ON SMALL DEFORMATIONS OF CURVES

BY C. E. WEATHERBURN

1. *Introduction.* This paper is concerned with small deformations of a single tortuous curve, of a family of curves on a given surface, and of a congruence of curves in space. In all cases, the displacement $\mathbf{s}$ is supposed to be a small quantity of the first order, quantities of higher order being negligible.*

2. *Single Twisted Curve.* Consider first a given curve in space. The position vector $\boldsymbol{r}$ of a point on the curve may be regarded as a function of the arc-length s of the curve, measured from a fixed point on it. Let $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ be the unit tangent, principal normal and binormal. These are connected with the curvature κ and the torsion τ as in the Serret-Frenet formulas. Imagine a small deformation of the curve, such that the point of the curve originally at $\boldsymbol{r}$ suffers a small displacement $\mathbf{s}$, its new position vector $\boldsymbol{r}_1$ being then

$$\boldsymbol{r}_1 = \boldsymbol{r} + \mathbf{s}. \tag{1}$$

Let a suffix unity be used to distinguish quantities belonging to the deformed curve, and let primes denote differentiations with respect to the arc-length s. Then the element $d\boldsymbol{r}_1$ of the deformed curve, corresponding to the element $d\boldsymbol{r}$ of the original, is given by $d\boldsymbol{r}_1 = d\boldsymbol{r} + d\mathbf{s}$, and its length ds_1 by

$$(ds_1)^2 = (d\boldsymbol{r}_1)^2 = (d\boldsymbol{r})^2 + 2d\boldsymbol{r} \cdot d\mathbf{s} = ds^2(1 + 2\boldsymbol{t} \cdot \mathbf{s}').$$

Consequently $ds_1 = ds(1 + \boldsymbol{t} \cdot \mathbf{s}')$.

The quantity $\boldsymbol{t} \cdot \mathbf{s}'$ represents the increase of length per unit length of the curve, or the extension of the curve at

*See also a paper by Perna, Giornale di Matematiche, vol. 36 (1898), pp. 286–299; and another by Salkowski, Mathematische Annalen, vol. 66 (1908), pp. 517-557.

the point considered. Let it be denoted by ϵ. Then

$$ds_1 = ds(1+\epsilon)\ ; \quad ds = ds_1(1-\epsilon). \tag{2}$$

The unit tangent t_1 to the deformed curve is given by

$$t_1 = \frac{dr_1}{ds_1} = (1-\epsilon)(r' + s') = (1-\epsilon)t + s'. \tag{3}$$

Consequently, if κ_1 is the curvature and n_1 the unit principal normal for the new curve,

$$\kappa_1 n_1 = \frac{dt_1}{ds_1} = (1-\epsilon)\frac{dt_1}{ds} = (1-2\epsilon)\kappa n - \epsilon' t + s''.$$

On "squaring" both members, and neglecting small quantities of the second order, we have

$$\kappa_1^2 = (1 - 4\epsilon)\kappa^2 + 2\kappa n \cdot s'',$$

and therefore

$$\kappa_1 = (1-2\epsilon)\kappa + n \cdot s''. \tag{4}$$

Inserting this value in the above product $\kappa_1 n_1$ we find

$$\left\{\begin{aligned} n_1 &= n - \frac{1}{\kappa}((n \cdot s'')n + \epsilon' t - s'') \\ &= n - (n \cdot s')t + \frac{1}{\kappa}(b \cdot s'')b. \end{aligned}\right. \tag{5}$$

The unit binormal b to the deformed curve is then given by

$$\left\{\begin{aligned} b_1 = t_1 \times n_1 &= (1-\epsilon)b + s' \times n - \frac{1}{\kappa}(b \cdot s'')n \\ &= b - (b \cdot s')t - \frac{1}{\kappa}(b \cdot s'')n. \end{aligned}\right. \tag{6}$$

The torsion τ_1 may then be found by differentiating the unit binormal, using the Serret-Frenet formulas. Thus on differentiating (6), and using the preceding results, we find on reduction

$$\tau_1 = (1-\epsilon)\tau + \kappa b \cdot s' + \frac{d}{ds}\left(\frac{1}{\kappa} b \cdot s''\right). \tag{7}$$

We have thus determined the geometric characteristics of the deformed curve in terms of those of the original curve and the small displacement $\mathbf{s}$. The rectangular trihedron, consisting of the unit tangent, the principal normal, and the binormal, undergoes a small rotation which is represented by the vector

$$R = \frac{1}{\kappa}(\boldsymbol{b} \cdot \mathbf{s}'')\boldsymbol{t} - (\boldsymbol{b} \cdot \mathbf{s}')\boldsymbol{n} + (\boldsymbol{n} \cdot \mathbf{s}')\boldsymbol{b} \tag{8}$$

or

$$R = \frac{1}{\kappa}(\boldsymbol{b} \cdot \mathbf{s}'')\boldsymbol{t} + \boldsymbol{t} \times \mathbf{s}'. \tag{9}$$

The coefficients of $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ in (8) represent the small rotations of the trihedron about the tangent, the principal normal, and the binormal, respectively.

An *inextensional deformation* of the curve is one for which ϵ vanishes identically.* If $\mathbf{s}$ is expressed in the form

$$\mathbf{s} = P\boldsymbol{t} + Q\boldsymbol{n} + R\boldsymbol{b},$$

the vanishing of $\boldsymbol{t} \cdot \mathbf{s}'$ gives $P' = \kappa Q$ as the necessary and sufficient condition for inextensional deformation.

3. *Family of Curves on a Surface.* Consider next a family of curves on a given surface. Suppose that the curves of the family suffer a small deformation such that they remain on the same surface. Then the displacement $\mathbf{s}$ at any point is tangential to the surface, and is a function of two parameters that specify the point considered. On any one curve $\mathbf{s}$ is a function of the arc-length s, and the formulas found above hold for the deformed curve.

Other results may be very neatly expressed in terms of the two-parametric differential invariants for the surface, introduced and examined by the author in a recent paper *On differential invariants in geometry of surfaces.*† If ϕ is the value of any function associated with the deformed curve at the point $\boldsymbol{r}+\mathbf{s}$, the new value of this function at

*This case has been considered in some detail by Sannia, Rendiconti di Palermo, vol. 21 (1906), pp. 229–256.

† Quarterly Journal, vol. 50 (1925), pp. 230–269.

the point $\boldsymbol{r}$ originally occupied by this point of the curve is $\phi - \mathbf{s} \cdot \nabla\phi$, where ∇ is the two-parametric differential operator for the surface. Thus after the deformation the unit tangent $\bar{\boldsymbol{t}}$ to the new curve through the point $\boldsymbol{r}$ is, by (3),

$$(10) \quad \left\{ \begin{aligned} \bar{\boldsymbol{t}} &= \boldsymbol{t}_1 - \mathrm{s} \cdot \nabla \boldsymbol{t}_1 = (1 - \epsilon)\boldsymbol{t} + \boldsymbol{t} \cdot \nabla \mathrm{s} - \mathrm{s} \cdot \nabla \boldsymbol{t} \\ &= (1 - \epsilon)\boldsymbol{t} + \operatorname{curl} (\mathrm{s} \times \boldsymbol{t}) - \mathrm{s} \operatorname{div} \boldsymbol{t} + \boldsymbol{t} \operatorname{div} \mathrm{s}. \end{aligned} \right.$$

We have shown elsewhere* that the line of striction of a family of curves with unit tangent $\boldsymbol{t}$ is given by div $\boldsymbol{t}=0$. Hence the line of striction of the deformed family has for its equation div $\bar{\boldsymbol{t}}=0$, which may be expressed in the form

$$(11) \qquad (1 - \epsilon) \operatorname{div} \boldsymbol{t} - \boldsymbol{t} \cdot \nabla \epsilon - \mathrm{s} \cdot \nabla \operatorname{div} \boldsymbol{t} + \boldsymbol{t} \cdot \nabla \operatorname{div} \mathrm{s} = 0,$$

since the divergence of the curl of the normal vector $\mathrm{s} \times \boldsymbol{t}$ vanishes identically.†

If the original family of curves is one of parallels, div $\boldsymbol{t}$ vanishes identically.‡ Hence *a family of parallels will remain parallels after the deformation provided*

$$(12) \qquad \boldsymbol{t} \cdot \nabla(\epsilon - \operatorname{div} \mathrm{s}) = 0.$$

The geodesic curvature of a curve of the original family§ is $\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{t}$, and that of one of the deformed curves is

$$\boldsymbol{n} \cdot \operatorname{curl} \left[(1 - \epsilon)\boldsymbol{t} + \boldsymbol{t} \cdot \nabla \mathrm{s} - \mathrm{s} \cdot \nabla \boldsymbol{t}\right].$$

If then the family of curves is a family of geodesics, it will remain so after the deformation provided

$$\boldsymbol{n} \cdot [\boldsymbol{t} \times \nabla \epsilon + \operatorname{curl} (\boldsymbol{t} \cdot \nabla \mathrm{s} - \mathrm{s} \cdot \nabla \boldsymbol{t})] = 0.$$

Similarly, the original family will be *lines of curvature*‖ if $\boldsymbol{t} \cdot \operatorname{curl} \boldsymbol{t}=0$; and they will remain lines of curvature after the change provided $\bar{\boldsymbol{t}} \cdot \operatorname{curl} \bar{\boldsymbol{t}} = 0$.

* *Some new theorems in geometry of a surface*, § 2, Mathematical Gazette, vol. 13 (1926), pp. 1–6.

† Quarterly Journal, loc. cit., § 7.

‡ See a paper by the author *On families of curves and surfaces*, § 7, recently communicated to the Messenger of Mathematics.

§ Quarterly Journal, loc. cit., § 8.

‖ Mathematical Gazette, loc. cit., § 4.

4. *Congruence of Curves in Space.* Consider finally a small deformation of a congruence of curves in three dimensions. On any one curve the displacement $\mathbf{s}$ is a function of a single parameter; but on the congruence it is a point-function in space. Let ∇ now represent the three-parametric differential operator for three dimensions. Then the extension ϵ is $\epsilon = \mathbf{t} \cdot \mathbf{s}' = \mathbf{t} \cdot (\nabla \mathbf{s}) \cdot \mathbf{t}$, and, by the same argument as above, the unit tangent $\bar{\mathbf{t}}$ to the curve of the congruence which passes through the point $\mathbf{r}$ after the deformation is $\bar{\mathbf{t}} = (1 - \epsilon)\mathbf{t} + \mathbf{t} \cdot \nabla \mathbf{s} - \mathbf{s} \cdot \nabla \mathbf{t}$, which may also be expressed in the same form as (10).

The surface of striction (or orthocentric surface) of a congruence with unit tangent $\mathbf{t}$ is given by* div $\mathbf{t}=0$. Hence the surface of striction of the deformed congruence has for its equation div $\bar{\mathbf{t}} = 0$, which may be expanded in the form $(1 - \epsilon) \operatorname{div} \mathbf{t} - \mathbf{t} \cdot \nabla\epsilon - \mathbf{s} \cdot \nabla \operatorname{div} \mathbf{t} + \mathbf{t} \cdot \nabla \operatorname{div} \mathbf{s} = 0$, since the divergence of the curl now vanishes identically. The limit surface† of the congruence before deformation is

$$(13) \qquad \operatorname{div} (\mathbf{t} \operatorname{div} \mathbf{t} - \mathbf{t} \cdot \nabla \mathbf{t}) = 0,‡$$

and that of the deformed congruence is given by a similar equation in which $\bar{\mathbf{t}}$ takes the place of $\mathbf{t}$. The deformed curves will constitute a normal congruence provided we have

$$\bar{\mathbf{t}} \cdot \operatorname{curl} \bar{\mathbf{t}} = 0,$$

and they will constitute an isometric congruence if, in addition, $\bar{\mathbf{t}}$ satisfies the equation§

$$(14) \qquad \operatorname{curl} (\bar{\mathbf{t}} \operatorname{div} \bar{\mathbf{t}} - \bar{\mathbf{t}} \cdot \nabla \bar{\mathbf{t}}) = 0$$

which may be expanded in terms of $\mathbf{t}$ and $\mathbf{s}$.

CANTERBURY COLLEGE,
CHRISTCHURCH, NEW ZEALAND.

* See a paper by the author *On congruences of curves*, § 6, recently communicated to the Tôhoku Mathematical Journal.

† Ibid., § 4.

‡ This equation for the limit surface of a curvilinear congruence was first given by the author in a paper *On isometric systems of curves and surfaces*, recently communicated to the American Journal.

§ Ibid., § 6.