HOBSON'S SECOND VOLUME

The Theory of Functions of a Real Variable and the. Theory of Fourier's Series. By E. W. Hobson. Second Edition, vol. II. Cambridge University Press, 1926. x+780 pp.

It is a long time since I previously had so much delight in examining a book as I have recently experienced in the rapid reading of the second volume of Hobson's *Functions of a Real Variable.* It is characterized by an extraordinary richness of content and by the remarkable indications which it affords of the marvelous vitality and fecundity of the current investigations in the general theory of functions of a real variable and in the theory of Fourier's series in particular. This second volume completes the author's extension and revision of the one-volume work which first appeared in 1907, the earlier part of his revision having been published in 1921 as volume I of the second edition. Almost the whole of the matter for the new volume II has been re-written and much new matter has been added so that the extent of the whole work is now twice as great as that of the first edition. The new matter is largely the fruit of investigations carried out in the last twenty years. The completed work now affords a most effective testimonial to current progress in one division of mathematics as well as the most important exposition yet given of knowledge in that domain.

The volume is beautifully printed in the style for which the Cambridge University Press has justly become famous among mathematicians. The volume is bound so as to be pleasing both to the eye and to the hand when new; but, as experience with earlier books so bound has shown, it has not the durability which one has a right to expect in the case of a volume of such permanent value and one likely to be so frequently used. In some places the author might have improved the appearance of the page by a happier choice of notation or a more suitable arrangement of formulas. A considerable number of minor errors escaped detection in the final proof-reading. But where there is so much excellence in regard to the matters of major importance, it is perhaps ungracious to do so much as to call attention to these minor blemishes.

There can be no doubt that the two volumes of the second edition of this work will be for a long time of constant use to every one who has to do with investigations touching the theory of functions of a real variable and the theory of Fourier series.

The first of the ten chapters of volume II deals with sequences and series of numbers. Into the 98 pages of this chapter the author compresses an excellent exposition of the theory of series with real constant terms, the earlier part of it being abbreviated by the use of some theorems from volume I. The reader will be interested in the compactness of the exposition and the comprehensiveness of the theory so far as the problem of convergence is concerned. It is in no sense a complete summary of what is known about series with constant real terms but it is an account which sets all the main facts in their proper light. Nearly forty pages of the chapter are given to the problem of the summability of series of constant terms.

In this part no effort is made to reach a comprehensive exposition. In fact attention is directed chiefly to an account of some of the main features of summability by the methods of Cesàro, Hölder, and Riesz; and the more general problems connected with the summability of series find here no systematic treatment.

An exhaustive exposition of the general theory of the summability of series has not yet appeared. The theory has now probably attained such a state of development as would render feasible such an exposition. It is one of the important desiderata which must be felt by every one who has to deal with the more delicate questions concerning the convergence of infinite processes. Such a work, if properly prepared, would be of constant use to a large number of investigators in many fields of analysis.

In Chapter II (pp. 99–171) is given a systematic account of the theories of convergence and oscillation of sequences and series of functions of one or more variables. This is done in an elegant and pleasing way and with such a comprehensiveness as one would expect in a general treatise. Special attention is given to the problems associated with uniform convergence and with the various modifications of the notion of uniformity of convergence. The question of the continuity of the sum-function of a series is treated in detail.

An application of the general results of Chapter II is made in Chapter III (pp. 172–227) to the special but important case of power series. Problems relating to interior points of the interval of convergence are disposed of briefly. Much attention is devoted to the behavior of the sum-function near the ends of the interval of convergence and to its relation to the series obtained by giving to the variable its value at an end-point of this interval. In this connection one finds an interesting and illuminating account of the Tauberian theorems which have been the subject of a considerable amount of recent investigation.

Chapter IV (pp. 228–288) contains an account of the theorem of Weierstrass concerning the representation of a continuous function (of one or more variables) by means of sequences of polynomials; of the theory of convergence of sequences on the average; of F. Riesz' classification of summable functions; and of Baire's classification of functions. The fundamental result of Baire, relating to the representation of a function as the limit of a sequence of continuous functions, is proved by a method due to de la Vallée Poussin, but with some modification and extension.

Chapter V (pp. 289–388) is on sequences of integrals. It contains an exposition of those parts of the theory of integration which were not already treated in volume I. In particular, considerable space is given to the theories of integration due to W. H. Young, to Tonelli, and to Perron. The chapter concludes with a very brief account of the summability of integrals, especially by the methods analogous to Cesàro and Hölder summability of series. The author expresses (p. 384) the opinion that Perron's recent definition of an integral may prove to be of great importance in the future developments of the concept of integration.

In Chapter VI (pp. 389–421) is to be found an account of the methods of construction of functions exhibiting assigned peculiarities of behavior and in particular of functions which are continuous but non-differentiable. An exposition is given of various methods of condensation of singularities.

Chapter VII (pp. 422–475) is devoted to the problem of the representation of functions as limits of integrals. A special feature of this chapter, and indeed of the whole volume, is the prominence which is here given to what the author calls the General Convergence Theorem and to its developments and consequences. This general theorem is treated fully, and with considerable novelty of exposition, in Chapter VII, and various applications of it to Fourier's series and integrals are given in the following chapters.

The remaining three chapters of this volume correspond to the final chapter of the first edition.

Chapter VIII (pp. 476–719) deals with trigonometric series. It is by far the longest chapter in the volume. It is here that we find the amplest evidence of the recent activity of investigators to be found anywhere in the volume,—a volume which is marked throughout with much evidence of rapid recent progress. Most of the recent progress in the development of the theory of trigonometric series has been due to the use of two recently constructed tools—that afforded by the theory of Lebesgue integration and various other more recent definitions of integration and that due to various definitions of the summability of series recently exploited with so much vigor. Though trigonometric series have had a remarkable history covering more than a century and a half, the extraordinary recent progress in our knowledge of them and the questions that still remain unanswered show unmistakably that the subject is capable of much further development.

The theory of trigonometric series has had a nearly continuous and remarkably intimate connection with the development of much of the general theory of functions of a real variable. This alone is sufficient to justify the author in analyzing it with some fulness in such a general treatise as that under review. The connection began early and its intimacy is reflected in much of the most recent development. Concerning the connection the author says (p. 476): "Historically, the questions which have arisen in connection with this theory have influenced the development of the theory of functions of a real variable to an extent which is comparable with the degree in which the theory of functions in general has been affected by the theory of power series. The theory of sets of points, which led later to the abstract theory of aggregates, arose directly from questions connected with trigonometric series. The precise formulation by Riemann of the conception of the definite integral, and the gradual development of the modern notion of a function as existent independently of any special mode of representation by an analytical expression, are further examples of the results of the study of the properties of these series upon Mathematical Analysis."

The chapter contains an adequate treatment both of the earlier and of the later theory of trigonometric series and especially of Fourier series, limitation being made to the case in which the arguments involved in the trigonometric functions are integral multiples of the independent variable.

A reference is made to the fact that there are important trigonometric series in which the arguments are differently determined, but these additional types are not treated. Notwithstanding the great richness of detail in the theory as here expounded, one comes from a reading of the chapter with the feeling that the whole theory is yet in a state of development which is far from complete. It can be predicted that much further development of the theory will take place in the near future, both in the way of deeper penetration into details and in the analysis of general methods of treatment and especially of more general types of summability of the series. The theory of summability is capable of much further development by aid of the theory of functions of a complex variable (and especially the theory of residues), a tool which is nowhere employed in the exposition as given by Hobson.

Chapter IX (pp. 720–752) is devoted to the representation of functions by Fourier integrals, together with the modern theory of Fourier transforms. A significant part of the chapter is given to the summability of Fourier integrals, a subject which appears to be capable of much further development.

The final Chapter X (pp. 753–772) contains a brief treatment of series of normal orthogonal functions, a subject which is included "not only on account of the intrinsic importance of the subject, but also because the processes which have been employed in various recent investigations in this domain afford excellent illustrations of ideas and methods which have been developed earlier in this work." In this chapter no attempt at a comprehensive treatment has been made. In fact the exposition scarcely contains so much as an introduction to this extended subject. Many of the properties of Fourier series can be carried over to certain classes of expansions in orthogonal functions—a fact which is but little more than indicated in the brief account to be found here. Just as the first chapter leaves one with the desire for a full exposition of a subject but partially treated so the final chapter leaves one wishing for an adequate and exhaustive treatise on the theory of expansions in orthogonal and biorthogonal functions. What the author gives is scarcely enough to suggest the vast richness of this subject. In this respect the chapter is different from most of those contained in the volume,—the exposition being in most cases adequate and satisfying.

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