HAUSDORFF'S GRUNDZÜGE DER MENGENLEHRE.

Grundzüge der Mengenlehre. By FELIX HAUSDORFF. Veit and Company, Leipzig, 1914. viii + 476 pp.

IF there are still mathematicians who hold the theory of aggregates under general suspicion, and are reluctant to grant it full recognition as a rigorous, mathematical discipline, they will find it hard to retain their doubts under fire of the logic of Hausdorff's treatise. It would be difficult to name a volume in any field of mathematics, even in the unclouded domain of number theory, that surpasses the Grundzüge in clearness and precision.

But it is only in a subsidiary rôle that the Grundzüge is an answer to the skeptics. Its most striking feature is that it is a work of art of a master. No one thoroughly acquainted with its contents could fail to withhold admiration for the happy choice and arrangement of subject matter, the careful diction, the smooth, vigorous and concise literary style, and the adaptable notation; above all things, however, for the highly pleasing unifications and generalizations and the harmonious weaving of numerous original results into the texture of the whole.

It is not an uncommon fault of authors of treatises on general subjects to expound their own researches with an unwarranted degree of detail; so that at times, if one has no other evidence, one may be rightly led to suspect particular portions, on account of their remoteness from the central ideas, of being the author's own handiwork. This fault is not shared by the Grundzüge. Few treatises on as comprehensive a subject as the theory of aggregates contain as large a proportion of the author's investigations; yet the parts that are distinctly Hausdorff's own contributions are properly inserted in view of their generality and in relation to other topics.

The author is endowed with a keen psychological and didactic instinct that prompts him to depart from his usual succinctness when engaged in the clarification of the more important ideas. The following quotation may serve to illustrate this pedagogic sense and the general lucidity of style, not without an occasional glimmer of humor. After speaking of the primitive stage of numerical comparison—by means of direct, successive mating—of a pile of apples and a pile of pears, he says (page 45): "But if the apples and the pears are in different places, and the transportation of one pile to the other is attended with difficulties, the inventive mind of man will in the next stage make use of an intermediary set of conveniently transportable objects, such as stones, shells, or chips, and infer the equivalence $A \sim B$ from the equivalences $A \sim C$, $B \sim C$. Finally, however, even this earthly residuum will be eliminated, and the intermediary set will be replaced by a system of spoken, written, or thought symbols, —the number symbols, 1, 2, $\cdots$. *Comparison* turns into *counting*, and equivalent sets now acquire a common property, the number of their elements.

"These remarks, for which no claim whatsoever is entered on psychological or cultural-historical grounds, are intended merely to make clear that equivalence is the natural foundation for the comparison of aggregates, and that by its means we may undertake even the seeming paradox of counting infinite sets."

That the author enjoyed himself while at work may be seen from such passages as the following (page 61): "From an 'alphabet,' i.e. a finite set of 'letters,' we may construct a countable assemblage of finite complexes [= ordered sets] of letters, i.e. 'words,' among which, of course, meaningless words such as abracadabra occur. If in addition to the letters, other elements are used, such as punctuation marks, type-spacings, numerals, notes, etc., we see that the assemblage of all books, catalogs, symphonies and operas is also countable, and would remain countable even if we were to employ a countable set of symbols (but for each complex only a finite number). On the other hand, if in the case of a finite number of symbols we restrict the complexes to a maximum number of elements, agreeing for example, to rule out words of more than one hundred letters and books of more than one million words, these assemblages become finite; and if we assume with Giordano Bruno an infinite number of heavenly bodies, with speaking, writing, and musical inhabitants, it follows as a mathematical certainty that on an infinite number of these heavenly bodies there will be produced the same opera with the same libretto, the same names of the composer, the author of the text, the members of the orchestra and the singers."

Several other examples of spirited and colorful language are the following. Speaking of the equivalence of the whole and the part of an infinite set, he says (page 34): "Of course, when asserted in the rather provocative form, A has as many elements as B, it is one of those 'paradoxes of the infinite' that shock the unprepared mind." Again, page 34: "A segment and an arbitrarily small partial segment, a kilometer and a millimeter, the sun's globe and a drop of water have in this sense the 'same number of points.'" Page 48: ". . . we shall have to desist from giving every proper subset a cardinal number $< \mathfrak{a}$; we must violate the hallowed axiom 'totum parte majus,' as we must in general expect that calculation with infinite cardinals will deviate in many respects from that with finite cardinals, without thereby espying the minutest objection against these infinite numbers." On page 63, after learning that $10^{\aleph_0} = \aleph$, we are led to the equations $\aleph = 2^{\aleph_0} = 3^{\aleph_0} = \cdots$ by the remark that "the fact that we have ten fingers is obviously without influence on the theory of aggregates." Page 60: "The equivalence of the set of whole numbers with the much more inclusive set of rational numbers belongs to those facts of the theory of aggregates which impress you on your first acquaintance with them as astonishing and even paradoxical; especially if you have before your eyes the geometric representation (of the correspondence between the numbers and the points of a straight line), and picture to yourself on the one hand, the 'integral' points, which lie isolated at finite distances from one another, and on the other hand, the 'rational' points, which are distributed over the entire line as dust of more than microscopic fineness."

One of the characteristic traits of the style is its continuity, brought about by the neat and confident conjunctional devices of the author. On page 335 there is need of the awful descent from the general spaces previously considered to the very special euclidean plane. The author is unwilling to take the plunge without assuring himself of the reader's good-will: "In the subsequent discussion we prefer to confine ourselves to the plane. The extension to three or more dimensional space presents in part not inconsiderable difficulties, because the rôle which polygons play in the plane falls there to the lot of the much less simple polyedra and hyper-polyedra. Even in the plane we shall find that the apparently most plausible, intuitive assertions require fairly complicated proofs. A certain prolixity is already produced by the fact that sets that are not compact, and are therefore unbounded, behave in many respects not like bounded sets, and indeed, less regularly. A radical remedy would be the adjunction of a point 'at infinity,' as in function theory; but thereby the character of a metrical space is destroyed, and if again, you get rid of this evil by means of stereographic projection on the sphere, you lose in exchange certain elementary geometric advantages of the plane. We must, therefore, come to terms as best we can with the 'dreary infinities of homaloidal space' as Clifford says."

We now turn to a more detailed description of the contents. In view of the great wealth of ideas—made possible by the concentrated style—we must refrain from attempting to discuss even all the important topics; only some of these can we describe in detail, and we shall give preference to the more novel or the less technical features.

The book is fittingly inscribed to Georg Cantor, "the creator of the theory of aggregates."

Chapter I (pages 1–31) deals with the sum (denoted by $\mathfrak{S}$), section (Durchschnitt, denoted by $\mathfrak{D}$), and difference of sets. Of the numerous topics treated in this chapter, we shall discuss the notion of aggregate, the principle of duality, and symmetric sets. Among the other topics may be mentioned difference chains, rings and fields (Ringe und Körper), sequences of sets and of real numbers, σ- and δ-systems, and the non-convergence points of a sequence of functions.

Aggregate is defined in the cantorean naïve fashion—as distinguished from the less debatable, but more restraining manner of Zermelo—as a whole constituted by the conceived assembling of individuals. The author makes it clear that it is inadvisable on pedagogical and other grounds to found everything upon Zermelo's Grundlagen; paradoxes are duly banished, however, by appropriate interpretation of the naïve definition (see page 106 for the disposal of the Burali-Forti antinomy).

If $A_1, A_2, \cdots$ are subsets of a set M and $\bar{A}_1 = M - A_1$, $\bar{A}_2 = M - A_2, \cdots$ their complements,—we denote generally by $\bar{X}$ the complement of X in M—, then the complement in M of the sum of the given sets is the section of their complements $[\overline{\mathfrak{S}}(A_1, A_2, \cdots) = \mathfrak{D}(\overline{A}_1, \overline{A}_2, \cdots)]$ and the complement of the section of the given sets is the sum of their complements $[\overline{\mathfrak{D}}(A_1, A_2, \cdots) = \mathfrak{S}(\overline{A}_1. \overline{A}_2, \cdots)]$. Since $P = Q$, $P \subset Q$ (i.e., P is a subset of Q) imply $\overline{P} = \overline{Q}$, $\overline{P} \supset \overline{Q}$, it follows that every equation between sets remains true if every set is replaced by its complement and the symbols $\mathfrak{S}$ and $\mathfrak{D}$ are interchanged; and every inequality remains true after the same changes and the additional interchange of $\supset$ and $\subset$. For example, $A \subseteqq \mathfrak{S}(A, B)$ leads to $\overline{A} \supseteqq \mathfrak{D}(\overline{A}, \overline{B})$ and hence to $A \supseteqq \mathfrak{D}(A, B)$. This simple property Hausdorff calls the "principle of duality"; he utilizes it in various connections to secure results through formulas usually obtained otherwise. We remark here that the author uses formulas to a much larger extent than is customary in the theory of aggregates, one of whose noticeable characteristics is the unusual freedom from calculational methods.

Let $X_1, X_2, \cdots, X_m$ be m given sets. Taking a cue from algebra, we seek a list of simple sets which like the sum $\mathfrak{S}(X_1, X_2, \cdots, X_m)$ and the section $\mathfrak{D}(X_1, X_2, \cdots, X_m)$ involve the given sets symmetrically. Hausdorff calls these sets "symmetrische Grundmengen," and defines them as follows: $A_i (i = 1, 2, \cdots, m)$, the ith such set, consists of the elements that occur in at least i X's. Thus $A_1 = \mathfrak{S}(X_1, X_2, \cdots, X_m)$, $A_m = \mathfrak{D}(X_1, X_2, \cdots, X_m)$. An essential property of these sets is that they are expressible in terms of the X's by means of sums and sections. Their introduction is due to Hausdorff. The author considers various properties of these sets, and in particular, utilizes them in an interesting theory of measure, which was at first planned as final but was later discarded for a more concise treatment; a sketch of the old theory appears in the appendix.

The second chapter (pages 32–45) deals with functions, products, and powers, and their laws of operation.

The third chapter (pages 45–69) treats of the cardinal numbers. Cardinal number is not defined with Cantor as what remains of a set after the individual nature and the order of the elements are abstracted; nor with Russell, as a class of classes. Hausdorff takes the simple and formal point of view, which is clearly the most satisfactory: We associate uniquely with a system of sets A a system of things a—of indifferent nature—in such a way that the same things correspond to two sets if and only if the sets are equivalent. These things or symbols we call cardinal numbers. Two proofs are given of the Bernstein equivalence theorem, the first essentially like that of Bernstein, and the second, according to Zermelo, without the use of the infinite set of integers. The rest of the chapter is devoted to the comparison of cardinal numbers, and includes such theorems as $\aleph^{\aleph_0} = \aleph$, $\aleph_0^{\aleph} = 2^{\aleph} > \aleph$ ($\aleph$ = cardinal number of the continuum), and the theorem of J. König. Famous theorems, such as those of the non-denumerability of the continuum, and of the equivalence of the plane of points and the line of points, frequently appear—as is also the case in other chapters—as very special cases of general considerations, or as side remarks in the current text. For the non-denumerability of the continuum, however, a special proof is added.

Chapter IV (pages 69–101) takes up order; a substantial portion of the ideas and results is due to Hausdorff. After various definitions of simple (= linear) order, the sum of an ordered set (the "argument") of any number of ordered sets, and the product of a finite number of sets are defined, and the laws for operating with these processes given. The subset M of the ordered set A is said to be "coinitial" with A if no element of A exists preceding every element of M. Similarly, "cofinal." M is "dense in A" if for every pair $a < b$ of elements of A there exists a pair of elements $m < n$ of M such that $a \leqq m < n \leqq b$. The decomposition $A = P + Q$, where $P (\neq 0)$ and $Q (\neq 0)$ have no elements in common and every element of P precedes every element of Q, is said to be a "jump" (Sprung) if P has a last element and Q a first; a "gap" (Lücke), if neither P has a last, nor Q a first. A "dense set" (in the "absolute," as contrasted with the "relative" sense) is one without jumps; a "continuous set," one without jumps or gaps. A "scattered set" is one possessing no dense subset. A sum $\Sigma_i^J A_i$ $(A_i \neq 0)$ is scattered when and only when the argument J is scattered and each A_i is scattered. Every ordered set is either scattered or the sum of scattered sets over a dense argument. The chapter closes with the discussion of types of order, in particular, of the class of countable dense types, and of continuous types.

The fifth chapter (pages 101–139) is devoted to normally ordered (wohlgeordnete) sets and the ordinal numbers. The treatment comprises comparability of cardinal numbers, transfinite induction, powers and products, alephs and the number classes, the initial numbers (Anfangszahlen) and Zermelo's Wohlordnungssatz. The proof of the general theorem $\aleph_a\aleph_a = \aleph_a$ for every ordinal number a, is given in elegant and brief form; the first proof of the theorem, given by Hessenberg in his Grundbegriffe der Mengenlehre, is long and roundabout. It may be remarked that frequently the simplest and most elegant proofs of important theorems are to be found in the Grundzüge, either directly or after appropriate modification of the generalized form in which they usually appear.

For the Wohlordnungssatz both of Zermelo's proofs with unessential but neat modifications are given. Hausdorff has no difficulty—neither has the reviewer—in accepting either of these proofs as rigorous. In fact, as is sometimes the case with the work of mathematicians who have misgivings about the theorem, the multiplicative principle (Prinzip der Auswahl) steals in noiselessly (cf. for example, page 54) before the Wohlordnungssatz is mentioned.

The sixth chapter (pages 139–209) contains a wealth of material mostly from the author's own researches. Unfortunately space will not permit—especially because of the more technical character of the subject matter—a description of these elegant and general results. We must content ourselves merely with mentioning the partially ordered sets, the distinction as related to coinitiality and cofinality of the element and gap characteristics and the consequent classification of ordered sets, the general products and powers of ordered sets and the interesting connection with non-archimedean number systems, as shown by the general theorem of Hahn (*Berichte der Wiener Akademie der Wissenschaften*, 1907).

The remaining chapters of the book (VII–X) will prove of more general interest because they are concerned with the applications of the abstract theory to the study of space relations. It is in these chapters especially that Hausdorff impresses you with his masterful exposition. The theory of point sets is cast into a new and more general mold, and the resulting treatment is characterized throughout by originality, naturalness, and beauty.

Chapter VII (pages 209–260) begins with the statement: "The theory of aggregates has celebrated its most beautiful triumphs in its application to point sets and in the clarification and heightened precision of the fundamental concepts of geometry; this is admitted even by those who demean themselves skeptically towards the abstract theory of aggregates." The subject matter of the chapter concerns point sets in general spaces. The author's justification of his abstract treatment is as follows (page 210): "Now a theory of spatial point sets would naturally have, in virtue of the numerous accompanying properties, a very special character, and if we wished to confine ourselves from the outset to this single case, we should be obliged to develop one theory for linear point sets, another for planar point sets, still another for spherical point sets, etc. Experience has shown that we may avoid this pleonasm and set up a more general theory comprehending not only the cases just mentioned but also other sets (in particular, Riemann surfaces, spaces of a finite or an infinite number of dimensions, sets of curves, and sets of functions). And, indeed, this gain in generality is associated not with increased complication, but on the contrary, with a considerable simplification, in that we utilize—at least for the leading features—only few and simple assumptions (axioms). Finally we secure ourselves in this logical-deductive way against the errors into which our so-called intuition may lead us; this alleged source of knowledge—the heuristical value of which, of course, no one will impugn—has, as it happens, shown itself so frequently insufficient and unreliable in the more subtle parts of the theory of aggregates, that only after careful examination may we have faith in its apparent testimony."

Hausdorff does not bind himself to a single set of assumptions. The center of interest lies, of course, in the theorems, and the assumptions are graded accordingly, a new assumption or a modification being adopted only when the mathematical content naturally calls for such a change. In the carefully planned march from the abstract in the direction of greater specialization, Hausdorff gives repeated evidence of his mathematical-esthetic insight.

The developments in Chapter VII are based entirely upon the following "neighborhood" postulates (Umgebungsaxiome, page 213). A "neighborhood" is a point set. The abstract set or space E in question is unrestricted except for the postulates:

(A) To every point x of E there corresponds at least one neighborhood U_x; every neighborhood U_x contains the point x.

(B) If U_x, V_x are two neighborhoods of x, there is a neighborhood W_x contained in both.

(C) If the point y lies in U_x, there is a neighborhood U_y lying in U_x.

(D) For $x \neq y$ there are two neighborhoods U_x, U_y with no point in common.

For the euclidean plane, the neighborhoods of a point P may be taken as the circles (exclusive of the boundary) having P as center.

A space satisfying the four neighborhood postulates is called "topological."

There are numerous concrete examples of topological spaces, among which may be mentioned the ordinary euclidean spaces (also after adjunction of the ideal point at infinity), certain spaces in which the distance from point to point is measured on a non-archimedean scale, space of a denumerable infinity of dimensions, and function space. Some of these spaces are also "metric" (see below).

An "inner point" of a set A—belonging to the entire space or "universe" E—is one possessing a neighborhood lying entirely in A. A point of A that is not one of its inner points is a "brink" point (Randpunkt), as distinguished from "boundary" point (Grenzpunkt), which need not belong to the given set. A "region" (Gebiet) is a set every point of which is an inner point of the set; a "brink aggregate" (Randmenge), one every point of which is a brink point. The inner points of the complement B of A ($A + B = E$) are called the "outer" points of A; the boundary points of A consist of the brink points of both A and B. The universe E and every neighborhood is a region. The inner points of an arbitrary set constitute a region; the brink points, a brink aggregate. The sum of any number, and the section of a finite number, of regions are regions.

Connected with the last statement, there is a simple but fruitful principle: If a system of sets M—like the system of regions—has the property that the sum of any number of sets of the system belongs to the system, we may, for any given set A containing at least one M as subset, define the largest M contained in A; if the section of any number of M's is an M, we may for every A contained in at least one M define the smallest M containing A. This principle is used in various connections, for example, in the definition of the "kernel" (see below).

The introduction of the sets A_α, A_β, A_γ—the first has been little used—leads to considerable formal simplification, and enables Hausdorff to give many short proofs through formal processes instead of direct reflection upon the nature of the hypotheses. x is an α-point of A if every neighborhood U_x contains at least one point of A (which may be x itself); a β-point, if every neighborhood U_x contains an infinite number of points of A; a γ-point, if every neighborhood U_x contains a non-denumerable set of points of A. A_α, A_β, A_γ are the respective totalities of these points (A_β, the derivative of A). That A is "closed" may be expressed by $A \supseteqq A_\beta$, or by $A = A_\alpha$; "dense-in-itself," by $A \subseteqq A_\beta$ or by $A_\alpha = A_\beta$; "perfect," by $A = A_\beta$. The sets A_α, A_β, A_γ are closed. The section of any number of closed sets and the sum of a finite number of closed sets are closed. The sum of any number of sets each dense-in-itself is dense-in-itself. The largest subset of A that is dense-in-itself exists according to the principle just mentioned; it is called the "kernel" of A.

An infinite set without β-points is said to be "divergent"; a set without divergent subsets, "compact." The set A converges to the limit x if every neighborhood of x contains all the points of A with the possible exception of a finite number. A decreasing sequence $A_1 \supseteqq A_2 \supseteqq \cdots$ of compact, closed, non-vanishing sets has a non-vanishing section (Cantor). A compact, closed set contained in the sum of a sequence of regions is contained in the sum of a finite number of these regions (Borel).

A clear and systematic treatment is given of the limits of a sequence of sets $\{A_n\}$. Six different kinds of limits are distinguished: (1) the "lower limit" consists of the points belonging to "nearly all" the A_n, i.e., all with the possible exception of a finite number; (2) the "upper limit," of the points belonging to an infinity of the A_n; (3) the "lower closed limit," of the points (belonging to the A_n or not) every neighborhood of which contains points of nearly every A_n; (4) the "upper closed limit," of the points every neighborhood of which contains points of an infinity of the A_n; (5) the "lower limit region," of the points for which a neighborhood exists belonging (in its entirety) to nearly all the A_n; and (6) "the upper limit region," of the points for which a neighborhood exists belonging to an infinite number of the A_n. Various modes of representation of these limits by means of sums and sections are given.

Another novel feature is the systematic introduction and use of the notion of "relativity." A is said to be (relatively) "closed in M" if it is the section of M and a closed set; a "relative region of M," if it is the section of M and a region. These notions are special cases of a complete theory of relativity, which arises by substituting for the universe E an arbitrary subset M of it. The neighborhoods U_x are replaced by their sections with M, which again satisfy the neighborhood assumptions; M may therefore be regarded as a new universe possessing all the properties of topological spaces. We thus have relative α-points, relative inner points, and so on.

The definition of connectivity differs from those heretofore given, but it is the most desirable in the opinion of the reviewer: A non-vanishing set M is said to be "connected" if it is not expressible as the sum of two sets ($\neq 0$) that are (relatively) closed in M and have no points in common. A "component" of a non-vanishing set is one of its largest connected subsets, i.e., a connected subset contained in no other such subset. The "quasi-component" of A belonging to the point p consists of the points belonging to the same summand as p in every decomposition of A as the sum of two sets closed in A and having the null-set as section. The quasi-components may differ from the components. After a series of theorems on connectivity, the chapter is devoted to density, and to the application of some of the results to sets of real numbers.

In Chapter VIII (pages 260–358) special topological spaces are considered. A stride towards ordinary space is made by the successive introduction of the denumerability postulates:

(E) The set of neighborhoods of x is denumerable for every x.

(F) The totality of all neighborhoods is denumerable.

With the aid of (E), it follows, for example, that every convergent set (= set having a limit) is countable; and that if x is a β-point of a set A, there is a convergent subset of A with x as limit. With (F) the γ-points begin to play an important rôle. If a set has no γ-points belonging to it, it is countable. The following equations hold ($A_{\alpha\alpha}$ = set of α-points of A_α, etc.): $A_{\alpha\alpha} = A_\alpha$, $A_{\alpha\beta} = A_\beta$, $A_{\alpha\gamma} = A_{\beta\gamma}$; $A_{\beta\alpha} = A_\beta$; $A_{\gamma\alpha} = A_\gamma$, $A_{\gamma\beta} = A_\gamma$, $A_{\gamma\gamma} = A_\gamma$. A set of regions no pair of which have common points is countable. There are in all $\aleph$ regions and $\aleph$ closed sets ($\aleph$ = cardinal number of the continuum). The sum of any number of regions is the sum of a countable number (at most) of them.

By means of this result, Borel's theorem may be extended and its complete converse given.

The connection of point set theory with the transfinite ordinals lies in such theorems. as the following (proved with the aid of postulate (F)): An ascending normally ordered set $\{S_\xi\}$—i.e., $S_\rho \supset S_\sigma$ for $\rho > \sigma$—of different regions S_ξ is at most countable. Similarly for a descending set of regions and for relative regions, and likewise for closed and relatively closed sets. It is at this juncture that Hausdorff introduces his generalization of "reducible" sets. It turns out that a Hausdorff reducible set is representable as the sum of differences of descending normally ordered closed sets, and conversely.

Further developments refer to metric spaces: "We believe the time has come when a continuation of the neighborhood theory would be accompanied with a loss of simplicity." A metric space satisfies the following postulates, where $\overline{xy}$ denotes the "distance"—Fréchet's écart—from x to y:

(α) (Postulate of symmetry) $\overline{yx} = \overline{xy}$.

(β) (Postulate of coincidence) $\overline{xy} = 0$ when and only when $x = y$.

(γ) (Triangle postulate) $\overline{xy} + \overline{yz} \geqq \overline{xz}$.

Space will not permit a discussion of the rich content of the rest of the chapter. We mention: distances between sets, connectivity properties of metrical spaces including ρ-connectivity, properties related to the various limits of a sequence of sets, Borel sets, conditions for compactness and "complete" (vollständige) spaces. After certain general theorems concerned chiefly with connectivity in euclidean n-space, the chapter turns to the euclidean plane—see quotation in early portion of this review—and after a succession of thirteen carefully graded theorems, culminates in a proof—modeled after that of Brouwer—of the Jordan theorem.

The ninth chapter (pages 358–399) deals with representations or functions. If the original set A and its image B (= totality of elements $f(a)$, where a ranges over A) are topological spaces, the continuity of $f(a)$ is equivalent to the condition that every relative region of B is the image of a relative region of A. If f is continuous and A connected, then B is connected. From this it follows, in particular, that a continuous, real function defined in a connected set—for example, in a linear interval—takes every value between any two of its values. Among other things, the chapter considers uniform continuity for metrical spaces, continuous curves including the space-filling curves, the character of the points of continuity of a discontinuous function, sequences of functions and the generalizations of theorems of Arzelà and Baire. There are novel results on the classification of functions and the set of convergence points of a sequence of functions.

The tenth and final chapter (pages 399–448) treats of Peano-Jordan content and Borel-Lebesgue measure; both theories are developed in a new, elegant and appreciably similar fashion. There are applications to decimal and to continued fractions. After the deduction of the important properties of Lebesgue integrals, the chapter closes with the proof that a function of limited variation possesses a derivative except in a set of measure zero.

The appendix (pages 449–473) gives references to the literature and contains numerous discussions of substantial content and interest.

Misprints are few in number; and errors, invariably of a minor character. The mention of most of them would be regarded as hypercritical, were it not for the high standards of the author. The careful examination of the reviewer has brought to light only the following: page 11, line 15, "Elemente" instead of "Punkte"; page 28, line 27, "Relationen" instead of "Gleichungen"; page 58, line 14 and page 105, line 20, "Denn" instead of "Dann"; page 272, G_{i_1} is not printed clearly; page 366, line 6, insert "Kap. VIII" after "nach"; page 442, line 5, insert Σ before $\mu_i\delta_i$. On pages 85 and 291, in the footnotes, occur the equations $m = \{m\}$, $x = \{x\}$; the author clearly intimates their objectionability, but it would be better to bar altogether such illogical statements. On page 106, the author speaks of numbers greater than W, when he means numbers greater than numbers of W. On page 229, he forgets to discuss A_k at the end of the section. On page 276, the restriction that λ shall be a limiting number does not enter the proof. This is an exception to prove the rule that in the Grundzüge every word counts. On page 444, the statement that X is normally ordered, although later cleared up, contains at first an ambiguity; for X may be normally ordered without being ordered according to magnitude. On page 470, $\psi^{-m_1}\delta\psi^{m_1}$ is not necessarily an α, but may be a γ; it is an easy matter, however, to fill out the gap.

As for more important criticism, one may quarrel with the author for his abstract style, for his euclidean manner of grading the proofs, so that no difficulties remain and none but mild climaxes are reached, for his finish that may excite admiration but hardly activity on the reader's part. One may crave for a book that is built like a drama around a single idea —a more sketchy book, leaving more to the reader's imagination, a book with a less diversified and more emphatic message. But such remonstrance would be like quarrelling with Beethoven for having written symphonies instead of operas. There is no such thing as *the* book. Hausdorff's Grundzüge is a treatise, and as a treatise it necessarily falls short of the summum bonum. But as a treatise it is of the first rank.

HENRY BLUMBERG.

SHORTER NOTICES.

The Casting-Counter and the Counting Board. By FRANCIS PIERREPONT BARNARD. Oxford, Clarendon Press, 1916. 358 pp. + 63 plates.

When we consider that Gerbert, the greatest mathematician living in Europe at the close of the tenth century, wrote upon the use of counters as an aid to computation; that Robert Recorde, who is often called the founder of the English school of mathematicians of the sixteenth century, did the same; and that nearly all computation in Europe before the year 1500 (in Italy before c. 1200) was performed by the aid of some type of abacus, we may well infer that the "casting-counter," as Professor Barnard calls it, has played an important rôle in the history of calculation. Indeed, our very word "calculate" is, it need hardly be said, due to this very fact, the word "calculus" meaning a pebble, calculi being used in numerical work in the classical period of the Greek and Roman civilizations.

When we also consider the fact that it was the bamboo rods, used by the early Chinese algebraists to express coefficients, that suggested to the Japanese the *sangi* which were used for the same purpose, and also suggested the idea of determinants which their scholars developed in the seventeenth