which shows immediately that it is possible to carry P_1', P_2', P_3', P_4' over into Q_1', Q_2', Q_3', Q_4' by successive projections. Hence these sets have the same cross ratio λ'. Consequently, λ' is a function $\phi(\lambda)$ of λ alone and independent of the special choice of P_1, P_2, P_3, P_4. Hence four points on a straight line with the cross ratio λ are always transformed by the given collineation into points with the cross ratio $\lambda'=\phi(\lambda)$.

If P_1 and P_2 coincide, so do P_1' and P_2'. Hence we have $\lambda'=1$, if $\lambda=1$, or $\phi(1)=1$. Similarly, we find $\phi(0)=0$, $\phi(\infty)=\infty$. The points P_1, P_3, P_2, P_4 in this order correspond to P_1', P_3', P_2', P_4'. The cross ratios being $1-\lambda$ and $1-\phi(\lambda)$, we derive

$$\phi(1-\lambda) = 1-\phi(\lambda). \tag{3}$$

Let P_5 be a point on the straight line through P_1, P_2, P_3, P_4 such that the cross ratio $P_1P_2P_4P_5$ has a given value μ. According to a simple formula in analytic geometry, the cross ratio $P_1P_2P_3P_5$ has the value $\lambda\mu$. If by the collineation P_5 is transformed into P_5', the sets P_1', P_2', P_3', P_4'; P_1', P_2', P_4', P_5';

P_1', P_2', P_3', P_5' have the cross ratios $\phi(\lambda)$, $\phi(\mu)$, $\phi(\lambda\mu)$, respectively. But the last one is the product of the preceding ones, that is,

$$\phi(\lambda\mu) = \phi(\lambda)\phi(\mu). \tag{4}$$

Equations (3) and (4) show that the mapping $\lambda \to \phi(\lambda)$ establishes an automorphism of the field F. For, if $\lambda \neq 0$, we have

$$\begin{aligned} \phi(\lambda - \mu) &= \phi(\lambda)\phi\left(1 - \frac{\mu}{\lambda}\right) = \phi(\lambda)\left(1 - \phi\left(\frac{\mu}{\lambda}\right)\right) \\ &= \phi(\lambda) - \phi(\mu). \end{aligned} \tag{5}$$

From (4), for $\lambda = \mu = -1$, we derive $\phi(-1) = \pm 1$. If $\phi(-1) = +1$ we obtain $\phi(2) = 0$ from (3) for $\lambda = -1$ and hence $\phi(2\lambda) = 0$ from (4) for any λ. If the characteristic of F is not 2, we find a contradiction by putting $\lambda = -1/2$. In the case of characteristic 2, we have $-1 = 1$. Therefore, $\phi(-1) = -1$ and then $\phi(-\mu) = -\mu$, according to (4) for $\lambda = -1$. Hence (5) holds in the case $\lambda = 0$. By replacing μ by $-\mu$ in (5), we obtain $\phi(\lambda + \mu) = \phi(\lambda) + \phi(\mu)$. Therefore, $\lambda \to \phi(\lambda)$ is really an automorphism. The inner reason for this is the fact that one is able to construct points with the cross ratios $\lambda + \mu$, $\lambda - \mu$, $\lambda\mu$, λ/μ by means of projective constructions if points with the cross ratios λ and μ are given.

We define a new collineation by associating

$$\begin{aligned} P &= (x_0, x_1, \cdots, x_n) \\ \to P^* &= (\phi(x_0), \phi(x_1), \cdots, \phi(x_n)) = (x_0^*, x_1^*, \cdots, x_n^*), \\ \epsilon &= (u^{(0)}, u^{(1)}, \cdots, u^{(n)}) \\ \to \epsilon_* &= (\phi(u^{(0)}), \phi(u^{(1)}), \cdots, \phi(u^{(n)})) \\ &= (u^{*(0)}, u^{*(1)}, \cdots, u^{*(n)}). \end{aligned} \tag{6}$$

This is a special collineation (2), the matrix A being the unit matrix. Since the cross ratio is a rational function of the coordinates, points with the cross ratio λ are transformed into points with the cross ratio $\phi(\lambda)$.

If the original collineation is given by (1a), (1b), the transformation $P^* \to P'$, $\epsilon^* \to \epsilon'$ defines a collineation (P and hence P^* ranges over all points, ϵ and hence ϵ^* over all hyperplanes). For this collineation, the cross ratio of four points is necessarily invariant. We see in the usual manner,† that a collineation with this particular property is given by a linear transformation

$$(7) \qquad x_i' = \sum_{k=0}^{n} a_{ik}x_k^*, \qquad u'^{(i)} = \sum_{k=0}^{n} \alpha_{ik}u^{*(k)},$$

where (a_{ik}) is a non-singular matrix and (α_{ik}) its contragredient matrix. On comparing (6) and (7), we obtain (2).

If the only automorphism of the field F is the identity as in the case of the field of the real numbers, then, of course, every collineation is given by a linear transformation $x_i' = \sum a_{ik}x_k$, $u'^{(i)} = \sum \alpha_{ik}u_k$.

4. *Correlation*. We speak of a *correlation* in a projective space whose number of dimensions is larger than 1, if to every point P corresponds a hyperplane $\tilde{\pi}$ in a one-to-one manner and to every hyperplane ρ corresponds a point $\tilde{R}$ in a one-to-one manner, such that $\tilde{\pi}$ passes through $\tilde{R}$ if and only if P lies in ρ.

We define a special correlation by associating the hyperplane $\tilde{u}^{(i)}$ to the point x_i, where

$$(8a) \qquad \tilde{u}^{(i)} = x_i, \qquad (i = 0, 1, 2, \cdots, n),$$

and by associating the point $\tilde{x}_i$ to the hyperplane $u^{(i)}$, where

$$(8b) \qquad \tilde{x}_i = u^{(i)}, \qquad (i = 0, 1, 2, \cdots, n).$$

The inverse of a correlation is a correlation again, the product of two correlations is a collineation. Hence the most general correlation is obtained by performing first a collineation and afterwards the special correlation (8a), (8b). Therefore the most general correlation is given by

† We determine the transformation (7) such that the $n+2$ points $P_0^* = (1, 0, \cdots, 0)$; $P_1^* = (0, 1, 0, \cdots, 0)$; $\cdots$; $P_n^* = (0, \cdots, 0, 1)$; $P_{n+1}^* = (1, 1, \cdots, 1)$ have the same images as they have in the case of the collineation $P^* \rightarrow P'$. Equation (7) sets up a collineation for which the cross ratio is invariant. If we perform the inverse transformation to (7) after the collineation $P^* \rightarrow P'$, we obtain a collineation for which the cross ratio is invariant and the $n+2$ special points P_ν^* are fixed. Then, however, every point Q is fixed, because the position of Q with respect to $P_0^*, P_1^*, \cdots, P_{n+1}^*$ can be described uniquely by means of cross ratios only (projective coordinates). Therefore, the collineation (7) and the collineation $P^* \rightarrow P'$ map all points exactly in the same manner, and both are identical.

$$\tilde{u}^{(i)} = \sum_{k=0}^{n} a_{ik}\phi(x_k), \qquad \tilde{x}_i = \sum_{k=0}^{n} \alpha_{ik}\phi(u^{(k)}), \tag{9}$$

where again (a_{ik}) and (α_{ik}) are non-singular contragredient matrices and $\lambda \rightarrow \phi(\lambda)$ is an automorphism of the underlying field F.

5. *Correlations Defining a Null System.*

THEOREM. *If a correlation associates to every point P a hyperplane $\tilde{\pi}$ passing through P, the number n of dimensions of the space is odd and the correspondence defines a null system. The equation of $\tilde{\pi}$ in point coordinates is given by the vanishing of a skew-symmetric bilinear form, one set of variables being the coordinates of P, the other one the coordinates of a given variable point of $\tilde{\pi}$.*

PROOF. If the correlation is given by (9), then

$$\sum_{i,k=0}^{n} a_{ik}x_i\phi(x_k) = 0$$

must hold for every point P since x_i lies on $\tilde{u}^{(i)}$. Every automorphism of a field F leaves 0 and 1 unaltered. We put $x_i = 1$, $x_j = 0$ for $j \neq i$ and obtain $a_{ii} = 0$ for all i. Hereafter we set $x_i = 1$ and leave x_k indeterminate for the moment for a fixed pair $i \neq k$. We set $x_j = 0$ for $j \neq i, k$. We get

$$a_{ik}\phi(x_k) + a_{ki}x_k = 0. \tag{10}$$

If we put $x_k = 1$, we see that $a_{ik} = -a_{ki}$, the matrix (a_{ik}) is skew-symmetric. Furthermore, there exists a pair i, k for which $a_{ik} \neq 0$. By using the result we have just found we derive from (10), $\phi(x_k) = x_k$ for all elements x_k in F. Hence $\lambda \rightarrow \phi(\lambda)$ is the identical automorphism.

Then the condition for a point $(y_0, y_1, \cdots, y_n)$ to lie in the hyperplane $\tilde{u}^{(i)}$ corresponding to the point x_i is given according to (9) by $\sum a_{ik}y_ix_k = 0$, where (a_{ik}) is skew-symmetric as required by the theorem. The number of dimensions n is odd because otherwise the determinant of (a_{ik}) would vanish.

A one-to-one correspondence between points and hyperplanes with the properties (1) and (2) defined in the introduction establishes a correlation of period 2. It follows that we have a null system.

6. *General Correlations of Period* 2. We consider more general correlations of period 2 and prove the following theorem.

THEOREM. *A correlation of period* 2 *associates to a point* $(x_0, \cdots, x_n)$ *a hyperplane whose equation in point coordinates* z_i *is given either by*

$$\sum_{i,k} a_{ik} z_i x_k = 0 \quad \textit{with} \quad a_{ik} = a_{ki}, \qquad (\textit{polarity}),$$

or by

$$\sum_{i,k} a_{ik} z_i x_k = 0 \quad \textit{with} \quad a_{ik} = -a_{ki}, \quad (\textit{null system for an odd number of dimensions only}),$$

or by the vanishing of a Hermitian form†

$$\sum a_{ik} z_i \phi(x_k) = 0, \qquad a_{ik} = \phi(a_{ki}),$$

where $\lambda \rightarrow \phi(\lambda)$ *is an automorphism of period* 2 *of the underlying field* F.

PROOF. If a correlation of period 2 is given by (9), we derive, by effecting the correlation twice,

$$\gamma x_i = \sum_{j=0}^{n} \alpha_{ij} \phi(\bar{u}^{(j)}) = \sum_{j,k=0}^{n} \alpha_{ij} \phi(a_{jk}) \phi\phi(x_k),$$

where $\gamma \neq 0$ is a factor depending on $(x_0, x_1, \cdots, x_n)$ but independent of i.

Besides the matrix $A = (a_{ik})$ we consider $A^* = (\phi(a_{ik}))$, and form

$$M = A'^{-1} A^*. \tag{11}$$

Then, by the linear transformation with this matrix,

$$(\phi\phi(x_0), \phi\phi(x_1), \cdots, \phi\phi(x_n))$$

is transformed into

$$(\gamma x_0, \gamma x_1, \cdots, \gamma x_n).$$

† In the general sense in which this word is used by L. E. Dickson in *Modern Algebraic Theories*, p. 66. The first case can, of course, be considered as a special case of the third case.

By putting all $x_k=0$ except one which has the value 1, we prove in the usual manner that M is a diagonal matrix. By putting all $x_i=1$, we see that $M=c\cdot\mathbf{1}$, where $\mathbf{1}$ denotes the unit matrix and $c\neq 0$. Now we set $x_1=1$, $x_2=\lambda$, $x_3=x_4=\cdots=0$, where λ is an arbitrary element of F. Then we find

$$\gamma = c\cdot 1, \qquad \gamma\cdot\lambda = c\cdot\phi\phi(\lambda),$$

and hence $\phi\phi(\lambda)=\lambda$; the automorphism $\lambda\rightarrow\phi(\lambda)$ is of period 2. Equation (11) in combination with $M=c\cdot\mathbf{1}$ gives

$$A^* = cA'. \tag{12}$$

We apply the automorphism $\lambda\rightarrow\phi(\lambda)$ to the equation (12). Then $A^*=(\phi(a_{ik}))$ is transformed into $(\phi\phi(a_{ik}))=(a_{ik})=A$ and hence

$$A = \phi(c)A^{*\prime}. \tag{13}$$

From (12) and (13) follows $A=\phi(c)\cdot c\cdot A$, and, since $A\neq 0$,

$$\phi(c)\cdot c = 1. \tag{14}$$

If $c\neq -1$, we set $l=1+c$ and obtain

$$c\cdot\phi(l) = c(1+\phi(c)) = c + c\phi(c) = c+1 = l,$$

so that

$$c = l/(\phi(l). \tag{15}$$

We then replace A by lA, as we may, and consequently A^* by $\phi(l)A^*$. Then in (12), c is to be replaced by $c\phi(l)/l=1$. Therefore we can assume $c=1$, and (12) shows that A is Hermitian with respect to the automorphism $\lambda\rightarrow\phi(\lambda)$ of period 2.

If $c=-1$, but if there exist elements μ in F with $\phi(\mu)\neq\mu$, (15) holds for $l=\mu-\phi(\mu)$ and we obtain the same result.

If, finally, $\phi(\mu)=\mu$ for all μ in F, we have $A^*=A$ and from (14) follows $c^2=1$, $c=\pm 1$, $A'=\pm A$. Therefore, A is either symmetric or skew-symmetric.

If $(z_0, z_1, \cdots, z_n)$ is a point of the hyperplane corresponding to $(x_0, x_1, \cdots, x_n)$, it follows from (9) that $\sum a_{ik}z_i\phi(x_k)=0$. We have shown that here ϕ is of period 2 and (a_{ik}) Hermitian, or $\phi(\lambda)=\lambda$ identically and (a_{ik}) symmetric or skew-symmetric.

THE INSTITUTE FOR ADVANCED STUDY