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Precanonical quantization and the Schrödinger wave functional revisited

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(*In loving memory of Pavel Efimov*)

We address the issue of the relation between the canonical functional Schrödinger representation in quantum field theory and the approach of precanonical field quantization proposed by the author, which requires neither a distinguished time variable nor infinite-dimensional spaces of field configurations. We argue that the standard functional derivative Schrödinger equation can be derived from the precanonical Dirac-like covariant generalization of the Schrödinger equation under the formal limiting transition $\gamma^0 \varkappa \rightarrow \delta(\mathbf{0})$, where the constant $\varkappa$ naturally appears within precanonical quantization as the inverse of a small "elementary volume" of space. We obtain a formal explicit expression of the Schrödinger wave functional as a continuous product of the Dirac algebra valued precanonical wave functions, which are defined on the finite-dimensional covariant configuration space of the field variables and space-time variables.

1. Introduction

Precanonical quantization of field theory was proposed in [1–3] as an analogue of canonical quantization which does not distinguish between the space and time variables, and hence is more compliant with the concept of relativistic space-time. It is inspired by the De Donder-Weyl (DW) analogue of the Hamiltonian formulation known in the calculus of variations [5–8]. This formulation can be viewed as a generalization of the Hamiltonian description to field theory, such that all space-time variables are treated on equal footing as analogues of the time parameter in mechanics. Such an approach allows us to view fields as finite-dimensional systems *changing* in space-time rather than infinite-dimensional mechanical systems *evolving* in time.

Specifically, given a first order Lagrangian function $L(y^a, y^a_\mu, x^\nu)$ on the space of field variables y^a, their first space-time derivatives y^a_μ, and the space-time variables x^μ, one can introduce the Hamiltonian-like variables: $p^\mu_a := \partial L/\partial y^a_\mu$ (*polymomenta*) and $H = H(y^a, p^\mu_a, x^\nu) := y^a_\mu p^\mu_a - L$ (*DW Hamiltonian function*), and write the Euler-Lagrange field equations in the De Donder-Weyl Hamiltonian-like form [5–8]:

$$\partial_\mu y^a(x) = \frac{\partial H}{\partial p^\mu_a}\ , \quad \partial_\mu p^\mu_a(x) = -\frac{\partial H}{\partial y^a}\ , \tag{1.1}$$

provided $\det\left(\frac{\partial^2 L}{\partial y^a_\mu \partial y^b_\nu}\right) \neq 0$.

In this formulation the analogue of the extended configuration space is the finite-dimensional space of field variables y^a and space-time variables x^μ, and the analogue of the extended phase space is the polymomentum phase space of variables (y^a, p^μ_a, x^μ). Note that when the number of space-time dimensions $n = 1$, this formulation reproduces the standard Hamiltonian formulation of mechanics. At the same time it provides an alternative to the standard extension of the Hamiltonian formulation to field theory, the canonical formalism, which is based on the space-time decomposition and infinite-dimensional spaces of field configurations and, therefore, is applicable only on globally hyperbolic space-times.

We will refer to the DW Hamiltonian formalism and the related quantization as "precanonical" based on the observation that the symplectic structure and other structures of the canonical Hamiltonian formalism in field theory can be related to, or derived from those of the De Donder-Weyl (or polysymplectic, or multisymplectic) formalism by restricting the latter to the surface of initial field configurations at a fixed moment of time (which we call the Cauchy surface) and integrating over it [9–15].

Though the idea of an approach to field quantization based on the DW Hamiltonian formulation was discussed already by Hermann Weyl himself in 1934 [16], it was not further developed, because an analogue of the Poisson bracket in the DW Hamiltonian formalism was not known until it was constructed using the polysymplectic structure on the polymomentum phase space in [9, 17, 18] (see also [19–25] for later discussions and generalizations).

Since the Poisson bracket in precanonical DW formulation [9, 17, 18] is defined on differential forms representing the dynamical variables of field theory, and it leads to the Gerstenhaber algebra structure generalizing the standard Poisson algebra structure, its quantization has been shown to lead to a quantum formalism with the space-time Clifford algebra valued operators and wave functions, and the following precanonical generalization of the Schrödinger equation [1–3]:

$$i\hbar\varkappa\gamma^\mu\partial_\mu\Psi = \widehat{H}\Psi, \tag{1.2}$$

where $\Psi = \Psi(y^a, x^\mu)$ is a Clifford-valued wave function on the covariant configuration space, $\widehat{H}$ is the operator of the DW Hamiltonian function, which contains the partial derivatives with respect to the field variables: ∂_{y^a}, and $\varkappa$ is a (large) constant of the dimension $\mathtt{length}^{-(n-1)}$ in n space-time dimensions, which appears on dimensional grounds when, e.g., the differential form corresponding to the infinitesimal volume element of space: $d\mathbf{x} := dx^1 \wedge \cdots \wedge dx^{n-1}$, is mapped to the corresponding element of the space-time Clifford (or Dirac) algebra using the "quantization map" q known in the theory of Clifford algebras (see e.g. [26]):

$$d\mathbf{x} \xrightarrow{q} \frac{1}{\varkappa}\gamma_0. \tag{1.3}$$

In a sense, the procedure of precanonical quantization effectively introduces in the theory an "elementary volume" scale $1/\varkappa$ without any a priori assumptions regarding the microscopic structure of space-time. The limit of the vanishing "elementary volume" scale corresponds to $1/\varkappa \to 0$ or, more precisely,

$$\frac{1}{\varkappa}\gamma_0 \xrightarrow{q^{-1}} d\mathbf{x}$$

or, equivalently,

$$\gamma^0\varkappa \xrightarrow{q^{-1}} \delta^{n-1}(\mathbf{0}), \tag{1.4}$$

where q^{-1} is the inverse of the Clifford algebraic "quantization map" from the Grassmann algebra of forms to the Clifford algebra of Dirac matrices.

The precanonical wave function $\Psi(y, x)$ can be interpreted as the probability amplitude of finding the value of the field in the interval $[y, y + dy]$ when observed in the vicinity of the space-time point $[x, x + dx]$. This interpretation is supported by the conservation law

$$\partial_\mu \int dy\, \mathsf{Tr}(\overline{\Psi}\gamma^\mu\Psi) = 0,$$

which follows from (1.2) and the self-adjointness of $\widehat{H}$ with respect to the indefinite scalar product $\int dy \mathsf{Tr}(\overline{\Phi}\Psi)$, $\overline{\Psi} := \gamma^0\Psi^\dagger\gamma^0$, and the positive definiteness of the Frobenius norm $\mathsf{Tr}(\Psi^\dagger\Psi)$.

Note that at $n = 1$ the formulation of precanonical quantization essentially reduces to the conventional quantum mechanics formulated in terms of complex wave functions.

The description of quantum fields based on precanonical quantization appears to be fundamentally different from the familiar formulations of quantum field theory. By abandoning the usual explicit treatment of fields as infinite-dimensional Hamiltonian systems and replacing the starting point of quantization with the De Donder-Weyl Hamiltonian formalism we lose an obvious connection with the concepts of the standard formulations of QFT, such as free particles, which straightforwardly follow from the conventional treatment and are crucial for the comparison of the results of quantum field theory with the experiments, at least in the perturbative regime. On the other hand, the construction of precanonical quantization is non-perturbative, explicitly compliant with the relativistic nature of space-time, and it seems to be potentially better or easier defined mathematically than the infinite-dimensional constructions of QFT.

In spite of the fact that there already have been attempts to apply precanonical quantization in quantum Yang-Mills theory [27], quantum gravity [28–33] and string theory [34], the lack of good understanding of the connections of the description suggested by precanonical quantization with the concepts of standard QFT has been hindering applications of precanonical quantization and its further development so far.

For the sake of completeness let us quote several other attempts of field quantization inspired by the covariant Hamiltonian-like formulations in the calculus of variations: see e.g. [35–40].

In our previous papers [14, 27] we tried to understand the connection of precanonical quantization and its description of quantum fields in terms of Clifford-valued wave *functions* $\Psi(y, x)$ with the standard canonical quantization in the functional Schrödinger representation [41–45], where the states of quantum fields are described by the wave *functionals* $\mathbf{\Psi}([y(\mathbf{x})], t)$ on the infinite-dimensional configuration space of field configurations $y(\mathbf{x})$ at the instant of time t. However, the discussion in [1, 14, 27] has not established a convincing connection because of the following shortcomings of the arguments:

(i) the simplifying ultra-locality assumption that the Schrödinger functional $\mathbf{\Psi}([y(\mathbf{x})], t)$ can be represented in the form of the continuous product of precanonical wave functions $\Psi(y, \mathbf{x}, t)$ over all points of the surface $y = y(\mathbf{x})$ (cf. Eq. (2.2) below) neglects the correlations between the field values at the space-like separated points, thus contradicting both the known explicit form of the exact solutions of the functional Schrödinger equation for free field theories [41, 42, 44] and the known behaviour of the Wightman functions of free fields;

(ii) an attempt to take those correlations into account by means of a functional "unitary transformation", which would transform the equation in functional derivatives satisfied by the ultralocal continuous product Ansatz of [14] into the equation for the wave functional in the Schrödinger picture, leads to a non-integrable equation in functional derivatives for the functional operator $e^{i\mathbf{N}}$ determining such a transformation: $\delta\mathbf{N}/\delta y(\mathbf{x}) = \gamma^i \partial_i y(\mathbf{x})$;

(iii) the representation of the continuous product $\prod_{\mathbf{x}} \Psi(\mathbf{x})$ in the form $e^{\varkappa \int d\mathbf{x} \ln(\Psi(\mathbf{x}))}$ used in [14] is questionable for Clifford-valued $\Psi(\mathbf{x})$, which may not commute at different points $\mathbf{x}$; besides, its functional differentiation implies the existence of the inverse $\Psi^{-1}(\mathbf{x})$ for all $\mathbf{x}$, which is too restrictive and even impossible to define for arbitrary Clifford algebra elements Ψ.

In this paper we revisit our earlier treatment [14] of the relation between the precanonical wave function and the Schrödinger wave functional using the example of scalar field theory. We follow the conventions and notations of that paper, and we also refer to it both for a brief outline of the elements of precanonical and canonical quantization of scalar field theory and an explanation of some constructions to be used here. It will be shown below that a minimal generalization of the ultra-local Ansatz used in [1, 14, 27] allows us to take into account the space-like correlations, which were neglected in the earlier treatment of [14], and to derive a formula expressing the Schrödinger wave functional in terms of precanonical wave functions.

2. Precanonical wave functions and the Schrödinger wave functional

We restrict ourselves to the example of the scalar field theory given by

$$L = \frac{1}{2}\partial_\mu y \partial^\mu y - V(y),$$

where the potential term $V(y)$ also includes the mass term $\frac{1}{2}m^2 y^2$ and $\hbar = 1$ henceforth.

The Schrödinger wave functional of the quantum scalar field obeys the Schrödinger equation in functional derivatives [41–45]:

$$i\partial_t \mathbf{\Psi} = \int d\mathbf{x} \left\{ -\frac{1}{2}\frac{\delta^2}{\delta y(\mathbf{x})^2} + \frac{1}{2}(\nabla y(\mathbf{x}))^2 + V(y(\mathbf{x})) \right\} \mathbf{\Psi}. \tag{2.1}$$

Our task is to clarify how the description of quantum fields in terms of the wave functional $\mathbf{\Psi}([y(\mathbf{x})], t)$ is related to the description in terms of the precanonical wave function $\Psi(y, x)$, and how the Schrödinger equation of the canonical quantization approach, Eq. (2.1), is related to the Dirac-like partial differential equation, Eq. (1.2), playing the role of the Schrödinger equation in the precanonical quantization approach.

Note that if the precanonical wave function $\Psi(y, x)$ has the probabilistic interpretation as the probability amplitude of finding the value y of the field at the space-time point x, then the Schrödinger wave functional $\mathbf{\Psi}([y(\mathbf{x})], t)$, which is the probability amplitude of observing the field configuration $y(\mathbf{x})$ on the space-like hypersurface of constant time t, should be given by a certain composition of precanonical amplitudes $\Psi(y, \mathbf{x}, t)$ taken along the Cauchy surface $\Sigma : (t = \text{const}, \ y = y(\mathbf{x}))$ in the covariant configuration space of variables (y, x).

If we assume that the probability amplitudes of observing the field values y are independent in space-like separated points, then $\mathbf{\Psi}([y(\mathbf{x})], t)$ is given by the product over all points of Σ of the wave function $\Psi(y, x)$ restricted to Σ: $\Psi(y, x)|_\Sigma = \Psi_\Sigma(y = y(\mathbf{x}), \mathbf{x}, t)$, i.e.

$$\mathbf{\Psi} \sim \prod_{\mathbf{x} \in \Sigma} \Psi_\Sigma(y = y(\mathbf{x}), \mathbf{x}, t). \tag{2.2}$$

This ultra-locality assumption is, however, unphysical and an improved representation of the Schrödinger wave functional in terms of $\Psi(y, x)|_\Sigma$ has to be found, which would take into account the correlations of the amplitudes $\Psi(y, x)$ at space-like separated points.

The task is similar to the probability theory, where the joint probability of two events A and B is given in general by $P(A, B) = P(A|B)P(B)$, where $P(A|B)$ is the conditional probability of A given B, which reduces to $P(A)$ only if the events A and B are independent. In our case we have a continuum of events of obtaining the values $y_\mathbf{x}$ of the field in the corresponding points $\mathbf{x}$ of the hypersurface of constant time t, and their respective probability amplitudes $\Psi(y_\mathbf{x}, \mathbf{x}, t)$ given by the precanonical wave function taken along the surface Σ: $\Psi(y = y(\mathbf{x}), \mathbf{x}, t)$.

As a minimal deviation from the simplest ultralocal product formula in (2.2) let us assume that the correlations between space-like separated points can be taken into account by a multiplication of $\Psi(y, x)|_\Sigma$ by some function of the field configuration denoted $U(y(\mathbf{x}))$, so that the Schrödinger functional can be given by a modified product formula $\mathbf{\Psi} \sim \prod_{\mathbf{x} \in \Sigma} U(y(\mathbf{x}))\Psi(y, x)|_\Sigma$. Below we show that this minimal generalization of the ultra-local product formula of [14] is general enough to formulate a connection between the precanonical description of quantum fields and the canonical description in the functional Schrödinger representation.

Thus, let us assume that the Schrödinger wave functional has the form

$$\boldsymbol{\Psi} = \mathsf{Tr}\left\{ \prod_{\mathbf{x}} U(y(\mathbf{x}))\Psi_\Sigma(y(\mathbf{x}),\mathbf{x},t) \right\}, \tag{2.3}$$

where $U(y(\mathbf{x}))$ is a matrix transformation which, in order to go beyond the ultra-locality assumption in (2.2), is supposed to depend on the value of the field $y(\mathbf{x})$ and its derivatives at the point $\mathbf{x}$.

The latter expression means that for any $\mathbf{x}$ the functional $\boldsymbol{\Psi}$ can be written as

$$\boldsymbol{\Psi} = \mathsf{Tr}\left\{ \Xi(\breve{\mathbf{x}},t) U(y(\mathbf{x}))\Psi_\Sigma(y(\mathbf{x}),\mathbf{x},t) \right\}, \tag{2.4}$$

where

$$\Xi(\breve{\mathbf{x}},t) := \prod_{\mathbf{x}'\neq\mathbf{x}} U(y(\mathbf{x}'))\Psi_\Sigma(y(\mathbf{x}'),\mathbf{x}',t) \tag{2.5}$$

and the continuous product here implies the cyclic permutations over all points $\mathbf{x}'$.

This observation facilitates the calculation of functional derivatives of $\boldsymbol{\Psi}$, viz.,

$$\frac{\delta\boldsymbol{\Psi}}{\delta y(\mathbf{x})} = \mathsf{Tr}\left\{ \Xi(\breve{\mathbf{x}})\frac{\delta U(\mathbf{x})}{\delta y(\mathbf{x})}\Psi_\Sigma(\mathbf{x}) + \Xi(\breve{\mathbf{x}})U(\mathbf{x})\delta(\mathbf{0})\partial_y\Psi_\Sigma(\mathbf{x}) \right\} \tag{2.6}$$

and

$$\begin{aligned} \frac{\delta^2\boldsymbol{\Psi}}{\delta y(\mathbf{x})^2} = \mathsf{Tr}\Bigg\{ & \Xi(\breve{\mathbf{x}})\frac{\delta^2 U(\mathbf{x})}{\delta y(\mathbf{x})^2}\Psi_\Sigma(\mathbf{x}) + \Xi(\breve{\mathbf{x}})U(\mathbf{x})\delta(\mathbf{0})^2\partial_{yy}\Psi_\Sigma(\mathbf{x}) \\ & +2\,\Xi(\breve{\mathbf{x}})\frac{\delta U(\mathbf{x})}{\delta y(\mathbf{x})}\delta(\mathbf{0})\partial_y\Psi_\Sigma(\mathbf{x}) \Bigg\}, \end{aligned} \tag{2.7}$$

where the shorthand notations $\Xi(\breve{\mathbf{x}})$ for $\Xi(\breve{\mathbf{x}},t)$, $U(\mathbf{x})$ for $U(y(\mathbf{x}))$, and $\Psi_\Sigma(\mathbf{x})$ for $\Psi_\Sigma(y(\mathbf{x}),\mathbf{x},t)$ are introduced. Note that the $(n-1)$-dimensional $\delta(\mathbf{0})$ appears here as a result of functional differentiation of a function with respect to itself at the same spatial point.

The time derivative of $\mathbf{\Psi}$ in (2.3) is given by the chain rule:

$$\partial_t \mathbf{\Psi} = \mathsf{Tr}\left\{ \int d\mathbf{x}\ \frac{\delta \mathbf{\Psi}}{\delta \Psi^T_\Sigma(\mathbf{x})} \partial_t \Psi_\Sigma(\mathbf{x}) \right\}, \tag{2.8}$$

where Ψ^T denotes the transpose of Ψ. Using (2.4) we obtain

$$i\partial_t \mathbf{\Psi} = \mathsf{Tr}\left\{ \int d\mathbf{x}\ \mathbf{\Xi}(\check{\mathbf{x}}) U(\mathbf{x}) \delta(\mathbf{0}) i\partial_t \Psi_\Sigma(\mathbf{x}) \right\}. \tag{2.9}$$

Hence, the time evolution of the wave functional $\mathbf{\Psi}$ is totally dictated by the time evolution of the precanonical wave function restricted to the Cauchy surface, $\Psi_\Sigma(\mathbf{x})$.

The time evolution of the restricted wave function $\Psi_\Sigma(\mathbf{x})$ is given by our Dirac-like precanonical Schrödinger equation on $\Psi(y, x)$, Eq. (1.2), restricted to the Cauchy surface Σ (cf. [14]):

$$i\partial_t \Psi_\Sigma(\mathbf{x}) = -i\alpha^i \frac{d}{dx^i} \Psi_\Sigma(\mathbf{x}) + i\alpha^i \partial_i y(\mathbf{x}) \partial_y \Psi_\Sigma(\mathbf{x}) + \frac{1}{\varkappa} \beta (\widehat{H}\Psi)_\Sigma(\mathbf{x}). \tag{2.10}$$

Here $\beta := \gamma^0$, $\alpha^i := \beta\gamma^i$, $\frac{d}{dx^i}$ denotes the total derivative along Σ in jet space $(y, y_i, y_{ij}, ...)$, such that $y_\Sigma = y(\mathbf{x})$, $y_{i\Sigma} = \partial_i y(\mathbf{x})$, $y_{ij\Sigma} = \partial_{ij} y(\mathbf{x})$, etc.:

$$\frac{d}{dx^i} := \partial_i + \partial_i y(\mathbf{x}) \partial_y + \partial_{ij} y(\mathbf{x}) \partial_{y_j} + \cdots, \tag{2.11}$$

and, in the specific case of the scalar field y (cf. [1–3, 14]),

$$(\widehat{H}\Psi)_\Sigma(\mathbf{x}) = -\frac{1}{2}\varkappa^2 \partial_{yy} \Psi_\Sigma(\mathbf{x}) + V(y(\mathbf{x})) \Psi_\Sigma(\mathbf{x}). \tag{2.12}$$

It is easy to see that if equation (2.10) is substituted in (2.9), the potential term in (2.12) yields

$$\mathsf{Tr}\left\{ \int d\mathbf{x}\ \mathbf{\Xi}(\check{\mathbf{x}}) U(\mathbf{x}) \delta(\mathbf{0}) \frac{1}{\varkappa} \beta V(y(\mathbf{x})) \Psi_\Sigma(\mathbf{x}) \right\}.$$

It will reduce to the potential term in the functional Schrödinger equation:

$$\int d\mathbf{x}\, V(y(\mathbf{x})) \mathbf{\Psi},$$

with $\mathbf{\Psi}$ given by (2.4) if, in some mathematical sense, $\varkappa\beta$ is replaced by, or goes over into $\delta(\mathbf{0})$:

$$\varkappa\beta \to \delta(\mathbf{0}). \tag{2.13}$$

We also notice that, quite remarkably, under the same condition (2.13) the term $\varkappa^2 \partial_{yy}\Psi_\Sigma$ in (2.12) reproduces the second term in the second functional derivative of $\mathbf{\Psi}$ in Eq. (2.7).

Thus, the condition (2.13) establishes a formal limiting map under which the transition from precanonical to the functional Schrödinger description is possible. As it was explained in Eq. (1.4), this map coincides with the inverse quantization map q in the limit of the vanishing elementary volume $1/\varkappa \to 0$.

Next, we note that in order to obtain a description in terms of the wave functional $\mathbf{\Psi}$ alone, without any reference to precanonical wave functions, the remaining terms in front of $\partial_y \Psi_\Sigma$ in (2.10) and (2.7) should cancel each other, at least in the limiting case (2.13). This requirement leads to the equation on $U(\mathbf{x})$:

$$U(\mathbf{x}) i\beta\gamma^i \partial_i y(\mathbf{x}) + \frac{\delta U(\mathbf{x})}{\delta y(\mathbf{x})} = 0, \tag{2.14}$$

which is not integrable. However, by taking into account the condition (2.13), in the corresponding limit we can re-write (2.14) as

$$U(\mathbf{x}) i\gamma^i \partial_i y(\mathbf{x}) \delta(\mathbf{0}) + \varkappa \frac{\delta U(\mathbf{x})}{\delta y(\mathbf{x})} = 0, \tag{2.15}$$

whose solution can be written in the form

$$U(\mathbf{x}) = e^{-iy(\mathbf{x})\gamma^i \partial_i y(\mathbf{x})/\varkappa} \tag{2.16}$$

up to a factor, which can be implicitly taken into account in a redefinition of $\Xi(\check{\mathbf{x}})$. Besides, from (2.15), under the limiting map (2.13), it also follows that

$$\frac{\delta^2 U(\mathbf{x})}{\delta y(\mathbf{x})^2} = (\nabla y(\mathbf{x}))^2 U(\mathbf{x}). \tag{2.17}$$

Hence, in the limiting case (2.13) the first term in (2.7) correctly reproduces the term $\int d\mathbf{x} (\nabla y(\mathbf{x}))^2 \mathbf{\Psi}$ in the functional derivative Schrödinger equation, Eq. (2.1).

The last term to be considered is the total derivative term in (2.10). When inserted in (2.8) with the condition (2.13) taken into account, and then integrated by parts, it yields a term proportional to

$$\mathsf{Tr}\left\{\int d\mathbf{x}\ \Xi(\check{\mathbf{x}})\frac{d}{dx^i}U(\mathbf{x})\gamma^i\Psi_\Sigma(\mathbf{x})\right\}.$$

Using the explicit form of $U(\mathbf{x})$ in (2.16) to evaluate $\frac{d}{dx^i}U(\mathbf{x})$ and recalling the representation of $\boldsymbol{\Psi}$ in (2.4), this term is transformed to the form

$$\int d\mathbf{x}\left(y(\mathbf{x})\gamma^i\gamma^j\partial_{ij}y(\mathbf{x})+\gamma^i\partial_i y(\mathbf{x})\gamma^j\partial_j y(\mathbf{x})\right)\boldsymbol{\Psi},$$

which obviously vanishes upon integration by parts. Consequently, the total derivative term in (2.10) does not contribute to the functional derivative Schrödinger equation on $\boldsymbol{\Psi}$.

Thus, we have demonstrated that the substitution of the precanonical Schrödinger equation restricted to the Cauchy surface Σ, Eqs. (2.10–2.12), into the expression (2.8) for the time derivative of the functional (2.3) constructed from precanonical wave functions allows us to reproduce, in the limiting case (2.13), all the terms which are present in the canonical functional derivative Schrödinger equation (2.1) and to cancel those which are missing there. The procedure fixes both the condition (2.13), under which the transition from the precanonical description to the canonical description in terms of a Schrödinger wave functional is possible, and the transformation U in (2.3), which allows us to reproduce the term $\frac{1}{2}(\nabla y(\mathbf{x}))^2$ in the canonical Hamiltonian in (2.1). This is the latter term which is responsible for the non-ultralocal behaviour of quantum fields (see e.g. [46]), whose natural derivation was problematic in our previous paper [14].

Moreover, the above consideration also yields an explicit limiting expression of the Schrödinger wave functional as the continuous product over all spatial points of U-transformed precanonical wave functions restricted to the Cauchy surface Σ. Namely, using (2.16) in (2.3), we obtain:

$$\boldsymbol{\Psi}([y(\mathbf{x})],t)=\mathsf{Tr}\left\{\prod_{\mathbf{x}}e^{-iy(\mathbf{x})\gamma^i\partial_i y(\mathbf{x})/\varkappa}\Psi_\Sigma(y(\mathbf{x}),\mathbf{x},t)\right\}. \tag{2.18}$$

This formula is valid in the limit of an infinitesimal "elementary volume" $1/\varkappa$ and under the formal limiting map $\varkappa\beta \xrightarrow{q^{-1}} \delta(\mathbf{0})$ in the expression under the continuous product sign. In a subsequent paper [48] we have shown that in this limit the formula in (2.18) can be transformed to the multidimensional product integral invented by Volterra [47] and used to construct the well known vacuum state wave functional of free scalar field theory from the ground state solutions of the precanonical Schrödinger equation.

3. Conclusions

We have shown how the canonical functional derivative Schrödinger equation (2.1) can be derived from the partial differential covariant precanonical Schrödinger equation (1.2) restricted to the Cauchy surface in the covariant configuration space of field theory. We also obtained an explicit limiting expression of the Schrödinger wave functional of the canonical quantization approach in terms of the precanonical wave functions defined on the finite dimensional space of field and space-time variables.

Our result suggests that quantum field theory originating from canonical quantization can be viewed as a limiting case $1/\varkappa \to 0$ of quantum field theory resulting from the precanonical quantization approach.

Note that the result of our earlier discussion in [14] comes amazingly close to the formula (2.18). However, there are important differences:

First, we found that the transformation which allows us to fully take into account the deviations from ultra-locality acts directly on precanonical wave functions rather than on the ultra-local functional constructed from them. This solves the problem with the non-integrability of the functional derivative equation for the corresponding transformation functional, which we tried to circumvent in [14, 27].

Second, we found that the Schrödinger functional description is possible only under the limiting condition (2.13), which essentially tells us that the description of quantum fields in terms of the Schrödinger wave functional corresponds to the limit of vanishing "minimal volume" $1/\varkappa$, or $\varkappa \to \delta(\mathbf{0})$, as it was already noticed in [14]. It should be noted that the condition (2.13) complies both with the relativistic transformation laws of $\delta(\mathbf{0})$ and the rules of precanonical quantization itself (cf. Eq.(1.3)). Namely, (2.13) unifies two requirements to be fulfilled when a transition from precanonical to the functional Schrödinger description is being made: the absolute value of $\varkappa$ should tend to $\delta(\mathbf{0})$ and the inverse of the "quantization map" in (1.3) should be applied so that the Clifford-valued operator of the "minimal volume" $\beta/\varkappa$ is mapped (at $\varkappa \to \delta(\mathbf{0})$) to the differential form $d\mathbf{x}$ representing the classical infinitesimal volume element.

Third, the identification of the condition when the transformation from precanonical to the functional Schrödinger representation is possible as the inverse quantization map in the limit $1/\varkappa \to 0$, Eq. (1.4), has made superficial the use of the projector $\frac{1}{2}(1+\beta)$ introduced in the expression of the wave functional in [14].

Mathematically speaking, by starting from the assumption (2.3) and showing how it allows us to derive the canonical functional derivative Schrödinger equation from the precanonical partial derivative Schrödinger equation, and to fix the form of the transformation $U(y(\mathbf{x}))$, we have proven that the Ansatz (2.3) is the sufficient condition, which establishes a connection of the Schrödinger wave functionals with precanonical wave functions. In a forthcoming paper [48] we have shown that this condition is also necessary.

Note that, although our result is obtained using the particular case of the scalar field theory, in the subsequent papers it will be demonstrated that it can be extended also to other fields, such as Yang-Mills and spinor fields.

One should underline that the nature of the ultra-violet parameter $\varkappa$ appearing in precanonical quantization is different from the "minimal length" scale introduced by hand in nonlocal field theories or field theories over discrete/microstructured/non-commutative space-times. This parameter does not arbitrarily modify the relativistic space-time at small scales, and it was found to disappear from the final results for free field theories. We still have to investigate in detail what the role of $\varkappa$ is in interacting field theories, renormalizable and non-renormalizable, and what is its possible role in the common renormalization techniques.

In conclusion, let us recall that the main motivation of the precanonical quantization approach has been its potential to be a better synthesis of relativity and quantum theory in the context of field theory, that could provide a better framework for quantization of gravity. We hope that the results of this paper can help to understand the physical content of precanonical quantization of gravity [28–31, 49] and, in particular, clarify its relation with the the canonical quantum gravity leading to the Wheeler-DeWitt equation, the ill defined quantum gravity analogue of the functional Schrödinger equation, which could emerge as a singular limit of infinite $\varkappa$ from a better defined precanonical quantization of gravity.

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