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A new formulation of general relativity

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Abstract

In Sections 1–5 of this paper an axiomatic formulation of general theory of rativity (GR) is given and studied. Here use is made of the concept of pre-radar charts. These charts have "infinitesimally" the same properties as the true radar charts used in space–time theory. Their existence in GR has far-reaching consequences which are discussed throughout the paper. For the sake of simplicity and convenience I consider only such material systems the state of which is defined by a velocity field, a mass density and a temperature field. But the main results hold also for more complex systems. It follows from the axiomatics that the pre-radar charts define an atlas for the space–time manifold and that, in addition, they generate the metric, the velocity field and the displacement of the matter. Therefore, they are called generating functions. They act like "potentials". In Section 6 it is shown that the existence of pre-radar charts allows to simplify the original axiomatics drasticly. But the two versions of GR are equivalent. In Sections 7 and 8 the so-called inverse problem is treated. This means the question whether it is possible to define pre-radar charts, i.e., generating functions in arbitrary space–times. This problem is subtle. A local general constructive solution of it is presented. Sufficient conditions for the existence of global solutions are given. The aim of the last sections is formulating GR as a scalar field theory. The

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basic structural elements of it are a generating function, a generalized density and a generalized temperature. One of the axioms of this theory is a generalized Einstein equation that determines the generating function directly. It is shown that basic concepts like orientation, time orientation and isometry are expressible in terms of generating functions. At the end of the paper six problems are formulated which are still unsolved and can act as a stimulant for further research.

1 Introduction

1.1 Some general features of GR

The subject of this treatise is the general theory of relativity (GR) in its classical form. In a strict sense, GR is not a single physical theory, rather it is a class of theories that share common features. From this point of view Schwarzschild space–time, Robertson–Walker space–times, etc., are counted as separate relativistic theories.

The common features of all these theories are "principles" they obey. More properly, these principles should be called "axioms" because they have the same status as the axioms in mathematical theories. An inspection of relativistic theories reveals that their axioms can be grouped into five classes as follows.

GK. Axioms concerning geometry and kinematics.
EM. Axioms concerning matter and its motion, e.g., equations of motion and constitutive equations.
ED. Axioms concerning electromagnetism, e.g., Maxwells equations and constitutive equations.
EE. Einstein equation and constitutive equations.
AC. Additional conditions, e.g., initial conditions.

Clearly, in vacuum theories the axioms EM and ED are empty. The above classification of axioms possibly includes redundancies.

1.2 Different forms of physical theories

Among the many forms of presenting physical theories there are two extreme forms which are of special interest in our context.

(i) The first form is characterized by the property that the fundamental terms as in our case a set of events M, an atlas $\mathcal{A}$, a metric g, a velocity field, etc., are implicitly determined solely by axioms. Examples of such formulations of theories are well known in many branches of physics.

(ii) The second form of a physical theory can be characterized as "model theory", also widely known under the label "solution." In this case the letters M, $\mathcal{A}$, g, etc., are replaced by explicit terms of mathematical analysis, and the axioms of case (i) occur as theorems, i.e., the axioms are satisfied by these explicit terms. Also this type of a theory is well known in physics. Clearly, mixed forms are on the market, too. In what follows, I will consider relativistic theories according to the first form. Formulating their axioms, I will make use of some results of the space–time theory (STT) which is developed in [1–3] and which is reviewed in [4]. A detailed account of this STT can be found in [5]. More specific, I will take some features of radar coordinates in order to define a weaker form of them which I call pre-radar coordinates. Using these coordinates in the context of GR is the essential new aspect of this treatise. They can be comprised into one function Ψ depending on two events p, q which is a generating function for the atlas $\mathcal{A}$, the metric g and the velocity v, and which, in addition, determines the integral curves of v.

1.3 Radar charts

A few remarks may illustrate the notion of radar coordinates as it is used in [1–5]. Let A be an observer, b an event and t_1, t_2 times measured by the clock of A. Finally, let $e = (e^1, e^2, e^3)$ be the direction of a light signal which leaves A at instant t_1 and comes back to A at t_2 after being reflected at b. Then, if

$$t := \tfrac{1}{2}(t_2 + t_1), \quad r := \tfrac{1}{2}(t_2 - t_1),$$

the radar coordinates x of b are given by $x = (re^1, re^2, re^3, t)$. Here and in the sequel all quantities are dimensionless, and $c = 1$.

If A describes this situation with the help of its own radar coordinates he or she will get in two dimensions of $\mathbb{R}^4$ the picture shown in Figure 1. Here the outgoing light signal is a straight line by definition, whereas the incoming signal is not necessarily straight, but it is a curve that is a subset of a Minkowskian backward light cone. Intuitively, a pre-radar chart of an observer A has the properties of a radar chart only in an "infinitesimal neighborhood" of the worldline of A. In each case, radar coordinates are also pre-radar coordinates as introduced in Definition 3.2 of Section 3.1. It is to be emphasized that the term radar coordinate is not uniformly used in literature. Two examples may illustrate it. Coleman and Korté [7] define

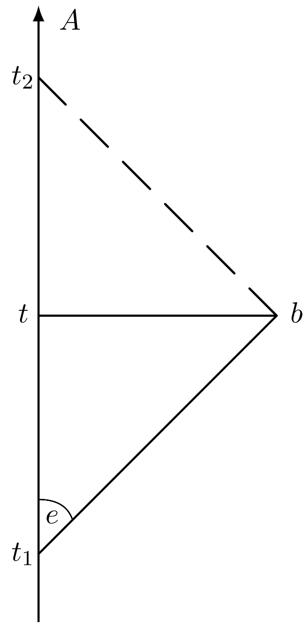

Figure 1.

radar charts that are similar to those introduced above. The difference is that the measurement of the direction e is performed on the incoming part of the radar signal. The radar coordinates used in EPS axiomatics [6] are more different from ours. Here the radar coordinates (u, v, u', v') of an event a are defined by two observers A and A' which send out radar signals starting at the times u, u' and arriving at v, v' after being reflected at a.

1.4 Types of continuum theories

Though the class of theories I want to consider are continuum theories the notions observer, particle or real point are employed. This does not contradict the continuum point of view. Rather it reflects only the fact that we have two possibilities to describe continuum systems, namely in the way of Lagrange as systems of particles or in the way of Euler by fields. Later on I will use a mixture of both these descriptions.

1.5 Matter and test particles

Throughout this paper I will use the following strategy: each material point, i.e., each point which contributes to gravitation, is a part of the system of pre-radar observers. However it is possible that there are gravitationally irrelevant test particles which are pre-radar observers.

1.6 Remarks on literature

As indicated in Section 1.2 I intend to give a new axiomatic formulation of general relativity. There is already a considerable amount of work done in axiomatics of relativity and space-time theory which cannot be reviewed here. Rather I refer to the paper of Schelb [5] where the reader will find an almost complete list of relevant papers. One of the most recent treatises in this field is that of Hehl and Obukhov [8] where a Lorentz metric on a manifold is constructed using electrodynamics without metric. But in what follows I do not adopt this general supposition, rather I will take into account only some special features of electrodynamics by using light signals as a basic concept. (Cf. also [9] and the literature quoted there.)

2 Description of the systems to be considered

2.1 Some basic suppositions

In what follows, I will not treat the most general continuum systems, rather I consider only material systems that can be described by one velocity field v, one mass density η and one empirical temperature ϑ. This means that mixtures of fluids, especially plasmas are not taken into account. (Mixtures of fluids are treated e.g., in [10, p. 13].) The above restriction is only a matter of convenience because the main results of this treatise are also valid for more complex systems.

Though each system considered is supposed not to have electromagnetic interaction it is assumed that all observers, i.e., particles of the material system and test particles can exchange light signals. It is assumed that these signals are irrelevant with respect to any kind of interaction. The only thing they can transport is information. They have the same status as test particles. The reason to take into account light signals is that we want to introduce pre-radar coordinates. This can be effected most simply if we have light signals at our disposal.

2.2 The primitive notions of GR

In order to formulate a physical theory in the sense of (i) Section 1.2 which is adequate to treat the systems described in Section 2.1, the fundamental mathematical terms have to be specified with the help of which the whole theory can be formulated and which are implicitly determined by the axioms of the theory. There are two kinds of fundamental terms, the so-called *base sets* and the so-called *structural terms*. The first ones contain the signs for the objects to be treated, whereas the latter, the structural terms define the basic properties of these objects. Expressed mathematically, the structural terms are relations that are elements of sets constructed from the base sets solely with the help of the operations "power set" and "Cartesian product."

Let us first specify the base sets. The most fundamental term in any relativistic theory is the set of signs for events. This is indispensable! In addition, we want to speak about particles some of which bear gravitationally active masses, and we take into account (light) signals. Finally, we need the real numbers because we have to express the values of some quantities by numbers.

Let us now come to the *structural terms*. They are the metric, the velocity field, the mass density and the (empirical) temperature field. As usual in GR the set of events should be a manifold. This means that there are coordinates defined on certain sets of events. The structural term that is introduced in the present context is a relation which assigns four coordinates to each event in some neighborhood of an observer.

It is convenient to assign a common name to the theories treated in this paper.

Notation 2.1. *The physical theories that are determined by the above-mentioned* (*and in Section 2.3 precisely described*) *base sets and structural terms which in turn are ruled by the axioms of Sections 3 and 4 are denoted* Φ_R. *Since* Φ_R *represents a class of physical theories* (*in a strict sense*) *it is called a frame theory.*

2.3 Basic mathematical terms

Summing up the considerations of Section 2.2 we arrive at the following result: The *base sets* of Φ_R are $M, P, S, \mathbb{R}$:

M is the set of signs $a, b, \ldots$, etc., for events;
P is the set of signs (i.e., indices) $A, B, \ldots$, etc., for particles;
S is the set of signs (i.e., indices) $s, s', \ldots$, etc., for signals;
$\mathbb{R}$ is the set of the real numbers as usual.

The *structural terms* of Φ_R are $\hat{\Psi}, g, v, \eta, \vartheta$:

$\hat{\Psi}$ determines the pre-radar coordinates, i.e., $(A, b, x) \in \hat{\Psi}$ means that observer A coordinatizes event b by x

g is the metric,
v is the velocity field,
η is the mass density and
ϑ is the temperature field.

3 Geometric and kinematic axioms

Following the notation of Section 1.1 the axioms of this section are denoted GK. Throughout Sections 3–5 the same natural numbers k resp. $r = k - 1$ are meant in phrases like "$\ldots \in C^k, k \geq 3$" or "$\ldots \in C^r, r \geq 2$."

3.1 Pre-radar charts

Intuitively, a pre-radar chart is a chart that has some (or perhaps all) of the properties of a true radar chart. Therefore, the following axioms can be motivated by pointing to the fact that true radar charts have a certain property. In Section 1.3 it was outlined how a radar observer coordinatizes a neighborhood of its worldline: he or she needs clocks and devices for measuring directions. In what follows, it is assumed that each observer uses one and only one clock and one and only one directional measuring instrument. Therefore, the radar charts of observers thus equipped are unique. Hence the following axiom is self-evident:

GK 1.1. the structural term $\hat{\Psi}$ is a function: $\hat{\Psi} : \bigcup_{A\in P}\{A\} \times V_A \to \mathbb{R}^4$, where $V_A \subset M$ and $V_A \neq \varnothing$.

It is useful to introduce some notation.

Definition 3.1. For short we write $\psi_A := \hat{\Psi}(A, \cdot)$ and $\mathcal{O}_A :=$ ran ψ_A; by definition dom $\psi_A = V_A$. Then let $\mathcal{A} = \{(V_A, \psi_A) : A \in P\}$.

The next axiom expresses that $\hat{\Psi}$ determines a manifold structure on M.

GK 1.2. $\mathcal{A}$ is a C^k-atlas, $k \geq 3$ such that $(M, \mathcal{A})$ is a connected Hausdorff manifold.

A motivation for further axioms comes from the fact that each radar observer A coordinatizes himself or herself by $(0, 0, 0, t)$, the parameter t being the time A measures. In addition, the fourth component of each quadruple of radar coordinates measured by A is a time at A.

GK 1.3. For each $A \in P$ there are two real numbers u_1, u_2 with $-\infty \leq u_1 < u_2 \leq \infty$ such that for each $\tau \in]u_1, u_2[=: J_A$ the relation $(0,0,0,\tau) \in \mathcal{O}_A$ holds; moreover, if $y_1 < u_1$ or $u_2 < y_2$, then $\{(0,0,0,\rho) : \rho \in]y_1, y_2[\} \not\subset \mathcal{O}_A$, i.e., J_A is maximal.

Finally, some additional notation is introduced by

Definition 3.2. (1) The term $\mathcal{D}$ is the differential structure containing all charts which are C^k-compatible, $k \geq 3$ with $\mathcal{A}$.
(2) The coordinates determined by the charts (V_A, ψ_A) of $\mathcal{A}$ will be called for short A-coordinates, etc. whereas the others are denoted by their coordinate functions χ, etc.
(3) Within the theory Φ_R the charts of $\mathcal{A}$ are called pre-radar charts.

3.2 Worldlines of particles

In our context there are two possibilities to define the worldline of an observer. First, the worldline of A is the set of events that occur at A. Second, the worldline of A is the set of events that A coordinatizes by $(0,0,0,t)$. This leads to the following

Definition 3.3. For each $A \in P$ the (surjective) function $\gamma_A : J_A \to W_A \subset M$ is defined by $\gamma_A(t) = \psi_A^{-1}(0,0,0,t)$, $t \in J_A$. The set $W_A :=$ ran γ_A is called the worldline of A. As usual $\dot{\gamma}_A$ denotes the velocity of A.

Definition 3.4. For each $A \in P$ the clock $\mathcal{U}_A$ of A is defined by $\mathcal{U}_A(a) = \psi_A^4(a), a \in W_A$.

Remark 3.5. (1) From Definition 3.3 it follows that $\psi_A \circ \gamma_A(t) = (0,0,0,t)$. Therefore, the function γ_A is of class $C^k, k \geq 3$ and bijective.
(2) From Definition 3.4 one concludes that

$$\mathcal{U}_A(\gamma_A(t)) = \psi_A^4 \circ \psi_A^{-1}(0,0,0,t) = t = \gamma_A^{-1}(\gamma_A(t)).$$

Hence $\mathcal{U}_A(a) = \gamma_A^{-1}(a)$ for all $a \in W_A$.

Since we want to describe a continuum, the set of particles should be a continuum. In addition, we want to describe the system by a smooth velocity field. Therefore, the worldlines of different particles cannot have common events. Hence the next two axioms are self-evident.

GK 2.1. The cardinality of P is that of a continuum and $\bigcup_{A \in P} W_A = M$.

GK 2.2. For all $A, B \in P$: if $A \neq B$ then $W_A \cap W_B = \varnothing$.

In other words, this axiom expresses that the set P of particles represents a congruence.

Remark 3.6. It follows directly from axioms GK 2.1 and 2.2 that there is a function $F : M \to P$ which is surjective and which is given by $F(a) = A$ for all $a \in W_A := \text{ran}\ \gamma_A$.

With the help of F we are now able to define a function Ψ which later on turns out to be a generating function for the metric g and the velocity v.

Definition 3.7. The function $\Psi : \bigcup_{q \in M} V_{F(q)} \times \{q\} \to \mathbb{R}^4$ is defined by $\Psi(p, q) = \hat{\Psi}(F(q), p) = \psi_A(p)$ for $A = F(q)$. The values of Ψ and $\hat{\Psi}$ are mostly written as row vectors: $\Psi = (\Psi^1, \ldots, \Psi^4)$, etc. But occasionally it is more convenient to write them as column vectors: $\Psi = (\Psi^1, \ldots, \Psi^4)^{\mathrm{T}}$, etc.

At this point, Ψ is nothing but another form of $\hat{\Psi}$ which has the advantage that it allows to express the intuitively reasonable property of pre-radar coordinates that "neighboring" observers attribute "neighboring" coordinates to the same event. This means that Ψ is "smooth" with respect to all of its arguments. Therefore, the following axiom is natural.

GK 2.3. Ψ is of class C^k with $k \geq 3$.

3.3 Axioms for the metric

Since the metric g is defined in the usual way it suffices to write down the axioms governing g.

GK 3.1. The structural term g is a function $g : \bigcup_{a \in M} \{a\} \times (T_a M \times T_a M) \to \mathbb{R}$, where $T_a M$ is the tangent space at $a \in M$.

It is convenient to introduce the

Notation 3.8. $g(a, w, w') = g(a)(w, w')$ *and* $g(a) = g_a = g(a)(\cdot, \cdot)$.

GK 3.2. $g = g(\cdot)$ is a $\binom{0}{2}$-tensor field of class $C^r, r \geq 2$.

GK 3.3. For each $a \in M : g_a$ is symmetric, non-degenerate and of signature 2.

With the help of g the clocks used by the observers now can be specified: they are to show proper time. This is the content of the next axiom that joins particles and metric.

GK 3.4. For each $A \in P$ and each $t \in J_A : g(\gamma_A(t))(\dot{\gamma}_A(t), \dot{\gamma}_A(t)) = -1$.

3.4 Properties of signals

In this subsection the relation between particles and signals is studied. The motivation for the following axioms comes from space–time theory (cf. [1–5]) where proper radar charts are treated in great detail. It is not possible to repeat all arguments of these papers here. One result may suffice. A true radar observer A describes outgoing light signals within his or her coordinate system by straight lines leaving the worldline of A by an angle of 45°. In what follows, one should also have in mind that S is merely a set of indices for signals.

GK 4.1. For each $s \in S$ there is a function $\sigma_s : K_s \to W_s^* \subset M, K_s \subset \mathbb{R}$ with the following properties:

σ_s is of class $C^k, k \geq 3$ and a null geodesic; moreover, there is an $A \in P$ such that

$$\begin{aligned} &K_s = [t_{s0}, t_{s1}] \subset J_A, \\ &\psi_A^4(\sigma_s(t)) = t \quad \text{for all} \quad t \in K_s, \end{aligned}$$

there is an $a \in W_A$ such that $\sigma_s(t_{s0}) = a$,

there is an $e = (e^1, e^2, e^3) \in S^2$ (the 2-sphere) such that

$$\frac{d}{dt}\psi_A(\sigma_s(t_{s0})) = (e^1, e^2, e^3, 1).$$

To a certain extent also the converse of this axiom is needed.

GK 4.2. For each $A \in P$, for each $a \in W_A$ and for each $(e^1, e^2, e^3) \in S^2$ there is an $s \in S$ such that the function σ_s determined by axiom GK 4.1 has the following properties:

$$\begin{aligned} \sigma_s(t_{s0}) &= a, \\ \frac{d}{dt}\psi_A(\sigma_s(t_{s0})) &= (e^1, e^2, e^3, 1). \end{aligned}$$

3.5 Velocity

According to Definition 3.3 the velocity of particle A is $\dot{\gamma}_A$. Therefore, the field v is determined by $\dot{\gamma}_A$. This is the content of axiom GK 5.

GK 5. The structural term v is a function $v : M \to TM$ which is defined for each $b \in M$ by $v(b) = \dot{\gamma}_{F(b)}(\gamma^{-1}_{F(b)}(b))$ and which is of class $C^r, r \geq 2$.

4 Further axioms

4.1 Equations of motion

Up to now only the geometric and the kinematic part of the theory Φ_R was treated, and this part can be characterized, roughly speaking, by the phrase: everything is smooth. For the formulation of the further axioms we have to take matter into account. The basic quantities describing matter can again be smooth, but there are many interesting systems that show material discontinuities. It is not possible to treat these different classes of systems by the same kind of axioms.

Therefore I restrict the further studies to the most simple class of systems which are those with smooth η and ϑ. In this case the following axiom is obvious.

EM1. The structural terms η and ϑ are functions $\eta : M \to \mathbb{R}, \vartheta : M \to \mathbb{R}$ which are of class $C^r, r \geq 2$.

In a next step the constitutive equations have to be specified. In a theory of type Φ_R as treated in this paper there is only one constitutive element, the energy–momentum tensor T, and the constitutive equation relates T to the structural terms g, v, η, ϑ. This is the content of the next axiom where the bundle of $\binom{n}{m}$-tensors is denoted T^n_m.

EM2. T is a function $T : M \to T^2_0 M$ which is of class $C^r, r \geq 2$ and which is defined by the functional $\mathcal{T}$ of g, v, η, ϑ for all $q \in M$ by $T(q) = \mathcal{T}(g, v, \eta, \vartheta)(q) \in T^2_{0q} M$. If $\eta(q) = 0$ for $q \in N \subset M$ then $\mathcal{T}(g, v, \eta, \vartheta)(q) = 0$.

This axiom is the point why Φ_R is a frame theory, for $\mathcal{T}$ is not explicitly specified. Each such specification of $\mathcal{T}$ defines a subclass of the systems governed by the (frame) theory Φ_R. The usual forms of the energy–momentum tensor for systems of dust or for general Euler fluids are examples of classifying systems with the help of $\mathcal{T}$.

The next two axioms are the equation of continuity and the balance of energy and momentum. Although the latter is a consequence of Einsteins equation it is written down here.

EM3. Throughout $M\colon \operatorname{div}(\eta v) = 0$.

EM4. Throughout $M\colon \operatorname{div}(T) = 0$.

4.2 Einstein equation

In our context there is no need to comment on Einstein's equation or to motivate it. It suffices to write it down. So let, as usual, R be the Ricci tensor, $\bar{R}$ the Ricci scalar, Λ_0 the (unspecified) cosmological constant and finally κ_0 Einsteins gravitational constant. Moreover, $T^\flat$ denotes the covariant energy–momentum tensor.

EE. Throughout $M\colon R - \frac{1}{2}g\bar{R} + \Lambda_0 g = \kappa_0 T^\flat$.

4.3 Additional conditions

The general term additional conditions (AC) comprises all those axioms that have to be imposed in order to get physically relevant and uniquely determined classes of models. In this context, by a model (or a "solution") the following is to be understood:

Notation 4.1. *Let M', P', S' and $\Psi', g', v', \eta', \vartheta'$ be terms defined within mathematical analysis or, more precisely, within the theory of sets such that the axioms GK, EM, EE and AC for a specified functional $\mathcal{T}$ and a specified cosmological constant Λ_0 are satisfied. Then we say that these terms define an analytical or a set theoretical model of the frame theory Φ_R.*

A model is a theory of the form described in (ii) of Section 1.2. In this sense Robertson–Walker space–times, the Schwarzschild space–time, etc., are models of Φ_R. But this is not proved here.

There is a great variety of AC. Three examples may illustrate the role AC play:

(i) Initial conditions, e.g., for solving a Cauchy problem.
(ii) Boundary conditions, e.g., for space–times which are to be asymptotically flat in a certain sense.

(iii) Symmetry conditions by which e.g., a general ansatz can be restricted to a more special form.

A more detailed discussion of this subject is outside the scope of this paper.

5 Some consequences of the axioms

5.1 Metric and velocity in B-coordinates

First of all let us fix some

Notation 5.1. *The components of v and g with respect to B-coordinates (cf. Definition 3.2) for each $B \in P$ are denoted*

$$v_B^\alpha(y) \quad \text{and} \quad g_{\alpha\beta}^B(y) \quad \text{where } y \in \psi_B[V_B].$$

For the general coordinates χ with $(W, \chi) \in \mathcal{D}$ we write

$$v_\chi^\alpha(x) \quad \text{and} \quad g_{\alpha\beta}^\chi(x) \quad \text{where } x \in \chi[W].$$

Then the following simple lemma holds:

Lemma 5.2. *For each $B \in P$ and for all $y \in \psi_B[W_B]$ it holds that*

$$v_B^\alpha(y) = \delta_4^\alpha.$$

Proof. By axiom GK5 and by making use of Definition 3.4 for all $b \in W_B$ one finds that $v(b) = \dot{\gamma_B}(t)$ where $t = \mathcal{U}_B(b)$. Let $\dot{\gamma_B}(t) = w_B^\alpha(t)\partial_{\psi_B^\alpha}$. Then

$$w_B^\alpha(t) = \frac{d}{dt}(\psi_B^\alpha \circ \gamma_B)(t).$$

From Remark 3.5 it follows that $w_B^\alpha(t) = \delta_4^\alpha$ for all $t = \mathcal{U}_B(b)$ and $b \in W_B$, i.e., for all $t \in J_B$. Let $y = \psi_B(b)$ and $t = \mathcal{U}_B(b)$. Then for each $b \in W_B$ we have $v_B^\alpha(y) = w_B^\alpha(t)$, so that the proposition is proved. $\square$

A similar result holds for the metric g:

Lemma 5.3. *For each $B \in P$ and for all $y \in \psi_B[W_B]$ it holds that*

$$g^B_{\alpha\beta}(y) = \eta_{\alpha\beta},$$

where $((\eta_{\alpha\beta})) := \mathrm{diag}(1,1,1,-1)$ is the Minkowski matrix.

Proof. In what follows, for the sake of simplicity the argument (y) is omitted.

1. From axioms GK 3.4 and GK 5 it follows that $g^B_{\alpha\beta} v^\alpha_B v^\beta_B = -1$.
   By Lemma 5.2 we have $v^\alpha_B = \delta^\alpha_4$ so that $g^B_{\alpha\beta} v^\alpha_B v^\beta_B = g^B_{44} = \eta_{44}$.
2. From axioms GK 4.1 and 4.2 one concludes that for all (e^1, e^2, e^3) the relation
   $$\sum_{jk}^{3} g^B_{jk} e^j e^k + 2 \sum_{j}^{3} g_{j4} e^j - 1 = 0$$
   holds. Now let $e^1 = \pm 1$, $e^2 = 0$, $e^3 = 0$. Then $g^B_{11} \pm 2 g^B_{14} = 1$ Hence $g^B_{11} = 1 = \eta_{11}, g^B_{14} = g^B_{41} = 0 = \eta_{14}$.

Similarly we find that $g^B_{22} = 1 = \eta_{22}$, $g^B_{24} = g^B_{42} = \eta_{24}$, $g^B_{33} = 1 = \eta_{33}$ and $g^B_{34} = g^B_{43} = \eta_{34}$.

Finally let $e^1 = \frac{1}{\sqrt{2}}, e^2 = \frac{1}{\sqrt{2}}, e^3 = 0$. Then $g^B_{12} = g^B_{21} = 0 = \eta_{12} = \eta_{21}$. By analogous arguments one finds that $g^B_{jk} = 0 = \eta_{jk}, k = 1,2,3, j \neq k$. □

Remark 5.4. (1) Lemma 5.3 is remarkable in so far as it states that for each observer B the metric in B–coordinates, i.e., $g^B_{\alpha\beta}$ is Minkowskian not only for one point but for each point of the whole worldline W_B. Therefore, by axiom GK 2.1 the metric g as well as the velocity v is completely determined on M once for all $B \in P$ the worldlines W_B are known. These in turn are determined by the coordinate function $\hat{\Psi}$ resp. Ψ.

(2) As already mentioned in Section 1.3, the proper radar coordinates are also pre-radar coordinates, i.e., B-coordinates for some observer B. Also the Fermi coordinates introduced by Synge in [13, p. 84] can be used to define pre-radar coordinates. But there are pre-radar charts which are neither radar charts nor Fermi charts.

(3) It is stated without proof that a pre-radar observer B is freely falling exactly if
$$\frac{\partial}{\partial y^j} g^B_{44} = 0$$
for all $y \in \psi_B[W_B]$ and $j = 1,2,3$.

5.2 Coordinate representations for metric and velocity

In this section we are looking for explicit expressions for $g^\chi_{\alpha\beta}$ and v^α_χ where χ are arbitrary coordinates. First of all we have to fix some notation.

Definition 5.5. Let $(W,\chi) \in \mathcal{D}$ and $(V_B,\psi_B) \in \mathcal{A}$. Then:

1. $\phi_{\chi B} := \psi_B \circ \chi^{-1}$, $\phi_{B\chi} := \phi^{-1}_{\chi B}$, $\phi_{AB} := \psi_B \circ \psi_A^{-1}$.
2. $G_\chi := F(\chi^{-1})$, where F is defined in Remark 3.6.
3. $\phi_\chi := \Psi(\chi^{-1},\chi^{-1})$ (cf. Definition 3.7).
4. $\Lambda_\chi = ((\Lambda_\chi{}^\alpha_\beta))$ with $\Lambda_\chi{}^\alpha_\beta(x) := \left.\frac{\partial\phi^\alpha_\chi(x,z)}{\partial x^\beta}\right|_{z=x}$ where α denotes the rows and β the columns.

Remark 5.6. Since $\Psi(p,q) = \psi_{F(q)}(p)$ it follows that

$$\phi_\chi(x,z) = \psi_{G_\chi(z)}(\chi^{-1}(x)) = \phi_{\chi G_\chi(z)}(x).$$

Next the representation for the metric is deduced.

Proposition 5.7. *If $(W,\chi) \in \mathcal{D}$ then for all $x \in$ ran χ we have*

$$g^\chi_{\alpha\beta}(x) = \Lambda_\chi{}^\kappa_\alpha(x)\Lambda_\chi{}^\lambda_\beta(x)\eta_{\kappa\lambda}. \tag{5.1}$$

Proof. Let $x = \chi(p)$ and $y = \psi_B(p)$. Then $y = \phi_{\chi B}(x)$. The matrix elements $g^\chi_{\alpha\beta}(x)$ and $g^B_{\kappa\lambda}(y)$ are related by

$$g^\chi_{\alpha\beta}(x) = \frac{\partial\phi^\kappa_{\chi B}(x)}{\partial x^\alpha}\frac{\partial\phi^\lambda_{\chi B}(x)}{\partial x^\beta} \quad g^B_{\kappa\lambda}(y).$$

Therefore, if $B = G_\chi(z)$ it follows from Remark 5.6 that

$$g^\chi_{\alpha\beta}(x) = \frac{\partial\phi^\kappa_\chi(x,z)}{\partial x^\alpha}\frac{\partial\phi^\lambda_\chi(x,z)}{\partial x^\beta} g^B_{\kappa\lambda}(y).$$

Now let $z = x$. Then $B = G_\chi(x) = F(\chi^{-1}(x)) = F(p)$, so that $p \in W_B$ and $y \in \psi_B[W_B]$. Hence by Lemma 5.3 we have $g^B_{\kappa\lambda}(y) = \eta_{\kappa\lambda}$ for all $y \in \psi_B[W_B]$, so that by use of Definition 5.5 the proposition holds. □

Also the components of velocity v can be expressed by Λ_χ in the following way.

Proposition 5.8. *If* $(W, \chi) \in \mathcal{D}$ *then for all* $x \in$ ran χ *we get*

$$v_\chi^\alpha(x) = \Lambda_\chi^{-1}\ {}^\alpha{}_4(x). \tag{5.2}$$

Proof. Let $x = \chi(p), \quad y = \psi_B(p)$ and $x = \phi_{B\chi}(y)$. Then

$$v_\chi^\alpha(x) = \frac{\partial \phi_{B\chi}^\alpha(y)}{\partial y^\beta} \quad v_B^\beta(y).$$

By Definition 5.5 we have $\Phi_{B\chi} = \Phi_{\chi B}^{-1}$. Therefore

$$v_\chi^\alpha(x) = \left[\left(\frac{\partial \phi_{\chi B}(x)}{\partial x}\right)^{-1}\right]_\beta^\alpha v_B^\beta(y).$$

If $B = G_\chi(z)$ then from Remark 5.6 it follows that

$$v_\chi^\alpha(x) = \left[\left(\frac{\partial \phi_\chi(x, z)}{\partial x}\right)^{-1}\right]_\beta^\alpha v_B^\beta(y).$$

Finally let $x = z$. Then $B = G_\chi(x) = F(p)$, hence $p \in W_B$ and $y \in \psi_B[W_B]$. Therefore, from Lemma 5.2 we have $v_B^\alpha(y) = \delta_4^\alpha$. Using Definition 5.5 the proposition is proved. □

Corollary 5.9. *For the covariant components of the velocity the relation*

$$v_\alpha^\chi(x) = -\Lambda_{\chi\alpha}{}^4(x). \tag{5.3}$$

holds. For, using the Propositions 5.7 and 5.8, we have

$$v_\alpha^\chi = \Lambda_\chi^{-1}\ {}^\beta{}_4 \Lambda_{\chi\alpha}{}^\kappa\, \Lambda_{\chi\beta}{}^\lambda\, \eta_{\kappa\lambda} = -\Lambda_{\chi\alpha}{}^4$$

Similarly, the contravariant components of the metric are

$$g_\chi^{\alpha\beta} = \Lambda_\chi^{-1}\ {}^\alpha{}_\kappa \Lambda_\chi^{-1}\ {}^\beta{}_\lambda \eta^{\kappa\lambda}. \tag{5.4}$$

This follows directly from $((g_\chi^{\alpha\beta})) = ((g_{\alpha\beta}^\chi))^{-1}$ *and* $\eta^{\kappa\lambda} = \eta_{\kappa\lambda}$.

5.3 Global representations

In this section the results of Section 5.2 will be shaped in a form which is independent of coordinates. These considerations show again the role the function Ψ plays. We start with some notation.

Definition 5.10. Let Ψ be the function introduced in Definition 3.7. Then

$$e_{q\alpha}(p) := \partial_{\Psi^\alpha}(\cdot, q)\big|_p\,, \tag{5.5}$$

$$\Theta_q^\beta(p) := d\Psi^\beta(\cdot, q)\Big|_p\,. \tag{5.6}$$

Remark 5.11. Since $\Psi(\cdot, q)$ is a coordinate function, $(e_{q1}(p), \ldots, e_{q4}(p))$ is a tetrad in T_pM and $(\Theta_q^1(p), \ldots, \Theta_q^4(p))$ is the dual tetrad in T_p^*M. Hence for all $p \in V_{F(q)}$ we have

$$\Theta_q^\beta(p)(e_{q\alpha}(p)) = \delta_\alpha^\beta = e_{q\alpha}(p)(\Theta_q^\beta(p)). \tag{5.7}$$

In what follows, we need only a special form of these bases.

Notation 5.12. (1) *For all $p \in M$ we write*

$$e_\alpha(p) := e_{p\alpha}(p), \Theta^\beta(p) := \Theta_p^\beta(p). \tag{5.8}$$

(2) *To simplify notation, the arguments $(p), (x)$, etc. and the index χ indicating a coordinate system are mostly omitted in the sequel.*

Proposition 5.13. *Let $(W, \chi) \in \mathcal{D}$. Then for all $p \in W, x = \chi(p)$ and $\Lambda = \Lambda_\chi$:*

$$\Theta^\beta(p) = \Lambda_\kappa^\beta(x)\, dx^\kappa, \tag{5.9}$$

$$e_\alpha(p) = \Lambda^{-1}{}_\alpha^\lambda(x)\partial_{x^\lambda}. \tag{5.10}$$

Proof. Let q be fixed and $z = \chi(q)$. Then $y = \Psi(\chi^{-1}(x), q) = \phi(x, z)$ is the transformation between χ-coordinates x and pre-radar coordinates y. Therefore

$$dy^\beta = \frac{\partial \phi^\beta}{\partial x^\kappa} dx^\kappa, \quad \partial_{x^\alpha} = \frac{\partial \phi^\lambda}{\partial x^\alpha}\partial_{y^\lambda}.$$

Hence

$$\partial_{y^\lambda} = \left[\left(\frac{\partial \phi}{\partial x}\right)^{-1}\right]_\lambda^\alpha \partial_{x^\alpha}.$$

With $z = x$ and Definition 5.5 the proposition is seen to hold. $\square$

Remark 5.14. The matrix elements $\Lambda^{\beta}_{\kappa}, \kappa = 1, \ldots, 4$ are the χ-components of Θ^{β} and the $\Lambda^{-1}{}^{\lambda}_{\alpha}, \lambda = 1, \ldots, 4$ are the χ-components of e_{α}, i.e., $\Lambda^{\beta}_{\kappa} = \Theta^{\beta}(\partial_{x^{\kappa}})$ and $\Lambda^{-1}{}^{\lambda}_{\alpha} = e_{\alpha}(dx^{\lambda})$. Hence under transformation of coordinates they transform like components of covectors and vectors.

Proposition 5.13 has an immediate consequence for g and v. Inserting formulae (5.9) and (5.10) into formulae (5.1)–(5.4) we arrive at the following result.

Proposition 5.15. *Throughout M we have*

$$g = \eta_{\alpha\beta}\Theta^{\alpha} \otimes \Theta^{\beta}, \quad g^{\sharp} = \eta^{\kappa\lambda} e_{\kappa} \otimes e_{\lambda}, \quad v^{\flat} = -\Theta^{4}, \quad v = e_{4}. \tag{5.11}$$

Therefore, it is justified to say that Ψ generates g and v.

The result can be stated thus: with respect to $(\Theta^{1}, \ldots, \Theta^{4})$ the tetrad components of g are $\eta_{\alpha\beta}$, and with respect to $(e_{1}, \ldots, e_{4})$ the tetrad components of v are δ^{α}_{4}.

Remark 5.16. Using Proposition (5.15) and equation (5.7) we find the orthogonality relations

$$g(e_{\kappa}, e_{\lambda}) = \eta_{\kappa\lambda}, \quad g^{\sharp}(\Theta^{\alpha}, \Theta^{\beta}) = \eta^{\alpha\beta} \tag{5.12}$$

and

$$\eta^{\kappa\lambda} g(e_{\lambda}, \cdot) = \Theta^{\kappa}, \quad \eta_{\alpha\beta} g^{\sharp}(\Theta^{\beta}, \cdot) = e_{\alpha}. \tag{5.13}$$

Roughly speaking, the result of Section 5 is this: the function Ψ is a "potential" such that the metric g and the velocity v are determined by the derivates of Ψ. Moreover, Ψ itself has a physical meaning, namely $\Psi(\cdot, q)$ is a coordinate function for each $q \in M$.

The existence of the fields $\Theta^{\alpha}, e_{\beta}, \alpha, \beta = 1, \ldots, 4$, hence the existence of the function Ψ which determines Θ^{α} and e_{β}, has a consequence concerning time:

Remark 5.17. (1) From axioms GK 3.4 and GK 5 and from Proposition 5.15 one concludes that $e_{4} = v$ is a timelike C^{r}-vector field, $r \geq 2$ on M so that it nowhere vanishes. Therefore, the space–time manifold $(M, \mathcal{A}, g)$ is time orientable (cf., e.g. [11, p. 26]). Already from Definition 3.4 where the clock $\mathcal{U}_{A}$ of a particle $A \in P$ is introduced we conclude that $(W_{A}, \mathcal{U}_{A})$ is a one-dimensional manifold with a global chart for each $A \in P$. Therefore, W_{A} cannot be a closed curve.

(2) Following Geroch [12], in a noncompact space time the existence of a smooth global field of tetrads is a necessary condition of a spinor structure.

Remark 5.18. The function Ψ determines not only g and v but also a worldfunction Ω (cf. [13]) by

$$\Omega(p,q) = \eta_{\kappa\lambda}(\Psi^\kappa(p,q) - \Psi^\kappa(q,q))(\Psi^\lambda(p,q) - \Psi^\lambda(q,q)).$$

Then it is easily seen that for $\bar{\Omega}(x,z) := \Omega(\chi^{-1}(x), \chi^{-1}(z))$ the relation

$$\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}\bar{\Omega}(x,z)|_{z=x} = g^\chi_{\alpha\beta}$$

holds. The same result is obtained if one takes the covariant derivatives.

6 Alternative axiomatics

In the investigation so far the set S of signals was introduced solely to ensure that Lemma 5.3 is provable. Thus the question arises if it is possible to avoid S by changing some axioms. This is indeed the case. To see this, let us introduce some

Notation 6.1. *Let Φ^*_R be the frame theory which has the base sets $M, P, \mathbb{R}$ and the structural terms $\hat{\Psi}, g, v, \eta, \vartheta$ which are subject to the following axioms: GK 1.1–1.4, GK 2.1–2.3, EM, EE and AC together with GK*3 and 4 which read:*

GK$*$3. *For all $p \in M$:*

$$g(p) = \eta_{\alpha\beta}\, d\Psi^\alpha(p,q)|_{q=p} \otimes d\Psi^\beta(p,q)|_{q=p}.$$

GK$*$4. *For all $p \in M : v(p) = \partial_{\Psi^4(p,q)}|_{q=p}$,*

Then the following theorem holds.

Proposition 6.2. *The theories Φ_R and Φ^*_R are equivalent in the following sense. The theory Φ_R is stronger than Φ^*_R, i.e., all axioms of Φ^*_R are theorems in Φ_R. Conversely, there is a term S^* defined in Φ^*_R such that all axioms of Φ_R are theorems in Φ^*_R if the letter S is replaced by the term S^*.*

Proof. From Proposition 5.15 one concludes that Φ_R is stronger than Φ^*_R. To see the converse, one has to show that axioms GK 3–GK 5 of Φ_R are theorems in Φ^*_R. The term g in axiom GK* 3 is a $\binom{0}{2}$-tensor field throughout M

and $g(p)$ acts on $T_pM \times T_pM$ for all $p \in M$. Moreover, g is of class $C^r, r \geq 2$, because Ψ is of class $C^k, k \geq 3$. Finally g is symmetric, non-degenerate and of signature 2 because $\hat{\eta} = \mathrm{diag}(1, 1, 1, -1)$ has these properties. Therefore, axioms GK 3.1–3.3 are theorems in Φ_R^*.

Within Φ_R^* the function γ_A is defined like in Φ_R. Consequently, by axiom GK$*$4 together with Definition 5.10 and Notation 5.12 one has $v(p) = e_4(p)$. Moreover, from Definition 3.3 it follows that $\dot{\gamma}_A(t) = e_4(\gamma_A(t))$. Therefore,

$$v(p) = \dot{\gamma}_{F(p)}(t), \quad t = \gamma_{F(p)}^{-1}(p) \tag{6.1}$$

so that axiom GK 5 holds in Φ_R^*. Because of $\Theta^\alpha(e_4) = \delta_4^\alpha$ also GK 3.4 is a theorem in Φ_R^*.

In order to prove axioms GK 4.1 and 4.2 in Φ_R^* one has to define a term S^* so that these axioms can be verified in Φ_R^* if S is replaced by S^*. The definition of S^* runs as follows. For each $A \in P$ let us consider the set of all null geodesics σ which start at W_A, and let $\zeta = \psi_A \circ \sigma$. Then σ obeys the equations

$$\nabla_{\dot{\sigma}}\dot{\sigma} = r\dot{\sigma} \tag{6.2}$$

and

$$g(\dot{\sigma}, \dot{\sigma}) = 0. \tag{6.3}$$

Let the parameter of σ be denoted λ. Then in ψ_A-coordinates let $\zeta(\lambda_0) = x$, where $x = \psi_A(p)$, $p \in W_A$. (Generally λ_0 depends on p!) Now for each $p \in W_A$, (6.3) reads

$$\eta_{\alpha\beta}\dot{\zeta}^\alpha(\lambda_0)\dot{\zeta}^\beta(\lambda_0) = 0. \tag{6.4}$$

Hence $\dot{\zeta}^4(\lambda_0) > 0$ because only starting null geodesics are considered. This means we can choose $t = \zeta^4(\lambda)$ as a new parameter. Thus, using the same symbol ζ, we have $\dot{\zeta}^4(t_0) = 1$. From (6.4) it follows that $\dot{\zeta}(t_0) = (e^1, e^2, e^3, 1)$ where $(e^1, e^2, e^3) \in S^2$(S^2 being the two-sphere). Now we define $\hat{S}_A$ to be the set of all null geodesics σ, i.e., solutions of (6.2) and (6.3) which in ψ_A-coordinates obey the initial conditions $\zeta(t_0) = x_0$, $t_0 = x_0^4$ for any $x_0 = \psi_A(p)$, $p \in W_A$ and $\dot{\zeta}(t_0) = (e^1, e^2, e^3, 1)$ for any $(e^1, e^2, e^3) \in S^2$.

Finally, let $\hat{S} = \cup_{A \in P}\hat{S}_A$ and let S^* be any set of the same cardinality as $\hat{S}$ which may serve as a set of indices for $\hat{S}$.

Then it is easily seen that Axioms GK 4 are satisfied if S is replaced by S^*. □

This axiomatics is the simplest one for the considered systems.

7 Generating functions

7.1 General characteristics

7.1.1 Introductory remarks

In Section 3.2 a function Ψ generating charts was introduced within the theory Φ_R (or Φ^*_R) by Definition 3.7, and in Proposition 5.15 it was stated that Ψ also generates g and v. In Sections 7 and 8 the question is to be treated whether the concept of a generating function, i.e., a family of pre-radar charts, is definable only with the help of some special axioms of the theory Φ_R (or Φ^*_R) or whether it can be introduced also for arbitrary space–times. The conditions for a positive answer will be given in Section 8. In Section 7 we do some necessary preliminary work.

7.1.2 Properties of generating functions

The goal of this section is summarizing *all* the properties Ψ has and to define the concept of a generating function also within relativistic theories which are different from Φ_R and Φ^*_R. In order to do so, at first the conditions are written down which later on will be seen to be satisfied by Ψ.

Let $(M, \mathcal{A}^+)$ be a connected Hausdorff C^k-manifold, $k \geq 3$. Moreover, let g be a Lorentzian metric and v be a velocity field on $(M, \mathcal{A}^+)$, i.e., $g(v, v) = -1$, and let g and v be of class $C^r, r \geq 2$. Then conditions P1–P5 governing a term Ψ read as follows:

P1. Ψ is a C^k-function, $k \geq 3$,

$$\Psi : \bigcup_{q \in M} V_q \times \{q\} \to \mathbb{R}^4 \tag{7.1}$$

with V_q an open subset of M and $q \in V_q$ such that $\mathcal{A} = \{(V_q, \Psi(\cdot, q)) : q \in M\}$ is a C^k-atlas, $k \geq 3$ which is C^k-compatible with $\mathcal{A}^+$.

P2. Ψ generates g, i.e.,

$$g(p) = \eta_{\kappa\lambda} d_p \Psi^\kappa(p, q)|_{q=p} \otimes d_p\ \Psi^\lambda(p, q)\Big|_{q=p} \tag{7.2}$$

for all $p \in M$.

P3. Ψ generates v , i.e.,

$$v(p) = \partial_{\Psi^4(p,q)}\big|_{q=p} . \tag{7.3}$$

for all $p \in M$.

P4. For each $q \in M$ let $\gamma_q : J_q \to M, J_q \subset \mathbb{R}$ be a solution of $\dot{\gamma}_q = v(\gamma_q)$ such that there is a $t_q \in J_q$ for which $\gamma_q(t_q) = q$. Then J_q is an interval and

$$\gamma_q(t) = \Psi(\cdot, q)^{-1}(0, 0, 0, t) \tag{7.4}$$

for all $t \in J_q$. Moreover, J_q is also the domain of the right-hand side of (7.4).

P5. For all $q' \in W_q := \mathrm{ran}\gamma_q$ the equation $\Psi(\cdot, q) = \Psi(\cdot, q')$ holds.

In other words, the function Ψ generates an atlas $\mathcal{A}$ of pre-radar charts, a metric g, a velocity field v and the integral curves γ_q of v. Loosely speaking, Ψ knows almost everything one is interested in GR.

Then the following proposition is true.

Proposition 7.1. *The function Ψ introduced within the theory Φ_R (or Φ_R^*) by Definition 3.7 satisfies conditions P1–P5.*

Proof. Since $\Psi(p, q) = \hat{\Psi}(F(q), p) = \psi_A(p)$ for each $q \in W_A$ Conditions P1, P4 and P5 follow directly from axioms GK 1, 2 and 5. Conditions P2 and P3 are satisfied because of Proposition 5.15. $\square$

7.1.3 Definitions

These considerations suggest generalizing the concept of a generating function also to relativistic theories different from Φ_R and Φ_R^* as follows.

Definition 7.2. (1) Let $(M, \mathcal{A}^+)$ be a C^k-manifold, $k \geq 3$, for which g is a Lorentzian metric and v a velocity field. Then any term Ψ satisfying conditions P1–P5 is called a (full) generating function. If Ψ satisfies P1 and perhaps some, but not all of conditions P2–P5 it is called a partial generating function. If Ψ satisfies only P1 and P2 the coordinates $\Psi(\cdot, q)$ are known as locally Minkowskian.

(2) If Ψ generates a C^k-atlas $\mathcal{A}, k \geq 3$, and if $\mathcal{D}$ is the differential structure of class C^k containing $\mathcal{A}$ we say that $\mathcal{D}$ is generated by Ψ.

(3) The coordinates generated by a full generating function are called pre-radar coordinates.

Remark 7.3. If Ψ is a partial generating function which satisfies P1 and P4 then it satisfies also P3. For from P4 we have the equations

$$\Psi(\gamma_q(t), q) = (0, 0, 0, t) \tag{7.5}$$

and $\dot{\gamma}_q(t) = v(\gamma_q(t))$ for each $t \in J_q$. Furthermore, it follows from (7.5) that

$$\dot{\gamma}_q(t) = \frac{d}{dt}\Psi^\alpha(\gamma_q(t), q)\partial_{\Psi^\alpha}(\cdot, q)\Big|_{\gamma_q(t)} = \partial_{\Psi^4(\cdot,q)}\big|_{\gamma_q(t)} . \tag{7.6}$$

Therefore, for each $q \in M$ we find

$$v(q) = v(\gamma_q(t_q)) = \dot{\gamma}_q(t_q) = \partial_{\Psi^4(\cdot,q)}|_q \tag{7.7}$$

so that (7.3) holds.

Hence P3 in Definition 7.2 is superfluous, but there are practical reasons to take P3 as a separate condition. This will be seen e.g., in Section 7.2.

Remark 7.4. (1) From condition P5 one concludes that $\gamma_q = \gamma_{q'}$ for each $q' \in W_q :=$ ran γ_q. Similarly we have $J_q = J_{q'}$ and $W_q = W_{q'}$ for $q' = W_q$.

(2) It follows from condition P4 that for each $q \in M$ the function $\Psi(\cdot, q)^{-1}(0, 0, 0, \cdot)$ is an integral curve of v.

(3) Condition P1 has the consequence that any two partial generating functions have a common domain because $q \in V_q$, and that they generate the same differential structure $\mathcal{D}$.

7.2 Relations between generating functions

In this section the problem is to be treated to which extent conditions P1–P5 determine the partial generating functions. In a first step two partial generating functions Ψ and Ψ' are considered which satisfy conditions P1 and P2. Then the following proposition holds.

Proposition 7.5. *Let Ψ, Ψ' satisfy P1. Moreover, let g be generated by Ψ. Then Ψ' generates g exactly if*

$$\Psi'(p, q) = L(q) \cdot \Psi(p, q) + R(p, q), \tag{7.8}$$

where $L(q) = ((L^\alpha_\beta)(q)))$ is a Lorentz matrix and $d_p R(p, q) \mid_{q=p} = 0$. (Here Ψ', Ψ and R are column vectors.)

Proof. Let $\Theta'^\kappa = d\Psi'^\kappa \mid_{q=p}$ and $\Theta^\alpha = d\Psi^\alpha \mid_{q=p}$. Then by definition we have $g = \eta_{\alpha\beta}\Theta^\alpha \otimes \Theta^\beta$. Moreover, let $g' := \eta_{\kappa\lambda}\Theta'^\kappa \otimes \Theta'^\lambda$. Now assume (7.8) to be valid. Then $\Theta'^\kappa = L^\kappa_\alpha \Theta^\alpha$ and because of $\eta_{\alpha\beta} = \eta_{\kappa\lambda} L^\kappa_\alpha L^\lambda_\beta$ we have $g = g'$. Conversely, let g and g' be as above and assume $g' = g$. Moreover, let e_β denote the duals of Θ^α. Then $g(e_\alpha, e_\beta) = \eta_{\alpha\beta} = \eta_{\kappa\lambda}\Theta'^\kappa(e_\alpha)\Theta'^\lambda(e_\beta)$. Therefore, the matrix $L = ((L^\kappa_\alpha))$ with $L^\kappa_\alpha = \Theta'^\kappa(e_\alpha)$ is a Lorentz matrix. Since $L^\kappa_\alpha = L^\kappa_\varrho \Theta^\varrho(e_\alpha)$ one concludes that

$$\Theta'^\kappa = L^\kappa_\varrho \Theta^\varrho. \tag{7.9}$$

This equation reads explicitly $d\Psi'^\kappa(p,q)\big|_{q=p} = L^\kappa_\varrho(p) d\Psi^\varrho(p,q)\big|_{q=p}$.

Hence $d(\Psi'^\kappa(p,q) - L^\kappa_\varrho(q)\Psi^\varrho(p,q)) \mid_{q=p} = 0$ so that (7.8) results. □

Remark 7.6. If (7.8) is true the elements L^κ_α of the Lorentz matrix L are given by $L^\kappa_\alpha = \Theta'^\kappa(e_\alpha)$. Hence this equation holds if Ψ' and Ψ both generate the metric g. Therefore, in each case, L^κ_α is of class $C^r, r \geq 2$. Consequently, $R(p, \cdot)$ is of class $C^r, r \geq 2$, too. Mixed derivatives and derivatives with respect to p alone exist up to third order.

In the next step we consider two partial generating functions which satisfy conditions P1–P3. Clearly the result of Proposition 7.5 holds. But it turns out that the Lorentzian matrix L has a more special form.

Proposition 7.7. *Let Ψ, Ψ' satisfy P1. Moreover let Ψ generate g and v, and let v' be a velocity field. Then Ψ' generates g and v' exactly if Ψ' is given by (*7.8*) where L satisfies the condition*

$$v'^\flat = -L^4_\alpha \Theta^\alpha. \tag{7.10}$$

Proof. If Ψ' generates g and v' relation (7.8) is valid and $v'^\flat = -\Theta'^4$. Moreover, equation (7.9) holds because Ψ' generates g. Because of $v'^\flat = -\Theta'^4$ equation (7.10) holds, too. Now suppose the converse to be true. If (7.8) holds then Ψ' generates g. Therefore there is a Lorentz matrix L for which equation (7.9) is valid. If equation (7.10) is true we find that $v'^\flat = -\Theta'^4$. Hence Ψ' generates g and v'. □

Corollary 7.8. *The unique solution of (*7.10*) is*

$$L^4_\alpha = -v'^\flat(e_\alpha). \tag{7.11}$$

*Therefore (*7.10*) is equivalent to (*7.11*). Since $v^{\flat\prime}(e_\alpha) = v'_\alpha$ at any point $q \in M$ are the covariant components of v' in $\Psi(\cdot, q)$-coordinates we have $\eta^{\alpha\beta} v'_\alpha v'_\beta = -1$.*

This equation has the consequence that there is a Lorentz matrix L such that (7.10) holds. The proof of this statement can be read off from steps 3 and 5 of the proof of Proposition 8.2, where a somewhat more general case is treated.

Remark 7.9. Since the velocity fields v and v' define time orientations, they define the same orientation if $g(v, v') < 0$. In this case $L^4_4 > 0$ so that finally $L^4_4 \geq 1$. For $g(v', v) = v'^\flat(v) = -L^4_4$ because $v = e_4$.

Corollary 7.10. *Let the suppositions be as in Proposition (7.7). Then Ψ' generates g and v exactly if Ψ' is given by (7.8) with*

$$L = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}, \tag{7.12}$$

where $Q(q)$ is an orthogonal 3×3 matrix for each $q \in M$.

Proof. Let (7.8) and (7.12) be valid. Then $g = g'$ and

$$\Psi'^4 = \Psi^4 + R^4. \tag{7.13}$$

Therefore, $v'^\flat = -\Theta'^4 = -\Theta^4 = v^\flat$ so that $v' = v$. Now let $g = g'$ and $v = v'$. Since g' and v' are generated by Ψ', equation (7.11) holds. It reads in this case: $L^4_\alpha = \Theta^4(e_\alpha) = \delta^4_\alpha$. Then it follows from general properties of Lorentz matrices that L has the form (7.12). $\square$

Now let Ψ and Ψ' be partial generating functions satisfying P1–P3. What can be said about Ψ' if in addition Ψ satisfies P4? The answer is given by

Proposition 7.11. *Assume that Ψ and Ψ' satisfy condition P1. Moreover, assume that Ψ generates g and v and that it satisfies P4 where γ_q* is related to Ψ *by (7.4). Then Ψ' generates g and v, and satisfies condition P4 exactly if Ψ' is given by (7.8) and (7.12) where the additional condition*

$$R(\gamma_q(t), q) = 0 \tag{7.14}$$

holds for all $t \in J_q$.

Proof. First assume that Ψ' generates g and v, and satisfies P4. Hence, in P4 the integral curves with respect to Ψ' are the same as with respect to Ψ. Then, because Ψ' and Ψ are related by (7.8) together with (7.12), we find

$$(0,0,0,t)^{\mathrm{T}} = \Psi'(\gamma_q(t), q) = \begin{pmatrix} Q(q) & 0 \\ 0 & 1 \end{pmatrix} \cdot (0,0,0,t)^{\mathrm{T}} + R(\gamma_q(t), q),$$

so that equation (7.14) holds.

Conversely, if Ψ' is given by (7.8) and (7.12) it generates g and v so that the integral curves in P4 with respect to Ψ' and Ψ are the same. Therefore

$$\Psi'(\gamma_q(t), q) = \begin{pmatrix} Q(q) & 0 \\ 0 & 1 \end{pmatrix} (0,0,0,t)^{\mathrm{T}} + R(\gamma_q(t), q)$$

so that condition P4 for Ψ' is satisfied because of (7.14). $\square$

A similar result holds if one takes condition P5 into account. At first some suppositions are specified.

It is supposed that Ψ and Ψ' satisfy condition P1. Moreover, it is assumed that Ψ generates g and v, and that for each $q \in M$ there is an integral curve γ_q of v where $J_q := \mathrm{dom}\ \gamma_q$ is an interval and where there is a $t_q \in J_q$ with $\gamma_q(t_q) = q$. Finally it is supposed that Ψ satisfies P5 with $W_q := \mathrm{ran}\ \gamma_q$.

Proposition 7.12. *If these assumptions are true then Ψ' generates g and v, and satisfies P5 exactly if Ψ' is given by (7.8) and (7.12) where the additional conditions*

$$Q(q') = Q(q) \quad \text{and} \quad R(\cdot, q') = R(\cdot, q) \tag{7.15}$$

hold for all $q' \in W_q$.

Proof. If Ψ' is given by (7.8) and (7.12) then it generates g and v. In addition, if (7.15) are true Ψ' satisfies also P5. Conversely, if Ψ' generates g and v, it is given by (7.8) and (7.12). If in addition Ψ' satisfies P5 the equations

$$\frac{d}{dt}\Psi(p, \gamma_q(t)) = 0, \quad \frac{d}{dt}\Psi'(p, \gamma_q(t)) = 0$$

hold for all $p \in V_q$. Hence one concludes from (7.8) that

$$\Psi^\beta(p, \gamma_q(t))\frac{d}{dt}L^\alpha_\beta(\gamma_q(t)) + \frac{d}{dt}R^\alpha(p, \gamma_q(t)) = 0.$$

Since R is a least of class C^2 one can apply the d-operator with respect to p at the point $p = \gamma_q(t)$. Then we obtain

$$\Theta^\beta(\gamma_q(t))\frac{d}{dt}L^\alpha_\beta(\gamma_q(t)) = 0. \tag{7.16}$$

For $\alpha = 4$ it is an identity, and for $\alpha = l = 1, 2, 3$ it reads

$$\Theta^j(\gamma_q(t))\frac{d}{dt}Q^l_j(\gamma_q(t)) = 0. \tag{7.17}$$

Since $\Theta^j, j = 1, 2, 3$ are linearly independent we find $Q(q') = Q(q)$ for all $q' \in W_q$. Inserting this result into (7.8) and (7.12) we obtain also the second part of (7.15). $\square$

8 Construction of generating functions

8.1 The inverse problem

In Sections 3 and 4 an axiomatic formulation of the frame theory Φ_R is given (and analogously for Φ_R^* in Section 6) which is based on the existence of pre-radar charts. The set of all these charts forms a generating function for the metric g and the velocity v. Since in the usual formulation of GR the existence of pre-radar charts or of a generating function is not postulated, the question arises whether it is possible to construct a generating function in this case. I call this question the *inverse problem.*

In order to formulate the problem precisely let us first describe a relativistic (frame) theory Φ^+ which represents the usual account of GR. The base sets of Φ^+ are the set of (signs for) events M and the reals $\mathbb{R}$ (and possibly other sets). The structural terms are an atlas $\mathcal{A}^+$, a metric g, a velocity v, a mass density η and an empirical temperature ϑ (and possibly other fields). In any case the axioms of Φ^+ contain geometrical and kinematical axioms, the equations of motion of matter, Einstein's equation and additional conditions. In vacuum theories the density of matter is zero so that the equations of motion are empty and the right-hand side of Einstein's equation is zero. Then the inverse problem can be stated thus:

Problem 8.1. (1) If Φ^+ is given, is there a (full) generating function Ψ in Φ^+ in the sense of Definition 7.2?

(2) If Φ^+ is given, is there a local solution to the problem in the following sense: there is a covering of M by open sets V so that a generating function Ψ exists for each open submanifold with base set V?

This problem can be solved in two ways: first, one shows that a solution exists, and second, one constructs an explicit term which represents a generating function for each given theory Φ^+. In what follows, I shall present a general constructive solution of the local problem. But in addition, this result allows to formulate sufficient conditions for solutions of the global problem.

8.2 Construction of a field of tetrads

8.2.1 Proof of a theorem

In this section, we assume that a theory of type Φ^+ is given. More specific, we consider a C^k-manifold $(M, \mathcal{A}^+), k \geq 3$ with a Lorentz metric g and a velocity v defined on M such that $g(v, v) = -1$ and such that g and v are of class $C^r, r \geq 2$. The construction of a local (full) generating function in Section 8.3 is based on the existence of a tetrad field the components of which determine the components of g and v in the sense of formulae (5.1) and (5.2). For this purpose let us take a chart $(V, \chi) \in \mathcal{D}$ where $\mathcal{D}$ is the differential structure of class $C^k, k \geq 3$ which contains $\mathcal{A}^+$, and let $g_{\alpha\beta}$ and v^κ be the χ-components of g and v. Then the following result holds.

Proposition 8.2. *For each $x \in \chi[V]$ there is a matrix $\Lambda(x) = ((\Lambda^\varrho_\sigma(x)))$ with det $\Lambda(x) \neq 0$ and such that*

$$g_{\alpha\beta}(x) = \Lambda^\kappa_\alpha(x)\Lambda^\lambda_\beta(x)\eta_{\kappa\lambda} \tag{8.1}$$

and

$$v^\kappa(x) = \Lambda^{-1}{}_4^{\ \kappa}(x). \tag{8.2}$$

Moreover, the covectors $\Theta^\alpha(p) = \Lambda^\alpha_\beta(x)dx^\beta, \ \alpha = 1, \ldots, 4$ and the vectors $e_\kappa(p) = \Lambda^{-1}{}_\kappa^{\ \lambda}(x)\partial_{x^\lambda}, \ \kappa = 1, \ldots, 4$ with $x = \chi(p), p \in V$ form orthogonal tetrads in T^*_pV resp. in T_pV.

The proof is effected in five steps.

(1) Since the matrix $((g_{\alpha p}))$ is nonsingular, symmetric and of signature 2 there is an orthogonal matrix $((Q^\varrho_\sigma))$ such that $g_{\alpha\beta} = Q^\varrho_\alpha Q^\sigma_\beta d_{\varrho\sigma}$ where

$$d_{\varrho\sigma} = \mathrm{diag}(a_1^2, a_2^2, a_3^2, -a_4^2)_{\varrho\sigma}, \quad a_\lambda > 0, \quad \lambda = 1, \ldots, 4.$$

Now let $f^\nu_\mu = \sum\limits_j^4 \delta^\nu_j \delta^j_\mu a_j$. Then $d_{\varrho\sigma} = f^\kappa_\varrho f^\lambda_\sigma \eta_{\kappa\lambda}$. Defining $K := ((K^\kappa_\alpha))$ by $K^\kappa_\alpha = Q^\varrho_\alpha f^\kappa_\varrho$ we obtain the relations det $K \neq 0$ and

$$g_{\alpha\beta} = K^\kappa_\alpha K^\lambda_\beta \eta_{\kappa\lambda}. \tag{8.3}$$

(2) From (8.1) and (8.3) it follows that K and the matrix Λ we are looking for, can differ at most by a Lorentz matrix L, i.e.,

$$\Lambda^\kappa_\alpha = L^\kappa_\varrho K^\varrho_\alpha. \tag{8.4}$$

Since (8.2) is equivalent to

$$\Lambda^4_\alpha = -g_{\alpha\kappa} v^\kappa \tag{8.5}$$

the next task is to find a Lorentz matrix L such that

$$L^4_\varrho K^\varrho_\alpha = -g_{\alpha\kappa} v^\kappa \tag{8.6}$$

for the given terms $K^\varrho_\alpha, g_{\alpha\kappa}$ and v^κ. The unique solution of (8.6) reads simply

$$L^4_\beta = -K^{-1}{}^\alpha_\beta\, g_{\alpha\kappa} v^\kappa =: h_\beta. \tag{8.7}$$

But up to now it is not clear whether the four components h_β are possible candidates for the fourth row of a Lorentz matrix. It is shown in the next three steps that such Lorentz matrix exists.

(3) In this step a special Lorentz matrix is introduced. By simple calculations one can see that the following equations hold:

$$h_\beta = -\eta_{\beta\sigma} K^\sigma_\kappa v^\kappa, \tag{8.8}$$

$$\eta^{\alpha\beta} h_\alpha h_\beta = -1. \tag{8.9}$$

From (8.9) we get $|h_4| \geq 1$. Since both K and $-K$ satisfy equation (8.3), because of (8.8) we can choose $h_4 \geq 1$. Now let $r := -(h_4^2 - 1)^{\frac{1}{2}}$ and $\bar{v} := (1 - h_4^{-2})^{\frac{1}{2}}$. Because of $h_4 \geq 1$ we get $\bar{v} \in [0, 1[$. This means that $\bar{v}$ is a speed parameter. Moreover, by a simple calculation we find that

$$h_4 = (1 - \bar{v}^2)^{-\frac{1}{2}}, r = -\bar{v}(1 - \bar{v}^2)^{-\frac{1}{2}}.$$

Thus, the matrix

$$S = \begin{pmatrix} h_4 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ r & 0 & 0 & h_4 \end{pmatrix} \tag{8.10}$$

is a special Lorentz matrix.

(4) In this step two orthogonal matrices are defined. Let

$$b^1_l := -(h_4^2 - 1)^{-\frac{1}{2}} h_l, \quad l = 1, 2, 3, \tag{8.11}$$

then $\sum_l^3 (b^1_l)^2 = 1$. Moreover, let $B^1 = (b^1_1, b^1_2, b^1_3)$ and in addition let $B^\varrho = (b^\varrho_1, b^\varrho_2, b^\varrho_3)$, $\varrho = 2, 3$ be vectors such that $\{B^1, B^2, B^3\}$ is an orthonormal basis in $\mathbb{R}^3$. Then define the orthogonal matrix B by $B = (B^1, B^2, B^3)^{\mathrm{T}}$. Finally, let A be an arbitrary orthogonal 3×3 matrix and define $\hat{A}, \hat{B}$ by

$$\hat{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}. \tag{8.12}$$

(5) In the last step of the proof let us consider the matrix $L = \hat{A} \cdot S \cdot \hat{B}$. By definition, L is a Lorentz matrix and

$$L^4_\beta = S^4_1 \hat{B}^1_\beta + S^4_4 \hat{B}^4_\beta\,. \tag{8.13}$$

Inserting $S^4_1 = r$, $\hat{B}^1_\beta = r^{-1} \sum_l^3 h_l \delta^l_\beta$, $S^4_4 = h_4$ and $\hat{B}^4_\beta = \delta^4_\beta$ we obtain $L^4_\beta = h_\beta$. Hence a matrix Λ satisfying (8.1) and (8.2) is given by $\Lambda = L \cdot K$. Then it is easily seen that the dual tetrads $\Theta^\alpha, \alpha = 1, \ldots, 4$ and $e_\kappa, \kappa = 1, \ldots, 4$ are orthogonal.

Corollary 8.3. *It is seen from the proof that there is not only one matrix Λ, but there are infinitely many Λ which generate $g_{\alpha\beta}$ and v^κ by (8.1) and (8.2). Moreover, if there is a Lorentz matrix L such that $\Lambda = L \cdot K$ satisfies (8.1) and (8.2) for a given K, then $L = \hat{A} \cdot S \cdot \hat{B}$ where $\hat{A}, S$ and $\hat{B}$ are the matrices introduced by (8.10) and (8.12). This can be seen quite easily starting with a general ansatz $L = \tilde{A} \cdot \tilde{S} \cdot \tilde{B}$ and showing that $\tilde{S} = S, \tilde{B}^1_\beta = \hat{B}^1_\beta$ and $\tilde{A}$ is any matrix of the form $\hat{A}$ in (8.12).*

Remark 8.4. In what follows, we have to differentiate Λ. If, as usual, $g_{\alpha\beta}$ and v^κ are of class C^r, $r \geq 2$ then the eigenvalues and eigenvectors at $g_{\alpha\beta}$ are of class $C^r, r \geq 2$ (cf. [14, p. 122]; due to a private communication of H. Sohr, Paderborn the result of Kato can be generalized to the present case). Hence the matrix K is of class $C^r, r \geq 2$, and consequently $h_\beta, \beta = 1 \ldots, 4$ and $b^1_l, l = 1, 2, 3$, too. Then the vectors $B^\varrho, \varrho = 2, 3$ can be adjusted to form an orthonormal basis together with B^1, so that the matrix B is of class $C^r, r \geq 2$. Finally, let A be any sufficiently smooth field of orthogonal matrices. Then L and Λ are of class $C^r, r \geq 2$.

8.2.2 Special coordinates

In this section as well as in the subsequent ones the chart (V, χ) is specialized. It is assumed that χ is a comoving coordinate system with respect to v. This supposition has the following consequences.

Remark 8.5. By definition of χ we get for the components v^α of the velocity: $v^\alpha = \delta_4^\alpha w$. Since $g_{44}w^2 = -1$, both g_{44} and w are unequal zero, and by a special choice of χ^4 we can attain the relation $w > 0$. Moreover, since $\Lambda^{-1}\,{}_4^\alpha = v^\alpha = \delta_4^\alpha\ w$ we obtain the equation $\Lambda^{-1}\,{}_4^4 = w$ whereas the other elements of the fourth column of Λ are zero. Since $\Lambda = (\Lambda^{-1})^{-1}$, a similar result holds for Λ, i.e., $\Lambda_4^\alpha = \delta_4^\alpha w^{-1}$.

In a further step the integral curves of v are determined. First of all we introduce

Notation 8.6. (1) *Let, as above, $\tilde{V} = \chi[V] = ran\chi$. Then define*

$$\bar{V} := \{(x^1, x^2, x^3) : \textit{ there is an } x^4 \textit{ such that } (x^1, \ldots, x^4) \in \tilde{V}\}. \tag{8.14}$$

If $x = (x^1, \ldots, x^4)$ the abbreviations $\bar{x} = (x^1, x^2, x^3)$ and $x = (\bar{x}, x^4)$ are used.

(2) *If $z \in \tilde{V}$ and if w is as in Remark 8.5 then f is defined for a fixed x_0^4 by*

$$f(\bar{z}, x^4) = \int_{x_0^4}^{x^4} \frac{d\xi}{w(\bar{z}, \xi)}. \tag{8.15}$$

(3) *For each $\bar{z} \in \bar{V}$ let*

$$\begin{aligned} T_{\bar{z}} &: = \{x^4 : (\bar{z}, x^4) \in \tilde{V}\}, \\ J_{\bar{z}} &: = \{\tau : \textit{there is an } x^4 \in T_{\bar{z}} \quad \textit{and} \quad \tau = f(\bar{z}, x^4)\}. \end{aligned}$$

Remark 8.7. In what follows, it is assumed that there is an x_0^4 such that $\bar{V} \times \{x_0^4\} \subset \tilde{V}$. By restriction of V such x_0^4 can always be found. Then f is defined with exactly one of these numbers x_0^4. For the same reason we may assume that $T_{\bar{z}}$ and $J_{\bar{z}}$ are intervals. Because of $w > 0$ the function $f(\bar{z}, \cdot)$ is strictly increasing and of class $C^{r+1}, r \geq 2$. Therefore the inverse function exists and is also of class $C^{r+1}, r \geq 2$. We write $\varphi(\bar{z}, t) := f(\bar{z}, \cdot)^{-1}(t)$.

Then the following proposition holds.

Lemma 8.8. *The path γ is a solution of*

$$\dot{\gamma}^\alpha = v^\alpha(\gamma) = \delta_4^\alpha w(\gamma) \tag{8.16}$$

exactly if there is a $\bar{z} \in \bar{V}$ such that

$$\gamma^l(t) = z^l, \quad l = 1, 2, 3, \tag{8.17}$$

$$\gamma^4(t) = \varphi(\bar{z}, t) \tag{8.18}$$

for all $t \in J_{\bar{z}}$.

Proof. If γ is a solution of (8.16), then $\gamma^l, l = 1, 2, 3$ is constant, and by separation of variables one gets (8.17). The converse holds because $\dot{\gamma}^l(t) = 0, l = 1, 2, 3$ and

$$\dot{\gamma}^4(t) = \frac{\partial}{\partial t}\varphi(\bar{z}, t) = w(\bar{z}, \gamma^4(t)).$$

□

To complete the tools needed in the next section we introduce the

Notation 8.9. *If $\gamma^l(t) = z^l, l = 1, 2, 3,$ we write $\gamma = \gamma_{\bar{z}}$ and $W_{\bar{z}} = ran\gamma_{\bar{z}}$. Sometimes it is convenient to write also $\gamma_{z'} := \gamma_{\bar{z}}$ and $W_{z'} := W_{\bar{z}}$ for $z' \in W_{\bar{z}}$.*

8.3 Solution of the local inverse problem

8.3.1 Suppositions

In what follows, we consider again a relativistic theory of type Φ^+ which is characterized by the base sets M and $\mathbb{R}$, the structural terms $\mathcal{A}^+$, g and v and possibly others, and the axioms ruling these terms (and possibly others). (Cf. Sections 8.1 and 8.2.1.)

Let us assume that a comoving chart (V, χ) with respect to the velocity v is given. Then Notations 8.6 and 8.9 are used as well as the results of Remark 8.7 and Lemma 8.8.

Finally, it is assumed that the construction of the tetrad components Λ_β^α determining g and v by (8.1) and (8.2) is carried through in the chart (V, χ). But with respect to differentiability we assume a stronger condition to be valid: the components $\Lambda_\beta^\alpha, \alpha, \beta = 1, \ldots, 4$ are of class $C^k, k \geq 3$.

8.3.2 First step

At first a function $\tilde{\Phi}$ is defined. Later on it turns out that a restriction of it determines the generating function we are looking for. In the definition Notation 8.6 is used.

Definition 8.10. The function

$$\tilde{\Phi}: \bigcup_{z\in\chi[V]} (\mathbb{R}^3 \times T_{\bar{z}}) \times \{z\} \to \mathbb{R}^4 \tag{8.19}$$

is defined by

$$\begin{aligned} \tilde{\Phi}^j(x,z) &= \sum_l^3 (x^l - z^l)\Lambda^j_l(\bar{z},x^4), j = 1,2,3, \\ \tilde{\Phi}^4(x,z) &= \sum_l^3 (x^l - z^l)\Lambda^4_l(\bar{z},x^4) + f(\bar{z},x^4). \end{aligned} \tag{8.20}$$

Hence $\tilde{\Phi} \in C^k, k \geq 3$ if Λ^α_β, $\alpha,\beta = 1,\cdots,4$ are of class $C^k, k \geq 3$.

Before proofing that a restriction of $\tilde{\Phi}$ is a generating function two lemmas have to be proved.

Lemma 8.11. *For each $z \in \chi[V]$ there is an open set $V_{\bar{z}} \subset \mathbb{R}^3 \times T_{\bar{z}}$ such that $\tilde{\Phi}(\cdot,z)$ restricted to $V_{\bar{z}}$ is bijective and $\{\bar{z}\} \times T_{\bar{z}} \subset V_{\bar{z}}$.*

Proof. (1) Let $y = \tilde{\Phi}(x,z)$ and $X = (x^1 - z^1,\ldots,x^3 - z^3, w(\bar{z},x^4)f(\bar{z},x^4))$. Then (8.20) is equivalent to $\Lambda^{-1}(\bar{z},x^4)\cdot y^{\mathrm{T}} = X^{\mathrm{T}}$, i.e.,

$$\sum_l^3 \Lambda^{-1}{}^j_l(\bar{z},x^4)y^l + z^j = x^j, \quad j = 1,2,3, \tag{8.21}$$

$$\Lambda^{-1}{}^4_\alpha(\bar{z},x^4)y^\alpha = w(\bar{z},x^4)f(\bar{z},x^4). \tag{8.22}$$

Now assume that equation (8.22) is solvable for x^4. This means there is a function $F(\bar{z},\cdot)$ defined on a set $U_{\bar{z}} \subset \operatorname{ran}\tilde{\Phi}(\cdot,z)$ such that $x^4 = F(\bar{z},y)$ satisfies (8.22) if $y \in U_{\bar{z}}$. Then $\tilde{\Phi}(\cdot,z)^{-1}$ exists. It is defined for all $y \in U_{\bar{z}}$ by

$$\sum_{l}^{3} \Lambda^{-1}{}^{i}_{l}\,(\bar{z}, F(\bar{z}, y))y^l + z^i = x^i, i = 1, 2, 3, \tag{8.23}$$

$$F(\bar{z}, y) = x^4. \tag{8.24}$$

Hence, the only thing we need to prove is that equation (8.22) is solvable for x^4 if y is an element of a certain set $U_{\bar{z}}$.

(2) It is helpful to simplify the notation. In this and the next steps of the proof the fixed vector $\bar{z}$ is omitted. Since $\Lambda^{-1}{}^{4}_{4} = w > 0$, equation (8.22) reads

$$\sum_{l}^{3} \sigma_l(x^4)y^l + y^4 = f(x^4), \tag{8.25}$$

where $\sigma_l = w^{-1}\Lambda^{-1}{}^{4}_{l}$. Since f is bijective by Remark 8.7 one defines $\varrho_l(t) = \sigma_l(f^{-1}(t))$, so that equation (8.25) becomes

$$\sum_{l}^{3} \varrho_l(t)y^l + y^4 = t, \tag{8.26}$$

and we arrive at the result, that (8.25) has a unique solution $x^4 \in T_{\bar{z}}$ exactly if (8.26) has a unique solution $t \in J_{\bar{z}}$.

Let $N_{\bar{z}} := \{(0, 0, 0, \tau) : \tau \in J_{\bar{z}}\}$ then (8.26) is solvable if $y \in N_{\bar{z}}$, namely $t = y^4$. In the next steps of the proof the solvability of (8.26) is extended to a neighborhood $U_{\bar{z}}$ of $N_{\bar{z}}$.

(3) Since each open interval is the union of a countable family of increasing closed finite intervals we need to prove the solvability of (8.26) only for closed finite intervals. Thus let $J_i \subset J_{\bar{z}}$ be such an interval, then ϱ_l and $\dot{\varrho}_l$ are bounded in J_i because ϱ_l is of class $C^r, r \geq 2$, i.e., there is a number m_i such that

$$\sum_{l}^{3} \varrho_l^2(t) < m_i \quad \text{and} \quad \sum_{l}^{3} |\dot{\varrho}_l(t)| < m_i \tag{8.27}$$

for all $t \in J_i$. If $J_i = [a_i, b_i]$ and $0 < \zeta < 1$ define

$$\delta_i(y^4) = \min\{|y^4 - a_i|, |y^4 - b_i|, \zeta\} \tag{8.28}$$

and

$$U_i = \{y : a_i < y^4 < b_i, \sum_{l}^{3}(y^l)^2 < \frac{1}{m_i}\delta_i(y^4)\}. \tag{8.29}$$

Then δ_i is continuous in J_i, and therefore U_i is open. Now let us consider the function H defined by

$$H(y,t) = \sum_l^3 \varrho_l(t) y^l + y^4 \tag{8.30}$$

for $y \in U_i$ and $t \in J_i$.

(4) It is shown that for each $y \in U_i$ the function $H(y,\cdot)$ maps J_i into J_i and is contracting. Let $t \in J_i$ and $t' = H(y,t)$ then

$$|t' - y^4| \leq \sum_l^3 \varrho_l(t)^2 \cdot \sum_r^3 (y^r)^2 < \delta_i(y^4). \tag{8.31}$$

From (8.31) one concludes that $t' \in J_i$. Moreover let $t, t' \in J_i$, $t < t'$ and $H(y,t) = \tau$, $\quad H(y,t') = \tau'$. Then there is a $\tilde{t} \in J_i$ with $t \leq \tilde{t} \leq t'$ such that

$$\tau - \tau' = \sum_l^3 \dot{\varrho}_l(\tilde{t}) y^l (t - t'). \tag{8.32}$$

Hence for all $y \in U_i$

$$|\tau - \tau'| < \delta_i(y^4)|t - t'| \leq \zeta |t - t'|, \tag{8.33}$$

where $0 < \zeta < 1$, so that $H(y,\cdot)$ is contracting. Then by Banach's fixed-point theorem (cf., e.g., [15, p. 251; 16, p. 151]) we get that for each $y \in U_i$ there is exactly one $t \in J_i$ so that the equation $H(y,t) = t$ holds. Consequently, there is a function G_i such that $G_i(y) = t$ for each $y \in U_i$ and G_i is uniquely determined.

(5) If $y \in U_i \cap U_j$, $i \neq j$ then $G_i(y) = G_j(y)$ because G_i and G_j are the solutions to the same equation, i.e., equation (8.26). Now we can define the function F we are looking for (cf. (8.24)) as follows: Define $U_{\bar{z}}$ by $U_{\bar{z}} = \bigcup_i U_i$ and G by $G(\bar{z}, y) = G_i(y)$ for $y \in U_i$. Then let F be defined by

$$F(\bar{z}, y) = f^{-1}(\bar{z}, G(\bar{z}, y)) \tag{8.34}$$

for all $y \in U_{\bar{z}}$. By construction $x^4 = F(\bar{z}, y)$ is a solution of (8.22).

(6) Therefore, by the considerations of step one we conclude that $\tilde{\Phi}(\cdot, z)$ restricted to

$$V_{\bar{z}} := \{x : \tilde{\Phi}(x,z) \in U_{\bar{z}}\} \tag{8.35}$$

is bijective. The set $V_{\bar{z}}$ is open because $\tilde{\Phi}(\cdot, z)$ is continuous and because $U_{\bar{z}}$ is open. $\square$

8.3.3 Second step

This lemma now enables us to define a function which later on turns out to be the coordinate representation of the generating function we are looking for.

Definition 8.12. The function Φ is defined by $\Phi(x,z) = \tilde{\Phi}(x,z)$ for all

$$(x,z) \in \bigcup_{z' \in \chi[V]} V_{\bar{z}'} \times \{z'\} =: \mathrm{dom}\ \Phi. \tag{8.36}$$

The second lemma we need concerns the differentiability of $\Phi(\cdot, z)^{-1}$.

Lemma 8.13. *If $\Lambda \in C^k, k \geq 3$ (cf. Section 8.3.1) then $\Phi(\cdot, z)^{-1} \in C^k$, $k \geq 3$.*

Proof. To simplify notation the fixed vector $\bar{z}$ is omitted. From (8.23) and (8.24) we see that $\Phi(\cdot, z)^{-1} \in C^k$, if $F := F(\bar{z}, \cdot)$ is of class C^k. Let us write (8.25) in the form

$$Z(y, x^4) := \sum_{l}^{3} \sigma_l(x^4) y^l + y^4 - f(x^4), \tag{8.37}$$

where $y \in U_{\bar{z}}$ and $x^4 \in T_{\bar{z}}$. Then $Z(y, F(y)) = 0$ by definition of F. Since $\sigma_l = w^{-1} \Lambda^{-1}{}^4_l$, $w = \Lambda^{-1}{}^4_4$, $w > 0$ the functions $\sigma_l, l = 1,2,3$ and f are of class C^k, so that Z is also of class C^k.
Now using the notation of (8.26) we obtain $\sigma_l(x^4) = \varrho_l(f(x^4))$ and therefore

$$\frac{\partial \sigma_l}{\partial x^4} = \frac{\partial \varrho_l}{\partial t} \frac{\partial f}{\partial x^4} = \dot{\varrho}_l \frac{1}{w}.$$

Hence

$$\frac{\partial Z}{\partial x^4} = \frac{1}{w} \sum (\dot{\varrho}_l y^l - 1). \tag{8.38}$$

For each $t \in J_{\bar{z}}$ and each $y \in U_{\bar{z}}$ there is an $i \in \mathbb{N}$ such that $t = f(x^4) \in J_i$ and $y \in U_i$. Therefore

$$\left| \sum_{l}^{3} \dot{\varrho}_l y^l \right| < \delta_i(y^4), \leq \zeta < 1,$$

so that $|\frac{\partial Z}{\partial x^4}| > 0$. Applying the Implicite Function Theorem (cf., e.g., [17, p. 117]) we conclude that $F = F(\bar{z}, \cdot)$ is also of class $C^k, k \geq 3$. Thus the proof is complete. □

8.3.4 Final results

With the help of Lemmas 8.11 and 8.13 we are now able to prove the main result of Section 8.3.

Proposition 8.14. *If the suppositions of Section 8.3.1 are fulfilled and if Φ is given by Definition 8.12 then $\Psi = \Phi(\chi, \chi)$ is a generating function in the submanifold V with the global coordinate system χ.*

Proof. (1) The function Ψ is of class $C^k, k \geq 3$, because Φ is of class $C^k, k \geq 3$, by the Definitions 8.10 and 8.12. It follows from Lemmas 8.11 and 8.13 that for each $z \in \chi[V]$ the function $\Phi(\cdot, z)$ is a diffeomorphism of class $C^k, k \geq 3$, i.e., it is a transformation of coordinates. Hence for each $q \in V$ the function $\Psi(\cdot, q)$ is a coordinate function, and the set of all these functions is an atlas on V which is compatible with the (global) chart (V, χ). Thus condition P1 is satisfied.

(2) By a simple calculation one can see that

$$\frac{\partial \Phi^\alpha}{\partial x^\beta}(x, z) \mid_{z=x} = \Lambda^\alpha_\beta(x).$$

Since by construction of Λ equations (5.1) and (5.2) hold, and because of formulae (5.9) and (5.10) one can see quite easily that Ψ generates g and v. Hence conditions P2 and P3 are fulfilled.

(3) The integral curves $\hat{\gamma}_q$ of v are determined by formulae (8.17) and (8.18) and by $\hat{\gamma}_q(t) = \chi^{-1}(\gamma_z(t))$ where $z = \chi(q)$. Then for each $z = \chi(q), q \in V$ and $t \in J_{\bar{z}}$ we have $\Psi(\hat{\gamma}_q(t), q) = \Phi(\gamma_z(t), z) = (0, 0, 0, t)$. Hence P4 is fulfilled.

(4) Let $\hat{W}_q :=$ ran $\hat{\gamma}_q$. Then $\hat{W}_q = \chi^{-1}[W_z]$ where $z = \chi(q)$. Moreover, $\Psi(\cdot, q') = \Psi(\cdot, q)$ for $q' \in \hat{W}_q$ exactly if $\Phi(\cdot, z') = \Phi(\cdot, z)$ for $z' \in W_z$. But the latter equation is true because Φ depends only on $\bar{z}$. Therefore condition P5 is satisfied, too. □

Corollary 8.15. *For a given theory Φ^+ of the type described in Section 8.2.1 the velocity field v is defined on M. Hence, for each $q \in M$ there is a neighborhood V such that a comoving coordinate system χ exists on V. Therefore, by Proposition 8.14 the second part of Problem 8.1, the local form of the inverse problem, is solved. If $V = M$ one has obtained a solution of the global Problem 8.1.*

8.4 Examples

(i) The simplest case for which the inverse problem can be solved explicitly is the following: let $(M, \mathcal{A})$ be a space-time manifold with metric g and velocity v, and let (V, χ) be a chart such that

$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} a_\alpha^2(x), \quad v_\alpha(x) = -\delta_\alpha^4 a_4(x),$$

where $x \in \chi[V]$. Then $a_\beta(x) \neq 0$ for all $x \in \chi[V]$, so that

$$\Lambda_\alpha^\kappa(x) = \delta_\alpha^\kappa a_\alpha(x), \quad \Lambda^{-1}\,{}^{\mu}_{\ \beta}(x) = \delta_\beta^\mu a_\beta^{-1}(x).$$

Therefore

$$g_{\alpha\beta} = \Lambda_\alpha^\kappa \Lambda_\beta^\kappa \eta_{\kappa\lambda}, \quad v_\alpha = -\Lambda_\alpha^4$$

and

$$\Phi^j(x, z) = a_j(\bar{z}, x^4)(x^j - z^j), \quad j = 1, 2, 3, \tag{8.39}$$

$$\Phi^4(x, z) = f(\bar{z}, x^4), \tag{8.40}$$

where f is defined by Notation 8.6.

(ii) These considerations can be applied to the Robertson–Walker space-time $(M, \mathcal{A})$ with $k = 0$. Here the same notation is used as in Section 8.4.1. In this case $M = \mathbb{R}^3 \times T$ with $T =]0, \infty[$ and $\mathcal{A} = \{(M, id_M)\}$. For the global chart (M, id_M) we have

$$g_{\alpha\beta}(x) = K^2(x^4) \sum_i^3 \delta_\alpha^i \delta_\beta^i - \delta_\alpha^4 \delta_\beta^4,$$

where $K(x^4) \neq 0$ for all $x^4 \in T$, and $v_\alpha(x) = -\delta_\alpha^4$. Therefore

$$\Phi(x, z) = (K(x^4)(x^1 - z^1), \ldots, K(x^4)(x^3 - z^3), x^4).$$

The covariant velocity $v = -dx^4$ is solely owing to massive particles.

(iii) A similarly simple case is the outer Schwarzschild space-time $(M, \mathcal{A})$ where $M = \{y \in \mathbb{R}^3 : \|y\| > r_0\} \times \mathbb{R}$ and $\mathcal{A} = \{(M, \mathrm{id}_M)\}$. Using the (nonglobal) Schwarzschild coordinates $x = (r, \vartheta, \varphi, t)$ we have

$$a_1^2(x) = \left(1 - \frac{r_0}{r}\right)^{-1}, \quad a_2^2(x) = r^2,$$
$$a_3^2(x) = r^2 \sin^2 \vartheta, \quad a_4^2(x) = 1 - \frac{r_0}{r}.$$

Since the outer Schwarzschild solution is a vacuum solution the velocity field can be chosen such that the set of testparticles is comoving. This means

$$v_\alpha(x) = -\delta^4_\alpha \left(1 - \frac{r_0}{r}\right)^{\frac{1}{2}}.$$

Now let $z = (r', \vartheta', \varphi', t')$. Then the result is

$$\begin{aligned}
\Phi^1(x,z) &= (1 - \frac{r_0}{r'})^{-\frac{1}{2}}(r - r'),\\
\Phi^2(x,z) &= r'(\vartheta - \vartheta'),\\
\Phi^3(x,z) &= r' \sin\vartheta'(\varphi - \varphi'),\\
\Phi^4(x,z) &= (1 - \frac{r_0}{r'})^{\frac{1}{2}} t.
\end{aligned}$$

(iv) The last result can be generalized. For each vacuum solution g which is diagonal in a chart χ a generating function Ψ is given by (8.39) and (8.40). Especially if χ is a global chart then Ψ is a global solution of Problem 8.1.

9 GR as a scalar field theory

9.1 Formulation of a problem

In Section 5 (cf., e.g., Remark 5.4, Proposition 5.15) we have seen that the function Ψ (cf. Definition 3.7) within the frame theory Φ_R (or Φ^*_R) (cf. Notation 2.1 and Section 6,) generates the metric g and the velocity v. It turned out (cf. Proposition 7.1) that Ψ is a generating function in the sense of Definition 7.2.

In Sections 9 and 10 I want to establish a frame theory Φ_{sc} which has only two base sets, the set of events M and the reals $\mathbb{R}$, and three scalar structural terms $\Psi, \tilde\eta$ and $\tilde\vartheta$, where the two latter terms are generalizations of the density and the temperature. Hence, the class of systems to be considered is the same as in Section 1.

The problem to be solved in this section is this: find axioms governing the base sets and the structural terms such that there are reasonable models of the theory Φ_{sc}.

The method applied for this purpose is a heuristic argumentation. Clearly, such reasoning is not logically compelling, i.e., the axioms can not be deduced in the strict sense.

9.2 Possible axioms for generating functions

In a first step the geometrical and kinematical axioms of Φ_{sc} are considered, i.e., all those axioms which solely refer to Ψ. Since Ψ is intended to be a generating function it should have properties P1–P5 of Section 7.1.2. But in our case these conditions cannot serve as axioms for Ψ because they contain a metric g and a velocity v besides Ψ.

Nevertheless, let us consider that part of P1–P5 which refers only to Ψ. The result are the following three conditions Q1–Q3:

Q1. Ψ is a function, $\Psi : \cup_{q\in M} V_q \times \{q\} \to \mathbb{R}^4$, where V_q is a subset of M and $q \in V_q$, such that $\mathcal{A} = \{(V_q, \Psi(\cdot, q)) : q \in M\}$ is a C^k-atlas, $k \geq 3$, on M and such that $(M, \mathcal{A})$ is a connected Hausdorff manifold; the function Ψ is of class $C^k, k \geq 3$.

Q2. For each $q \in M$ define γ_q by

$$\gamma_q(t) = \Psi(\cdot, q)^{-1}(0, 0, 0, t) \tag{9.1}$$

for all $(0, 0, 0, t) \in \operatorname{ran} \Psi(\cdot, q)$. Then $\operatorname{dom} \gamma_q =: J_q$, is an interval and there is a $t_q \in J_q$ such that $\gamma_q(t_q) = q$.

Q3. For all $q' \in W_q := \operatorname{ran} \gamma_q$ the equation $\Psi(\cdot, q') = \Psi(\cdot, q)$ holds.

Now these conditions determine Ψ to the following extent:

Proposition 9.1. *If Ψ satisfies condition Q1 then there is exactly one metric g and one velocity field v such that Ψ is a generating function satisfying the conditions* P1–P3. *If, in addition, Ψ satisfies* Q2 *and* Q3 *then Ψ fulfills also* P4 *and* P5.

Proof. First of all, because of Q1 condition P1 is satisfied for $\mathcal{A}^+ = \mathcal{A}$. Next define

$$\Theta^\alpha(p) := d_p\Psi^\alpha(p, q)|_{q=p} \quad \text{and} \quad e_\beta(p) := \partial_{\Psi^\beta(p,q)}|_{q=p}. \tag{9.2}$$

Then, in order that Ψ is a generating function, the metric g and the velocity v have to be given uniquely by

$$g = \eta_{\alpha\beta}\Theta^\alpha \otimes \Theta^\beta \quad \text{and} \quad v = e_4. \tag{9.3}$$

Hence conditions P2 and P3 are fulfilled. From Q2 it follows that $\Psi^\alpha(\cdot, q) \circ \gamma_q(t) = t\delta^\alpha_4$. Hence $\dot\gamma_q(t) = \partial_{\Psi^4(\cdot,q)}|_{\gamma_q(t)}$. Using Q3 we have $\Psi(\cdot, q) = \Psi(\cdot, \gamma_q(t))$ so that $\dot\gamma_q(t) = e_4(\gamma_q(t))$. Hence, for each q the path γ_q is a solution of the differential equation $\dot{\gamma}_q = v(\gamma_q)$ which has unique solutions because v is of class $C^r, r \geq 2$. Since for each $q \in M$ we have $\gamma_q(t_q) = q$ all integral curves are of the form 9.1. This means that also P4 and P5 are satisfied. $\square$

This result suggests that the axioms governing Ψ we are looking for are conditions Q1–Q3 or some equivalents of them.

9.3 Heuristic steps for further axioms

In order to complete the axioms of the theory Φ_{sc} one has to set up equations that determine the fields $\Psi, \tilde{\eta}$ and $\tilde{\vartheta}$ where up to now we only know that $\tilde{\eta}$ and $\tilde{\vartheta}$ must have something to do with density and temperature. Clearly, the starting point for our heuristic search are the axioms EM and EE of Sections 4 and 6 . At the same time it is clear that the equations of motion and Einstein's equation written in terms of g, v, η and ϑ are not suitable to determine Ψ directly. But, since g and v are generated by Ψ or, more precisely, since they can be expressed in terms of the tetrads Θ^α and $e_\beta, \alpha, \beta = 1, \ldots, 4$, the equations of motion and the Einstein equation are also expressible in these terms. Thus, the problem of determining Ψ can be split up into two parts: first solve these equations for Θ^α, η and ϑ, and then determine Ψ from the equations $d_p\Psi^\alpha(\cdot, q)|_{p=q} = \Theta^\alpha(p)$, e.g., via the methods developed in Section 8. Such procedure is possible. But the theory Φ_{vs} thus obtained is not a scalar theory, rather it is a mixed one having vector fields and scalar fields as basic structural terms. Moreover the generating function Ψ is not a basic structural term, it is a derived quantity.

Since we want to establish a theory which has no other structural terms than $\Psi, \tilde{\eta}$ and $\tilde{\vartheta}$ the following heuristic idea is helpful: write down the equations of motion and the Einstein equation in terms of the tetrad components Λ^α_β for arbitrary coordinates χ, and in terms of density η and temperature ϑ. For this purpose the abbreviation $\Phi(x, z) = \Psi(\chi^{-1}(x), \chi^{-1}(z))$ is used.

Then remove in $\Lambda^\alpha_\beta(x) = \frac{\partial \phi^\alpha}{\partial x^\beta}(x, z)|_{z=x}$ the restriction $x = z$, i.e., substitute

$$\Lambda^\alpha_\beta(x) \quad \text{by} \quad \Pi^\alpha_\beta(x, z) := \frac{\partial \Phi^\alpha}{\partial x^\beta}(x, z),$$

and generalize $\eta(x)$ by $\tilde{\eta}(x, z)$ and $\vartheta(x)$ by $\tilde{\vartheta}(x, z)$.

The equations thus gained are taken for the remaining axioms of the theory Φ_{sc}. This program is carried through more detailed in the next section.

10 Generalized field equations

10.1 The tetrad form of the field equations

In this section the field equations, i.e., the equation of continuity, the balance of energy and momentum and Einstein's equation are formulated in terms of the components $\Lambda^{\alpha}_{\kappa}, {\Lambda^{-1}}^{\lambda}_{\beta}$ of Θ^{α} and $e_{\beta}, \alpha, \beta = 1, \ldots, 4$ with respect to an arbitrary coordinate system χ (cf. Remark 5.14). These equations are obtained from the usual formulation in terms of the χ-components $g_{\alpha\beta}$ and v_{α} by inserting (cf. formulae (5.1) and (5.2))

$$g_{\alpha\beta} = \Lambda^{\kappa}_{\alpha}\Lambda^{\lambda}_{\beta}\eta_{\kappa\lambda} \quad \text{and} \quad v_{\alpha} = -\Lambda^{4}_{\alpha}. \tag{10.1}$$

In what follows, for the sake of convenience the abbreviations

$$\Lambda := ((\Lambda^{\alpha}_{\beta})), \quad \bar{\Lambda}^{\alpha}_{\beta} := {\Lambda^{-1}}^{\alpha}_{\beta} \quad \text{and} \quad \bar{\Lambda} = ((\bar{\Lambda}^{\kappa}_{\lambda})) = \Lambda^{-1} \tag{10.2}$$

are used. Then the following proposition holds.

Proposition 10.1. *The equation of continuity,* $\operatorname{div}(\eta v) = 0$*, reads*

$$\bar{\Lambda}^{\beta}_{4}\frac{\partial \eta}{\partial x^{\beta}} + \eta(\bar{\Lambda}^{\beta}_{4}\bar{\Lambda}^{\alpha}_{\sigma} - \bar{\Lambda}^{\beta}_{\sigma}\bar{\Lambda}^{\alpha}_{4})\frac{\partial}{\partial x^{\beta}}\Lambda^{\sigma}_{\alpha} = 0. \tag{10.3}$$

The proof is based on the formula:

$$\Gamma^{\alpha}_{\beta\gamma} = \left[\begin{array}{c|c} \alpha & \kappa\lambda \\ \beta\gamma & \sigma \end{array}\right] \frac{\partial}{\partial x^{\kappa}}\Lambda^{\sigma}_{\lambda}, \tag{10.4}$$

where

$$\begin{aligned} 2\left[\begin{array}{c|c} \alpha & \kappa\lambda \\ \beta\gamma & \sigma \end{array}\right] &= \bar{\Lambda}^{\alpha}_{\sigma}\left(\delta^{\kappa}_{\beta}\delta^{\lambda}_{\gamma} + \delta^{\lambda}_{\beta}\delta^{\kappa}_{\gamma}\right) - \bar{\Lambda}^{\alpha}_{\nu}\bar{\Lambda}^{\kappa}_{\varrho}\eta^{\nu\varrho}\eta_{\sigma\mu}\left(\Lambda^{\mu}_{\gamma}\delta^{\lambda}_{\beta} + \Lambda^{\mu}_{\beta}\delta^{\lambda}_{\gamma}\right) \\ &\quad + \bar{\Lambda}^{\alpha}_{\nu}\bar{\Lambda}^{\lambda}_{\varrho}\eta^{\nu\varrho}\eta_{\sigma\mu}\left(\Lambda^{\mu}_{\gamma}\delta^{\kappa}_{\beta} + \Lambda^{\mu}_{\beta}\delta^{\kappa}_{\gamma}\right). \end{aligned}$$

It can be derived by a straightforward but lengthy calculation.

Since the balance of energy and momentum depends strongly on the constitutive equation

$$T = \mathcal{T}(g, v, \eta, \vartheta) \tag{10.5}$$

it cannot be written down explicitly for all the different functionals $\mathcal{T}$. One case of major interest is that of a Eulerian or ideal fluid. In this case T is given by its components:

$$T_{\alpha\beta} = p\, g_{\alpha\beta} + h\, v_\alpha v_\beta, \tag{10.6}$$

where $p = \hat{p}(\eta, \vartheta)$ and $h = \hat{h}(\eta, \vartheta)$.

Proposition 10.2. *The balance of energy and momentum,* $\mathrm{div}(T) = 0$, *reads in terms of* Λ *for a Eulerian fluid:*

$$\begin{aligned} &\bar{\Lambda}_j^\beta \frac{\partial p}{\partial x^\beta} + h(\bar{\Lambda}_4^\kappa \bar{\Lambda}_j^\beta - \bar{\Lambda}_4^\beta \bar{\Lambda}_j^\kappa) \frac{\partial}{\partial x^\beta} \Lambda_\kappa^4 = 0, \quad j = 1, 2, 3, \\ &\bar{\Lambda}_4^\beta \frac{\partial p}{\partial x^\beta} - \bar{\Lambda}_4^\beta \frac{\partial h}{\partial x^\beta} - h(\bar{\Lambda}_4^\beta \bar{\Lambda}_\sigma^\kappa - \bar{\Lambda}_4^\kappa \bar{\Lambda}_\sigma^\beta) \frac{\partial}{\partial x^\beta} \Lambda_\kappa^\sigma = 0. \end{aligned} \tag{10.7}$$

Again, the proof is based on (10.4) together with some lengthy calculations.

Since the right-hand side of Einstein's equation (in its usual form!) depends on the constitutive equation (10.5) one can write down only the left-hand side in an explicite way.

Proposition 10.3. *The Einstein equation* $G + \Lambda_0 g = \kappa_0 T$ *reads*

$$E_{jk\mu}^{\kappa\lambda\sigma} \left(\frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\kappa} \Lambda_\sigma^\mu \right) + D_{jk\mu\nu}^{\kappa\varrho\lambda\sigma} \left(\frac{\partial}{\partial x^\kappa} \Lambda_\varrho^\mu \right) \left(\frac{\partial}{\partial x^\lambda} \Lambda_\sigma^\nu \right) + \Lambda_0 \eta_{\kappa\lambda} \Lambda_j^\kappa \Lambda_k^\lambda = \kappa_0 T_{jk}, \tag{10.8}$$

where all indices run from 1 to 4. Here Λ_0 *is an (unspecified) cosmological constant and* κ_0 *is Einstein's gravitational constant as usual. The coefficients* E:: *and* D:: *are explicitly given. They are polynomials in* Λ: *and its inverse* $\bar{\Lambda}$:.

The proof is extremely lengthy, but also straightforward.

For later purposes it be noticed that the components of the Ricci tensor have a similar form as (10.8). They read

$$R_{jk} = S_{jk\mu}^{\kappa\lambda\sigma} \left(\frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\kappa} \Lambda_\sigma^\mu \right) + T_{jk\mu\nu}^{\kappa\varrho\lambda\sigma} \left(\frac{\partial}{\partial x^\kappa} \Lambda_\varrho^\mu \right) \left(\frac{\partial}{\partial x^\lambda} \Lambda_\sigma^\nu \right). \tag{10.9}$$

Hence

$$E^{\kappa\lambda\sigma}_{jk\mu} = S^{\kappa\lambda\sigma}_{j\varrho\mu} - \frac{1}{2}\eta_{\alpha\beta}\Lambda^{\alpha}_{j}\Lambda^{\beta}_{k}\eta^{\iota\gamma}\Lambda^{n}_{\iota}\Lambda^{m}_{\gamma}S^{\kappa\lambda\sigma}_{nm\mu} \tag{10.10}$$

and similarly for D:: and T::.

10.2 The generalizing procedure

In this section the heuristic ideas presented at the end of Section 9.3 are to be worked out in detail. For this purpose let us again consider a chart (V, χ), and let the terms $\Lambda^{\alpha}_{\beta}, \eta, \vartheta$ be functions of $x = \chi(p), p \in V$. Moreover, let $\Phi(x, z) := \Psi(\chi^{-1}(x), \chi^{-1}(z))$ for all $x, z \in \chi[V]$ for which the right-hand side is defined. Finally it is assumed that the constitutive equation (10.5) is given in the form

$$T_{\alpha\beta} = \mathcal{T}'_{\alpha\beta}(\Lambda, \eta, \vartheta). \tag{10.11}$$

Especially for a Eulerian fluid it follows from (10.6) that

$$T_{\alpha\beta} = p\Lambda^{\kappa}_{\alpha}\Lambda^{\lambda}_{\beta}\eta_{\kappa\lambda} + h\Lambda^{4}_{\alpha}\Lambda^{4}_{\beta}. \tag{10.12}$$

Then the generalized field equations are obtained from (10.3), (10.8) and from $\operatorname{div}(T) = 0$, e.g., from (10.7), by omitting the restriction $z = x$ in $\Lambda(x) = \frac{\partial}{\partial x}\Phi(x, z)\big|_{z=x}$. More precisely, this means we have to carry out in (10.3), (10.8) and (10.7) (or more general in $\operatorname{div}(T) = 0$) the following

Substitution 10.4.

1.

$$\Lambda(x) = \frac{\partial}{\partial x}\Phi(x, z)\bigg|_{z=x} \longrightarrow \Pi(x, z) := \frac{\partial}{\partial x}\Phi(x, z), \tag{10.13}$$

$$\bar{\Lambda}(x) := \Lambda^{-1}(x) \longrightarrow \bar{\Pi}(x, z) := \Pi^{-1}(x, z). \tag{10.14}$$

2.

$$\eta(x) \longrightarrow \tilde{\eta}(x, z) \quad \text{and} \quad \vartheta(x) \longrightarrow \tilde{\vartheta}(x, z), \tag{10.15}$$

where

$$\tilde{\eta}(x, x) = \eta(x) \quad \text{and} \quad \tilde{\vartheta}(x, x) = \vartheta(x). \tag{10.16}$$

3. For the derivatives of Λ it is natural to set

$$\frac{\partial}{\partial x^\varrho}\Lambda^\kappa_\alpha \longrightarrow \frac{\partial}{\partial x^\varrho}\Pi^\kappa_\alpha + \frac{\partial}{\partial z^\varrho}\Pi^\kappa_\alpha = \frac{\partial}{\partial x^\varrho}\frac{\partial}{\partial x^\alpha}\Phi^\kappa + \frac{\partial}{\partial z^\varrho}\frac{\partial}{\partial x^\alpha}\Phi^\kappa, \qquad (10.17)$$

$$\frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial x^\varrho}\Lambda^\kappa_\alpha \longrightarrow \frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial x^\varrho}\Pi^\kappa_\alpha + \frac{\partial}{\partial z^\sigma}\frac{\partial}{\partial x^\varrho}\Pi^\kappa_\alpha + \frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial z^\varrho}\Pi^\kappa_\alpha + \frac{\partial}{\partial z^\sigma}\frac{\partial}{\partial z^\varrho}\Pi^\kappa_\alpha$$
$$= \frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial x^\varrho}\frac{\partial}{\partial x^\alpha}\Phi^\kappa + \cdots + \frac{\partial}{\partial z^\sigma}\frac{\partial}{\partial z^\varrho}\frac{\partial}{\partial x^\alpha}\Phi^\kappa. \qquad (10.18)$$

4. Likewise the derivatives of η and ϑ are replaced by

$$\frac{\partial}{\partial x^\alpha}\eta \longrightarrow \frac{\partial}{\partial x^\alpha}\tilde\eta + \frac{\partial}{\partial z^\alpha}\tilde\eta, \quad \frac{\partial}{\partial x^\alpha}\vartheta \longrightarrow \frac{\partial}{\partial x^\alpha}\tilde\vartheta + \frac{\partial}{\partial z^\alpha}\tilde\vartheta. \qquad (10.19)$$

Remark 10.5. By definition we have $\Pi(x,x) = \Lambda(x), \tilde\eta(x,x) = \eta(x)$ and $\tilde\vartheta(x,x) = \vartheta(x)$. The same holds for (10.17)–(10.19). This means, if $z = x$ on the right-hand side the arrow $\to$ can be replaced by $=$.

10.3 Results

10.3.1 First step

Applying the rules of substitution (10.4) to equation (10.3) we arrive at the following

Proposition 10.6. *The generalization of the equation of continuity yields*

$$\mathcal{C}(\Phi,\tilde\eta) = 0,$$

where

$$\mathcal{C}(\Phi,\tilde\eta) = \bar\Pi^\beta_4\left(\frac{\partial\tilde\eta}{\partial x^\beta} + \frac{\partial\tilde\eta}{\partial z^\beta}\right) + \tilde\eta\left(\bar\Pi^\beta_4\bar\Pi^\kappa_\lambda - \bar\Pi^\kappa_4\bar\Pi^\beta_\lambda\right)\frac{\partial}{\partial z^\beta}\frac{\partial}{\partial x^\kappa}\Phi^\lambda. \qquad (10.20)$$

The proof is trivial.

Since the generalizing procedure cannot be carried through for arbitrary constitutive equations (10.11) we confine again to the case of a Eulerian fluid.

Proposition 10.7. *The generalization of the balance of energy and momentum yields for a Eulerian fluid*

$$\mathcal{E}_\varrho(\Phi, \tilde{\eta}, \tilde{\vartheta}) = 0, \quad \varrho = 1, \ldots, 4,$$

where

$$\mathcal{E}_n(\Phi, \tilde{\eta}, \tilde{\vartheta}) = \bar{\Pi}^\beta_n \left(\frac{\partial \tilde{p}}{\partial x^\beta} + \frac{\partial \tilde{p}}{\partial z^\beta} \right) - \tilde{h} \left(\bar{\Pi}^\beta_4 \bar{\Pi}^\kappa_n - \bar{\Pi}^\kappa_4 \bar{\Pi}^\beta_n \right) \frac{\partial}{\partial z^\beta} \frac{\partial}{\partial x^\kappa} \Phi^4 \quad (10.21)$$

for $n = 1, 2, 3$ and

$$\begin{aligned} \mathcal{E}_4(\Phi, \tilde{\eta}, \tilde{\vartheta}) = \bar{\Pi}^\beta_4 & \left(\frac{\partial \tilde{h}}{\partial x^\beta} - \frac{\partial \tilde{p}}{\partial x^\beta} + \frac{\partial \tilde{h}}{\partial z^\beta} - \frac{\partial \tilde{p}}{\partial z^\beta} \right) \\ & + \tilde{h} \left(\bar{\Pi}^\beta_4 \bar{\Pi}^\kappa_\lambda - \bar{\Pi}^\kappa_4 \bar{\Pi}^\beta_\lambda \right) \frac{\partial}{\partial z^\beta} \frac{\partial}{\partial x^\kappa} \Phi^\lambda; \end{aligned} \quad (10.22)$$

*moreover, using equation (*10.6*) we have*

$$\tilde{p} = \hat{p}\,(\tilde{\eta}, \tilde{\vartheta}) \text{ and } \tilde{h} = \hat{h}(\tilde{\eta}, \tilde{\vartheta}). \quad (10.23)$$

The proof is again simple.

Finally the replacement rules are applied to equation (10.8).

Proposition 10.8. *If the replacement rules applied to the constitutive equation (*10.11*) make sense, the generalization of the Einstein equation yields*

$$\mathcal{G}_{jk} + \Lambda_0 \Pi^\lambda_j \Pi^\kappa_k \eta_{\lambda\kappa} = \kappa_0 \tilde{T}_{jk}, \quad (10.24)$$

where

$$\tilde{T}_{jk} = \mathcal{T}'_{jk}(\Pi, \tilde{\eta}, \tilde{\vartheta}) \quad (10.25)$$

and

$$\begin{aligned} \mathcal{G}_{jk}(\Phi) = & \overset{1}{M}{}^{\lambda\kappa\sigma}_{jk\mu} \frac{\partial^3 \Phi^\mu}{\partial z^\lambda \partial z^\kappa \partial x^\sigma} + \overset{2}{M}{}^{\lambda\kappa\sigma}_{jk\mu} \frac{\partial^3 \Phi^\mu}{\partial z^\lambda \partial x^\kappa \partial x^\sigma} \\ & + \overset{1}{K}{}^{\lambda\kappa\varrho\sigma}_{jk\mu\nu} \left(\frac{\partial^2 \Phi^\mu}{\partial z^\lambda \partial x^\kappa} \right) \left(\frac{\partial^2 \Phi^\nu}{\partial z^\varrho \partial x^\sigma} \right) \\ & + \overset{2}{K}{}^{\lambda\kappa\varrho\sigma}_{jk\mu\nu} \left(\frac{\partial^2 \Phi^\mu}{\partial z^\lambda \partial x^\kappa} \right) \left(\frac{\partial^2 \Phi^\nu}{\partial x^\varrho \partial x^\sigma} \right). \end{aligned} \quad (10.26)$$

(All indices run from 1 to 4.)

Again the proof is simple but very lengthy.

The properties of the substituted terms mentioned in Remark 10.5 have some consequences which are important later on.

Remark 10.9. It follows from the construction of the terms $\mathcal{C}, \mathcal{E}_\varrho$ and $\mathcal{G}_{jk}$, $\varrho, j, k = 1, \ldots, 4$ that

$$\mathcal{C}(\Phi, \tilde{\eta})(x, x) = \operatorname{div}(\eta v)(x), \tag{10.27}$$

where $\operatorname{div}(\eta v)(x)$ is the left-hand side of (10.3), and that

$$\mathcal{G}_{jk}(\Phi)(x, x) = G_{jk}(x) = R_{jk}(x) - \frac{1}{2} g_{jk}(x) \bar{R}(x), \tag{10.28}$$

where R_{jk} is given by (10.9) and G_{jk} by the first and the second term of the left-hand side of (10.8).

Moreover, $\mathcal{E}_\varrho(\Phi, \tilde{\eta}, \tilde{\vartheta})(x, x)$ is equal to the left-hand side of (10.7)

10.3.2 Final form of the basic equations

In this section the results of Propositions 10.6–10.8 are generalized further. In order to do this let us introduce the following

Notation 10.10. *For a given chart (V, χ) let $\mathcal{V} \subset \chi[V] \times \chi[V]$ be the domain of $\Phi = \Psi(\chi^{-1}, \chi^{-1})$. Then a real function $\mathcal{O}$ defined on $\mathcal{V}$ such that $\mathcal{O}(x, x) = 0$, is called quasi-zero.*

Now the following proposition holds:

Proposition 10.11. *Let the functions $\Phi, \tilde{\eta}, \tilde{\vartheta}$ be defined on $\mathcal{V}$, and let $\mathcal{C}, \mathcal{E}_\varrho$, $\mathcal{G}_{jk}$ be given by formulae (10.20)–(10.22) and (10.26). If $\Phi, \tilde{\eta}, \tilde{\vartheta}$ satisfy any subset of the equations*

$$\mathcal{C}(\Phi, \tilde{\eta}) = \mathcal{O}, \tag{10.29}$$

$$\mathcal{E}_\varrho(\Phi, \tilde{\eta}, \tilde{\vartheta}) = \mathcal{O}_\varrho \tag{10.30}$$

$$\mathcal{G}_{jk}(\Phi) + \Lambda_o \Pi_j^\lambda \Pi_k^\kappa \eta_{\lambda k} = \kappa_0 \mathcal{T}'_{jk}(\Pi, \tilde{\eta}, \tilde{\vartheta}) + \mathcal{O}_{jk}, \tag{10.31}$$

where $\mathcal{O}, \mathcal{O}_\varrho$ and $\mathcal{O}_{jk}, \varrho, j, k = 1, \ldots, 4$ are quasi-zeros, then the terms Λ, η, ϑ defined by $\Lambda(x) = \Pi(x, x)$, $\eta(x) = \tilde{\eta}(x, x)$, $\vartheta(x) = \tilde{\vartheta}(x, x)$ satisfy the corresponding subset of equations (10.3), (10.7) and (10.8).

The proof results immediately from Remark 10.9.

Notation 10.12. *Equations* (10.29)*–*(10.31) *are called the generalized equation of continuity, the generalized balance of energy and momentum and the generalized Einstein equation.*

Remark 10.13. If one has a solution $\Phi, \tilde{\eta}, \tilde{\vartheta}$ of the generalized Einstein equation such that $\tilde{\eta}$ is a quasi-zero, then Λ is a vacuum solution of Einstein's equation.

To a certain extent also the converse of Proposition 10.11 holds:

Proposition 10.14. *Let be given a solution Λ, η, ϑ of equations (10.3) and (10.8), i.e., of the equation of continuity and of Einstein's equation in tetrad form. Moreover, let Φ be given by Definition 8.12 and define $\tilde{\eta}, \tilde{\vartheta}$ by*

$$\tilde{\eta}(x,z) = \eta(z^1, z^2, z^3, x^4), \tilde{\vartheta}(x,z) = \vartheta(z^1, z^2, z^3, x^4). \tag{10.32}$$

Finally, let the energy–momentum tensor T be defined by a functional $\mathcal{T}'$ such that for the generalization $\tilde{T}$ of T the following relations hold for $j, k = 1, \ldots, 4$:

$$\tilde{T}_{jk}(x,z) = \mathcal{T}'_{jk}(\Pi, \tilde{\eta}, \tilde{\vartheta})(x,z) = \mathcal{T}'_{jk}(\Lambda, \eta, \vartheta)(\bar{z}, x^4) + \mathcal{O}_{jk} \tag{10.33}$$

with $\mathcal{O}_{jk}$ being quasi-zeros and $\bar{z} = (z^1, z^2, z^3)$. Then the triple $\Phi, \tilde{\eta}, \tilde{\vartheta}$ is a (local) solution of the generalized equations (10.29) and (10.31).

The proof is again very lengthy but straightforward.

11 The theory Φ_{sc}

11.1 Geometry and kinematics

In this section the theory Φ_{sc} is formulated in the sense of Section 1.2(i), i.e., the terminology of Section 2.2 is used. Since the inductive procedure for setting up Φ_{sc} is described extensively in Sections 9 and 10 it suffices to write down the elements of Φ_{sc} without any further comment.

The *base sets* of Φ_{sc} are M and $\mathbb{R}$, and $\Psi, \tilde{\eta}, \tilde{\vartheta}$ are its *structural terms.*

The physical interpretation of these terms is this:

M is the set of signs for events;

- Ψ determines an atlas of pre-radar charts;
- $\tilde{\eta}$ is a generalized density which determines the density η by $\tilde{\eta}(p,p) = \eta(p)$;
- $\tilde{\vartheta}$ is a generalized temperature which determines the (empirical) temperature ϑ by $\tilde{\vartheta}(p,p) = \vartheta(p)$.

At first the axioms for geometry and kinematics are formulated:

GK$_{\rm sc}$1. Ψ is a function, $\Psi : \mathcal{M} \longrightarrow \mathbb{R}^4$ where $\mathcal{M} := \cup_{q \in M} V_q \times \{q\}$ with $V_q \subset M, \quad V_q \neq \emptyset$ and $q \in V_q$.

GK$_{\rm sc}$2. The term $\mathcal{A} := \{(V_q, \Psi(\cdot, q)) : q \in M\}$ is a C^k-atlas, $k \geq 3$ on M so that $(M, \mathcal{A})$ is a connected Hausdorff manifold.

GK$_{\rm sc}$3. Ψ is of class $C^k, k \geq 3$.

GK$_{\rm sc}$4. For each $q \in M$ there is a maximal open interval J_q such that $(0,0,0,\tau) \in$ ran $\Psi(\cdot, q)$ for each $\tau \in J_q$.

It is useful to introduce the following

Notation 11.1. (1) *For each $q \in M$ the function $\gamma_q : J_q \longrightarrow M$ is defined by $\gamma_q(t) = \Psi(\cdot, q)^{-1}(0,0,0,t)$. Moreover, we write $W_q :=$ ran γ_q.*
(2) *$\mathcal{D}$ is the differential structure which contains all charts (V, χ) which are C^k-compatible, $k \geq 3$, with $\mathcal{A}$.*

GK$_{\rm sc}$5. For each $q \in M$ there is a $t_q \in J_q$ such that $\gamma_q(t_q) = q$.

GK$_{\rm sc}$6. For all $q' \in W_q$ it holds that $\Psi(\cdot, q') = \Psi(\cdot, q)$.

11.2 Field equations

In the next step the axioms of the motion of matter are formulated. For this purpose a notation is used which depends on coordinates. Moreover, for any function F depending on $p, q \in M$ and its coordinate form the same symbol is used, i.e., we write $F(p,q) = F(x,z)$ for $x = \chi(p)$ and $z = \chi(q)$.

EM$_{\rm sc}$1. The terms $\tilde{\eta}$ and $\tilde{\vartheta}$ are functions, $\tilde{\eta} : \mathcal{M} \to \mathbb{R}$, $\tilde{\vartheta} : \mathcal{M} \to \mathbb{R}$ which are of class $C^r, r \geq 2$ and where $\mathcal{M}$ is the set introduced in GK$_{\rm sc}$1.

EM$_{\rm sc}$2. Let $(V, \chi) \in \mathcal{D}$ be any chart and let $(x,y) \in \mathcal{V} := (\chi \times \chi)[\mathcal{M} \cap (V \times V)]$. Then the χ-components of the generalized energy-momentum tensor $\tilde{T}_{jk}$ are given by

$$\tilde{T}_{jk}(x,z) = \mathcal{T}'_{jk}(\Pi, \tilde{\eta}, \tilde{\vartheta})(x,z), \tag{11.1}$$

where $\mathcal{T}'_{jk}$ is the same functional as in formula (10.11) for which substitution (10.4) makes sense.

EM$_{\text{sc}}$3. For each chart $(V, \chi) \in \mathcal{D}$ there is a quasi-zero $\mathcal{O}$ such that the equation $\mathcal{C}(\Phi, \tilde{\eta}) = \mathcal{O}$ holds.

EM$_{\text{sc}}$4. For each chart $(V, \chi) \in \mathcal{D}$ there are quasi-zeros $\mathcal{O}^\alpha, \alpha = 1, \ldots, 4$ so that the generalization of the balance of energy and momentum holds for $\tilde{T}^{\alpha\beta}$ and with $\mathcal{O}^\alpha$ at the right-hand side. Especially, for Euler fluids the resulting equation reads $\mathcal{E}_\varrho(\Phi, \tilde{\eta}, \tilde{\vartheta}) = \mathcal{O}_\varrho, \varrho = 1, \ldots, 4$. where $\mathcal{O}_\varrho$ are quasi-zeros and where $\mathcal{E}_\varrho$ is defined in Proposition 10.7.

Likewise, the generalized Einstein equation is introduced by an axiom:

EE$_{\text{sc}}$. For each chart $(V, \chi) \in \mathcal{D}$ there are quasi-zeros $\mathcal{O}_{jk}, j, k = 1, \cdots, 4$ so that the equations

$$\mathcal{G}_{jk}(\Phi) + \Lambda_0 \Pi_j^\lambda \Pi_k^\kappa \eta_{\lambda\kappa} = \kappa_0 \mathcal{T}'_{jk}(\Pi, \tilde{\eta}, \tilde{\vartheta}) + \mathcal{O}_{jk} \tag{11.2}$$

hold where Λ_0 is an unspecified cosmological constant and where κ_0 is Einstein's gravitational constant.

11.3 Additional conditions

Finally, in order to complete the axioms of Φ_{sc} the additional conditions (AC) have to be formulated. As in Section 4.3 we only illustrate the subject by three examples:

(i) initial conditions;
(ii) boundary conditions;
(iii) symmetry conditions.

In Section 12.2 we come back to the formulation of symmetry conditions for the function Ψ.

11.4 Some consequences of $\mathbf{\Phi_{sc}}$

(i) If one defines the tetrads Θ^α and $e_\beta, \alpha, \beta = 1, \ldots, 4$ by

$$\Theta^\alpha(p) = d\Psi^\alpha(\cdot, p)|_p \quad \text{and} \quad e_\beta(p) = \partial_{\Psi^\beta(\cdot,p)}|_p \tag{11.3}$$

and the metric g and the velocity v by

$$g = \eta_{\alpha\beta}\Theta^\alpha \otimes \Theta^\beta \quad \text{and} \quad v = e_4, \tag{11.4}$$

then Ψ is a full generating function in the sense of Definition 7.2, i.e., Ψ satisfies conditions P1–P5 of Section 7.1.2. This follows directly from axioms GK_{sc} 1–6 and from Proposition 9.1. If one is interested only in a partial generating function Ψ satisfying conditions P1–P3 then Φ_{sc} can be weakened by omitting axioms GK_{sc} 4–6.

(ii) Let γ_q be defined by Notation 11.1. Then γ_q is bijective and of class $C^k, k \geq 3$. This follows directly from $\Psi(\cdot, q) \circ \gamma_q(t) = (0, 0, 0, t)$.

(iii) From axiom $\text{GK}_{\text{sc}}5$ and from Notation 11.1 it follows that $q \in W_q$ for each $q \in M$, and therefore $\cup_{q\in M} W_q = M$. Moreover, from axiom $\text{GK}_{\text{sc}}6$ we conclude that $\gamma_{q'} = \gamma_q$ for $q' = W_q$. Hence $J_{q'} = J_q$ and $W_{q'} = W_q$ for $q' \in W_q$.

(iv) If $q' \notin W_q$ then $W_q \cap W_{q'} = \emptyset$. For assume that $\bar{q} \in W_q \cap W_{q'}$, then $\bar{q} \in W_q$ and $\bar{q} \in W_{q'}$, so that $W_q = W_{\bar{q}} = W_{q'}$.

(v) For each $q \in M$ let us select exactly one element $\hat{q}$ from W_q so that $\hat{q}$ is also selected from $W_{q'}$ for $q' \in W_q$, and let $N \subset M$ be the set of all these selected $\hat{q}$. Now, let P be any set of the same cardinality as N. Then, identifying a particle in Φ_{sc} with a worldline W_q the set P is a set of indices for particles as is used in the theories Φ_R and Φ_R^* (cf. Notation 2.1 and 6.1). Let $A \in P$ denote a particle and let $A \leftrightarrow \hat{q}$. Then by $\psi_A := \Psi(\cdot, \hat{q}), \gamma_A := \gamma_{\hat{q}}$ and $W_A := W_{\hat{q}}$ the notation used in Φ_R and Φ_R^* is regained. Also the function F (cf. Remark 3.6) is defined in Φ_{sc} by $F(q) = A$ for all $q \in W_A$.

11.5 Remarks concerning models of Φ_{sc}

By Notation 4.1 the concept of a model was explicitly introduced for the theory Φ_R. It can be transferred quite easily to each theory $\tilde{\Phi}$ which is formulated according to scheme (i) in Sections 1.2 and 2.2 as follows:

Notation 11.2. *If one replaces the base sets and the structural terms of $\tilde{\Phi}$ by explicit terms of mathematical analysis (or of the theory of sets) such that these terms satisfy the axioms of $\tilde{\Phi}$ within mathematical analysis (or the theory of sets), then we say that these terms define an analytical (or a set theoretical) model of $\tilde{\Phi}$.*

In the usual formulations of GR the AC are chosen such that the models are unique (or unique up to some diffeomorphisms). In case of the theory Φ_{sc} the situation is different. Uniqueness is not needed for the models. Rather *the generalized density $\tilde{\eta}$ and the generalised temperature $\tilde{\vartheta}$ only have to be unique up to quasi-zeros. Then different $\tilde{\eta}$ and $\tilde{\vartheta}$ define the same physically interpretable fields η and ϑ by $\tilde{\eta}(p,p) = \eta(p)$ and $\tilde{\vartheta}(p,p) = \vartheta(p)$. Likewise, the function Ψ need not be unique. Any two model functions Ψ and Ψ' describe the same physical situation if they generate the same differential structure $\mathcal{D}$, the same metric g and the same velocity v. This suggests the following*

Definition 11.3. Any two arrays of terms $M, \Psi, \tilde{\eta}, \tilde{\vartheta}$ and $M, \Psi', \tilde{\eta}', \tilde{\vartheta}'$ forming models of the theory Φ_{sc} are called *physically equivalent* if Ψ and Ψ' generate the same $\mathcal{D}, g$ and v and if $\tilde{\eta}, \tilde{\vartheta}$ and $\tilde{\eta}', \tilde{\vartheta}'$ differ only by a quasi-zero.

Clearly, physical equivalence is an equivalence relation within the models of Φ_{sc}. Consequently, axioms AC should be such that they determine uniquely a class of physically equivalent models of Φ_{sc}. But up to now, the mathematical question is still open how to formulate a well-posed initial value problem for the generalized Einstein equation and the generalized equation of continuity so that an equivalence class is uniquely determined.

Remark 11.4. (1) If Ψ and Ψ' belong to two physically equivalent models they both satisfy conditions P1–P5 of Section 7.1.2 and are related by formula (7.8) where the Lorentz matrix L and the function R obey the relations (7.12), (7.14), (7.15) and $d_p R(p,q)|_{q=p} = 0$.

This follows directly from Propositions 9.1 and Propositions 7.5, (7.11) and (7.12) together with Corollary 7.10. Especially from formula (7.14) one concludes that R is a quasi-zero because there is a $t_q \in J_q$ such that $\gamma_q(t_q) = q$ for each $q \in M$.

(2) Since one is only interested in a class of physically equivalent models the theory Φ_{sc} is a gauge theory.

12 Further properties of generating functions

12.1 Orientation and time orientation

In this section a connected Hausdorff manifold $(M, \mathcal{A}^+)$ is considered where $\mathcal{A}^+$ is of class $C^k, k \geq 3$. Moreover, it is assumed that a partial generating function Ψ in the sense of Definition 7.2 is defined on M. This implies that Ψ satisfies condition P1 of Section 7.1.2, i.e., that the atlas $\mathcal{A}$ generated by Ψ is C^k-compatible with $\mathcal{A}^+$. In other words, Ψ generates a differential structure $\mathcal{D}$ which contains $\mathcal{A}^+$. These assumptions are satisfied by the theories Φ_R and Φ_R^* treated in Sections 3, 4, 6 and the theory Φ_{sc} introduced in Section 11. Later on some additional assumptions are needed. The above assumptions allow to introduce the tetrads $\Theta^\alpha, e_\beta, \alpha, \beta = 1, \ldots, 4$ in the usual way by

$$\Theta^\alpha(p) = d\Psi^\alpha(\cdot, p)|_p \quad \text{and} \quad e_\beta(p) = \partial_{\Psi^\beta(\cdot,p)}|_p \tag{12.1}$$

as well as the fields g and v by (9.3). Then the following simple result holds:

Proposition 12.1. *The four-form ω defined by*

$$\omega = \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge \Theta^4 \tag{12.2}$$

determines an orientation on M.

For the proof one has to show that ω nowhere vanishes. This follows directly from $\omega(e_1, \ldots, e_4) = 1$ throughout M.

For the next step we need the assumption that a Lorentz metric g and a velocity v is defined on M and that $g(v, v) = -1$.

Then the manifold $(M, \mathcal{A}^+, g)$ is time orientable (cf., e.g., [11, p. 26]). In a further step it is assumed that Ψ in addition satisfies conditions P2 and P3. This means that the differential structure $\mathcal{D}$ defined by $\mathcal{A}$ is generated by Ψ and that Ψ also generates g and v. In this case a time orientation is given by

Definition 12.2. Let $u \in T_pM$ be timelike or lightlike. Then u is called future pointing if $\Theta^4(u) = -g(v, u) > 0$ and past-pointing if $\Theta^4(u) = -g(v, u) < 0$.

Remark 12.3. Independently of the fact that $(M, \mathcal{A}^+, g)$ is time orientable if a velocity field v exists on M it can be seen that Definition 12.2 makes sense. For, it can be shown that a future-pointing vector can not be transferred into a past-pointing only by parallel transport, and vice versa.

12.2 Isometries

The aim of this section is defining the concept of isometry solely in terms of a generating function Ψ, i.e., without using explicitly a metric g. For this purpose we use again the notation and the suppositions introduced at the beginning of Section 12.1.

Moreover, throughout this section it is assumed that a bijective function $f : M \to M$ is given. Then let $\Psi' := \Psi(f, f)$. Finally, let the term $\mathcal{A}'$ be defined by

$$\mathcal{A}' = \{(V'_{q'}, \Psi'(\cdot, q')) : q' \in M, V'_{q'} = f^{-1}[V_q], V_q = \text{dom } \Psi(\cdot, q), q = f(q')\}. \tag{12.3}$$

Then we obtain the following result:

Remark 12.4. If f is a homeomorphism then $\mathcal{A}'$ is a C^k-atlas, $k \geq 3$ on M. For, $V'_{q'}$ is open because V_q is open and f is continuous. Moreover, $\Psi'(\cdot, q') = \Psi(\cdot, f(q')) \circ f$ is a homeomorphism. Hence $(V'_{q'}, \Psi'(\cdot, q'))$ is a chart. Each two charts are C^k-compatible because

$$\Psi'(\cdot, q'_1) \circ \Psi'^{-1}(\cdot, q'_2) = \Psi(\cdot, f(q'_1)) \circ \Psi^{-1}(\cdot, f(q'_2)). \tag{12.4}$$

In the next step it is assumed that f is differentiable.

Proposition 12.5. *The function f is a C^k-diffeomorphism, $k \geq 3$ exactly if $\mathcal{A}$ and $\mathcal{A}'$ are C^k-compatible atlases, $k \geq 3$.*

Proof. (1) If f is diffeomorphic then $\mathcal{A}'$ is an atlas. Let $\Psi'(\cdot, q'), \Psi(\cdot, q)$ be arbitrary coordinate functions from $\mathcal{A}'$ resp. from $\mathcal{A}$. Then the function

$$\Psi'(\cdot, q') \circ \Psi^{-1}(\cdot, q) = \Psi(\cdot, f(q')) \circ f \circ \Psi^{-1}(\cdot, q) \tag{12.5}$$

is of class $C^k, k \geq 3$, and likewise its inverse, so that $\mathcal{A}$ and $\mathcal{A}'$ are C^k-compatible.

(2) If $\mathcal{A}$ and $\mathcal{A}'$ are C^k-compatible atlases the left-hand side of (12.5) and its inverse are of class C^k, so that f is a C^k-diffeomorphism. □

In a last step the main result of this section is formulated. For this purpose it is assumed that a metric g is defined on the manifold $(M, \mathcal{A}^+)$. Then a diffeomorphism f is called an isometry if $g(p') = f^*_{p'} g(p)$ where $p = f(p')$ and $f^*_{p'}$ is the pull back of f at p'.

Proposition 12.6. *Let Ψ satisfy conditions* P1 *and* P2 *of* Section 7.1.2, *i.e., Ψ generates the differential structure $\mathcal{D}$ of class $C^k, k \geq 3$ containing $\mathcal{A}^+$, and the metric g. Then the (bijective) function f is a C^k-isometry, $k \geq 3$ exactly if Ψ' satisfies also P1 and P2, i.e., if Ψ' generates $\mathcal{D}$ and g.*

Proof. (1) First of all some auxiliary formulae are proved. Let f be a diffeomorphism, and let $p = f(p')$ and $q = f(q')$. Then

$$d\Psi'(\cdot, q')|_{p'} = d\Psi(f, q)|_{p'} = f^*_{p'} d\Psi(\cdot, q)|_p. \tag{12.6}$$

This equation is seen to be true by the following short calculation. Let $w' \in T_{p'}M$, then

$$\begin{aligned} d_{p'}{\Psi'}^{\alpha}(\cdot, q')(w') &= w'({\Psi'}^{\alpha}(\cdot, q')) \\ &= w'(\Psi^{\alpha}(\cdot, q) \circ f) = f_{*p'}w'(\Psi^{\alpha}(\cdot, q)) \\ &= d_p\Psi^{\alpha}(\cdot, q)(f_{*p'}w') = f^*_{p'} d_p\Psi^{\alpha}(\cdot, q)(w'). \end{aligned} \tag{12.7}$$

Now let g' be the metric which is generated by Ψ':

$$g'(p') := \eta_{\alpha\beta} d{\Psi'}^{\alpha}(\cdot, p')|_{p'} \otimes d{\Psi'}^{\beta}(\cdot, p')|_{p'}. \tag{12.8}$$

Then from (12.6) it follows that

$$g'(p') = f^*_{p'} g(p). \tag{12.9}$$

(2) If f is a C^k-isometry then by Proposition 12.1 the atlases $\mathcal{A}$ and $\mathcal{A}'$ are C^k-compatible. Hence Ψ and Ψ' satisfy P1. Moreover, the isometry f satisfies the relation

$$g(p') = f^*_{p'} g(p). \tag{12.10}$$

Hence from (12.9) it follows that $g' = g$. This means that condition P2 is satisfied by Ψ and Ψ' for the same metric g.

(3) If Ψ and Ψ' satisfy P1 generating the same $\mathcal{D}$ then f is a diffeomorphism by Proposition 12.1 so that (12.9) is true. If Ψ and Ψ' satisfy P2 with $g' = g$ we find that (12.10) holds. Hence f is an isometry. □

Corollary 12.7. *Let Ψ be a partial generating function which satisfies conditions P1 and P2 of Section 7.1.2, i.e., Ψ generates the metric g and the differential structure $\mathcal{D}$, and let f be a C^k-diffeomorphism, $k \geq 3$. Then f is an isometry exactly if*

$$\Psi(f(p), f(q)) = L(q) \cdot \Psi(p, q) + R(p, q), \tag{12.11}$$

where L is a field of Lorentz matrices and where $dR(\cdot, q)|_{p=q} = 0$.

This follows directly from Propositions 12.6 and 7.5.

Equation (12.11) is the symmetry condition of Ψ for a given isometry f which we are looking for in this section. It can be helpful to derive a special form for Ψ which reduces the very complicated generalized Einstein equation. Similar results hold for conformal mappings, too.

13 Final Remarks

13.1 Results

The main point of this treatise is the existence of a function Ψ which generates an atlas $\mathcal{A}$ of pre-radar charts, a metric g, a velocity field v and the integral curves of v. It is shown that also an orientation and a time orientation is defined by Ψ. Finally, the concept of isometry can be formulated directly with the help of Ψ, i.e., without using the metric g.

This illustrates the significance Ψ has: it determines like a "potential" almost all of the fundamental concepts considered in GR and it is itself physically interpretable as a set of coordinate functions.

Since the existence of Ψ guarantees the existence of a smooth field of tetrads, it imposes restrictions on space–times. On the other hand, by space–time theory the existence of pre-radar or even radar charts is indispensiable for space–times so that the restrictions imposed on them by Ψ are physically motivated and natural.

Finally, in Section 11 it is shown that GR can be formulated as a scalar field theory. But the price is doubling the independent variables and a generalized Einstein equation which is of third order.

13.2 Open problems

(i) The main problem of any axiomatic formulation of GR, e.g., of the theories Φ_R, Φ_R^* and Φ_{sc}, is how to get models. For this purpose the additional conditions have to be concreted. This can be done for Φ_R and Φ_R^* in the usual way, but for Φ_{sc} it is not known up to now how to formulate a well-posed Cauchy problem for the generalized Einstein equation together with the generalized equation of continuity. In both cases solving Einstein's equation is the most difficult step in obtaining models, but it is not all one has to do. The other axioms must be satisfied, too.

(ii) An open practical problem is the exploitation of equation (12.11), the definition of isometry in terms of Ψ. The aim is obtaining a restricted form of Ψ for a given symmetry f. For this purpose one needs a sufficiently large set of representation theorems. But, little is known in this field.

(iii) A solution of the inverse problem described in Section 8.1 is of physical significance because the existence of a generating function imposes restrictions on a space–time. Therefore, it would be of great interest to find necessary and sufficient conditions for the solution of the (nonlocal) inverse problem.

(iv) A more principal problem is the formulation of the equations $\mathrm{EM}_{\mathrm{sc}}$ 3, 4 and $\mathrm{EE}_{\mathrm{sc}}$ in geometrical terms without use of coordinates. It seems to me that for this goal the product manifold $M \times M$ must be considered. Up to now the problem is unsolved.

(v) The generalized equation of continuity and the generalized Einstein equation form a system of 11 equations for the six unknown functions $\Psi, \tilde{\eta}, \tilde{\vartheta}$. This fact seems to be a hint that there are internal dependencies between these equations which are not known up to now. It could also be the case that the generalized equations of continuity and of motion together with a reduced version of the generalized Einstein equation, e.g., its trace, suffice to determine $\Psi, \tilde{\eta}, \tilde{\vartheta}$.

(vi) It is a hard task to calculate the coefficients M and K that occur in equation (10.26) for a given special ansatz of Ψ based on symmetries. A well adapted computer algebra could be helpful in this field. I think that this problem is solvable, though it is not yet solved.

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