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$\mathbb{Z}_2 \times \mathbb{Z}_2$ graded superconformal algebra of parafermionic type

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Abstract

We present a new conformal algebra. It is $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded and generated by three $N = 1$ superconformal algebras coupled to each other by nontrivial relations of parafermionic type. The representation theory and unitary models of the algebra are briefly discussed. We also conjecture the existence of infinite series of parafermionic algebras containing many $N = 1$ or $N = 2$ superconformal subalgebras.

Contents

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1 Introduction

Superconformal algebras are of great importance in theoretical physics. Probably the best known ones are the $N = 1$ and the $N = 2$ superconformal algebras, which play an important role in superstring theory. Here the N is the number of supersymmetry generators. The $N = 1$ algebra was introduced in [1] and [2], the $N = 2$ algebra first appeared in [3]. The mathematical meaning of the term "superalgebra" is that the algebra is $\mathbb{Z}_2$ graded. There are even (bosonic) generators and odd (fermionic) generators. The algebraic relations respect the $\mathbb{Z}_2$ grading.

In this paper we introduce a new algebra, the superconformal algebra graded by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ group is a finite abelian group containing four elements: the identity $(0, 0)$, and three more elements (1,0), (0,1), (1,1). The product of two different non-identity elements gives the third one. The square of a non-identity element gives identity, hence there are three different $\mathbb{Z}_2$ subgroups in $\mathbb{Z}_2 \times \mathbb{Z}_2$.

We take one superconformal generator field of conformal dimension 3/2 for each non-identity element of $\mathbb{Z}_2 \times \mathbb{Z}_2$: $G^{(\alpha)}, \alpha = 1, 2, 3$. Each one of them generates the standard $N = 1$ superconformal algebra:

$$G^{(\alpha)}(z)G^{(\alpha)}(w) = \frac{1}{(z-w)^3} + \frac{\frac{3}{c}T^{(\alpha)}(w)}{z-w} + O\big((z-w)^0\big), \tag{1.1}$$

$$T^{(\alpha)}(z)G^{(\alpha)}(w) = \frac{\frac{3}{2}G^{(\alpha)}(w)}{(z-w)^2} + \frac{\partial G^{(\alpha)}(w)}{z-w} + O\big((z-w)^0\big), \tag{1.2}$$

$$T^{(\alpha)}(z)T^{(\alpha)}(w) = \frac{c/2}{(z-w)^4} + \frac{2T^{(\alpha)}(w)}{(z-w)^2} + \frac{\partial T^{(\alpha)}(w)}{z-w} + O\big((z-w)^0\big). \quad (1.3)$$

The Virasoro fields $T^{(\alpha)}(z), \alpha = 1,2,3$, belong to the $(0,0)$ grading. The operator product expansion of two different superconformal generators should give the third one:

$$G^{(\alpha)}(z)G^{(\beta)}(w) \sim \frac{G^{(\gamma)}(w)}{(z-w)^{3/2}}, \quad \alpha \neq \beta \neq \gamma. \quad (1.4)$$

The power of the singularity $(3/2)$ is obtained by a simple dimensional analysis. The crucial point is that it is not integer. So our algebra is not a standard chiral algebra (vertex algebra in mathematical literature), but a parafermionic-type algebra (generalized vertex algebra). The full algebra as we show in this paper is formed by 10 generating fields. In addition to the six generating fields $G^{(\alpha)}(z), T^{(\alpha)}(z),\ \alpha = 1,2,3$, mentioned above, one has three dimension-5/2 fields $U^{(\alpha)}(z),\ \alpha = 1,2,3$, and one dimension-3 field $W(z)$. We call this algebra "the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra".

The first example of parafermionic algebra was introduced by Fateev and Zamolodchikov in [4]. This $\mathbb{Z}_N$ graded algebra is generated by $N-1$ fields of conformal dimensions $\Delta_i = i(N-i)/N,\ i = 1,2,\dots,N-1$. For a fixed N the algebra has no free parameters. In their next paper [5] the same authors presented another $\mathbb{Z}_3$ graded parafermionic algebra, generated by the Virasoro field and two dimension-4/3 fields. This algebra has a continuous free parameter — the central charge. Later Gepner [6] introduced new parafermionic theories through coset construction of the type $\mathfrak{g}_k/u(1)^r$, where $\mathfrak{g}_k$ is the affine Lie algebra on level k and r is its rank. The mathematical treatment of parafermionic algebras was developed in [7] (see also the recent paper [8]).

In our previous work [9] we applied the algebraic approach to calculate the structure constants of the $\mathrm{sl}(n)_2/u(1)^2$ and the $\mathrm{sl}(2|1)_2/u(1)^2$ coset parafermions. We called the generators of the former theory the $sl(n)$ fermions. The $\mathrm{sl}(3)_2/u(1)^2$ and the $\mathrm{sl}(2|1)_2/u(1)^2$ parafermionic algebras are also $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded.

The current paper is the direct continuation of [9], we use the same setting and the same tools to derive the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra. This algebra resembles in many aspects the sl(3) fermion algebra from [9]. But the new algebra is more complicated: it has more generating fields and has one free continuous parameter.

The paper is organized as following. First, in Section 2, we recall the main points of the algebraic approach to conformal algebras of parafermionic type. In Section 3, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra is derived. The full set of lengthy operator product expansions defining the algebra is listed in Appendix A. In Section 4 we convert the operator product expansions to the generalized commutation relations between the modes of the basic fields, preparing the ground to the study of representation theory of the algebra (in Section 5). The unitarity restrictions are discussed in Section 6. Two explicit realizations of unitary models possessing the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal symmetry are presented in Section 7. The last Section (8) contains a brief summary and the ideas for the further study. In particular, we announce the $N = 2$ superconformal analogue of the algebra described in this paper and also announce the existence of two series of more complicated $N = 1$ and $N = 2$ superconformal algebras of parafermionic type and speculate about their unitary minimal models.

2 Parafermionic conformal algebras

In this section, we briefly recall the main points of the algebraic formalism for parafermionic conformal algebras. We will follow here Ref. [9, Sections 2 and 3]. This algebraic approach in fact goes back to 1993 [7]. See also the recent paper [8], where parafermionic algebras are defined using the notion of polylocal fields.

An operator product expansion of parafermionic type has the following form:

$$A(z)B(w) = \frac{1}{(z-w)^\alpha}\Big(\big[A,B\big]_\alpha(w) + \big[A,B\big]_{\alpha-1}(w)(z-w) + \big[A,B\big]_{\alpha-2}(w)(z-w)^2 + \cdots\Big), \tag{2.1}$$

i.e., it is a general operator product expansion with one important restriction that except the overall singularity $(z-w)^{-\alpha}$ the integer powers of $(z-w)$ only are present on the right hand side of the equation. But the singularity α does not have to be integer. Here we also introduced a notation $\big[A,B\big]_n$, the n-product of fields A and B. It is the field, arising at the $(z-w)^{-n}$ term of the operator product expansion of the fields $A(z)$ and $B(w)$ around w as it appears in (2.1).

When $\alpha \notin \mathbb{Z}$ it is not clear a priori how to exchange the fields in the operator product expansion, since some phases are involved. The following axiom, which is the crucial point of the definition of parafermionic algebras, tells us how to exchange the fields in the operator product expansion (2.1):

$$A(z)B(w)(z-w)^{\alpha} = \mu_{AB} B(w)A(z)(w-z)^{\alpha}. \tag{2.2}$$

Here μ_{AB} is a *commutation factor* which is a complex number different from zero. The exponent α in (2.2) is usually chosen to be equal to the singularity of the operator product expansion. However one can add to α an integer number. If the integer is even, then the commutation factor is not changed, if we shift α by odd integer, then the sign of the commutation factor is flipped.

By exchanging the fields in (2.2) second time one shows that the commutation factor should satisfy the following consistency conditions:

$$\mu_{AB}\mu_{BA} = 1, \tag{2.3}$$

and if we assume that the term $\big[A, A\big]_{\alpha_{AA}} \neq 0$, then it follows that

$$\mu_{AA} = 1. \tag{2.4}$$

If the operator product expansion of two basic fields $B(w)$ and $C(v)$ gives a third one $D(v)$:

$$B(w)C(v) = \frac{D(v)}{(w-v)^{\alpha_{BC}}} + \cdots, \tag{2.5}$$

then exchanging another basic field $A(z)$ with $B(w)$ and then with $C(v)$ is essentially the same as exchanging $A(z)$ with $D(v)$. Therefore, μ_{AD} is proportional to $\mu_{AB}\mu_{AC}$:

$$\mu_{AB}\mu_{AC} = \mu_{AD}(-1)^{\alpha_{AB}+\alpha_{AC}-\alpha_{AD}} \tag{2.6}$$

It is also implicitly stated here that $\alpha_{AB} + \alpha_{AC} - \alpha_{AD} \in \mathbb{Z}$.

The most important tool in the study of parafermionic conformal algebras is the generalized Jacobi identities. This identities involve the operator product expansions between three fields:

$$\begin{aligned}
&\sum_{j\geq 0}(-1)^j \binom{\gamma_{AB}}{j} \big[A, \big[B, C\big]_{\gamma_{BC}+1+j}\big]_{\gamma_{AB}+\gamma_{AC}+1-j} \\
&\qquad - \mu_{AB}(-1)^{\alpha_{AB}-\gamma_{AB}} \sum_{j\geq 0}(-1)^j \binom{\gamma_{AB}}{j} \big[B, \big[A, C\big]_{\gamma_{AC}+1+j}\big]_{\gamma_{AB}+\gamma_{BC}+1-j} \\
&\quad = \sum_{j\geq 0} \binom{\gamma_{AC}}{j} \big[\big[A, B\big]_{\gamma_{AB}+1+j}, C\big]_{\gamma_{BC}+\gamma_{AC}+1-j},
\end{aligned} \tag{2.7}$$

The sums are finite, the upper bound is given by the order of singularity of the corresponding fields. The parameters γ differ from the corresponding singularity exponents α by an integer number: $\alpha_{AB} - \gamma_{AB}, \alpha_{AC} - \gamma_{AC}, \alpha_{BC} - \gamma_{BC} \in \mathbb{Z}$.

3 Derivation of the algebra

As we have already mentioned in the introduction, we start from three copies of the $N = 1$ superconformal algebra, associated to the three non-identity elements of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ abelian group. The generators are $G^{(\alpha)}(z), T^{(\alpha)}(z)$, $\alpha = 1, 2, 3$. The algebraic relations inside the $N = 1$ superconformal algebra are given by the operator product expansions (1.1) to (1.3). Note the unusual normalization of superconformal generators $G^{(\alpha)}(z)$. The parameter c is the central charge of the three $N = 1$ superconformal algebras.

Now we want to couple the fields $G^{(\alpha)}(z)$ to each other. The operator product expansion of two superconformal generators G gives the third one:

$$G^{(\alpha)}(z)G^{(\beta)}(w) = \frac{\kappa_{\alpha,\beta}G^{(\gamma)}(w)}{(z-w)^{3/2}} + O\big((z-w)^{-1/2}\big), \tag{3.1}$$

where $\kappa_{\alpha,\beta}$ are yet unknown structure constants and α, β, γ are all different.

The fields in these operator product expansions are exchanged as following:

$$G^{(\alpha)}(z)G^{(\beta)}(w)(z-w)^{3/2} = \mu_{\alpha,\beta}G^{(\beta)}(w)G^{(\alpha)}(z)(w-z)^{3/2}, \tag{3.2}$$

leading to the relations between the structure constants and the commutation factors:

$$\kappa_{\alpha,\beta} = \mu_{\alpha,\beta}\kappa_{\beta,\alpha}. \tag{3.3}$$

The commutation factors are easily determined using the relation (2.6) between them. Taking $A = G^{(1)}, B = G^{(1)}, C = G^{(2)}$, we get

$$\mu_{1,1}\mu_{1,2} = -\mu_{1,3}, \tag{3.4}$$

since the singularities are equal to $\alpha_{1,1} = 3, \alpha_{1,2} = \alpha_{1,3} = 3/2$, and so $(-1)^{\alpha_{1,1}+\alpha_{1,2}-\alpha_{1,3}} = -1$. Substituting $A = G^{(1)}, B = G^{(2)}, C = G^{(3)}$ in (2.6) we get

$$\mu_{1,2}\mu_{1,3} = \mu_{1,1}. \tag{3.5}$$

Taking into account that $\mu_{1,1} = 1$, one obtains

$$\mu_{1,2} = -\mu_{1,3} = \pm \mathrm{i}. \tag{3.6}$$

To resolve the formal ambiguity we fix $\mu_{1,2} = \mathrm{i}$. Using the cyclic permutations of indices we determine all the commutation factors:

$$\mu_{1,2} = \mu_{2,3} = \mu_{3,1} = -\mu_{2,1} = -\mu_{3,2} = -\mu_{1,3} = \mathrm{i}. \tag{3.7}$$

To determine the structure constants we use the generalized Jacobi identities (2.7). Take $A = G^{(1)}, B = G^{(2)}, C = G^{(3)}$ and two parameters from the set of three $\gamma_{AB}, \gamma_{BC}, \gamma_{AC}$ equal to $1/2$ and the third one equal to $3/2$. Then the corresponding Jacobi identities require that the structure constants are equal to each other:

$$\kappa_{1,2} = \kappa_{2,3} = \kappa_{3,1} = g\,\mathrm{e}^{\mathrm{i}\pi/4}, \tag{3.8}$$

where we introduced the new phase shifted structure constant g in order to avoid appearance of i in the formulas below.

We will assign the following $\mathbb{Z}_2 \times \mathbb{Z}_2$ charges to the fields: $G^{(1)}$ has charge $(1,0)$, $G^{(2)}$ — $(0,1)$, $G^{(3)}$ — $(1,1)$. Then the identity field and all the Virasoro generators $T^{(\alpha)}$ carry the charge $(0,0)$. The commutation factors of the fields from the $(0,0)$ sector with all the fields are equal to 1, and the commutation factors between other sectors are given by (3.7).

Now it is easy to derive the leading terms in the operator product expansions of different generating fields using the dimensional and $\mathbb{Z}_2 \times \mathbb{Z}_2$ charge analysis:

$$T^{(\alpha)}(z)G^{(\beta)}(w) = \kappa\left(T^{(\alpha)}, G^{(\beta)}\right)\frac{G^{(\beta)}(w)}{(z-w)^2} + O\big((z-w)^{-1}\big), \tag{3.9}$$

$$T^{(\alpha)}(z)T^{(\beta)}(w) = \frac{\kappa\left(T^{(\alpha)}, T^{(\beta)}\right)}{(z-w)^4} + O\big((z-w)^{-2}\big), \tag{3.10}$$

where $\alpha \neq \beta$ and $\kappa\left(T^{(\alpha)}, G^{(\beta)}\right), \kappa\left(T^{(\alpha)}, T^{(\beta)}\right)$ are structure constants to be determined by the Jacobi identities in the following way. Insert $A = G^{(\alpha)}, B = G^{(\alpha)}, C = G^{(\beta)}$ $(\alpha \neq \beta)$ and $\gamma_{BC} = \gamma_{AC} = 1/2$, $\gamma_{AB} = 0$ to the Jacobi identities (2.7) to get

$$\kappa\left(T^{(\alpha)}, G^{(\beta)}\right) = \frac{(1+16g^2)\,c}{24}. \tag{3.11}$$

Choose $A = T^{(\alpha)}$, $B = G^{(\beta)}$, $C = G^{(\beta)}$ $(\alpha \neq \beta)$ and $\gamma_{BC} = 0$, $\gamma_{AC} = 2$, $\gamma_{AB} = 1$, the Jacobi identity then enforces

$$\kappa\left(T^{(\alpha)}, T^{(\beta)}\right) = \frac{c}{3}\kappa\left(T^{(\alpha)}, G^{(\beta)}\right) = \frac{(1+16g^2)\,c^2}{72}. \tag{3.12}$$

All the Jacobi identities for the fields $G^{(\alpha)}, T^{(\alpha)}, \alpha = 1,2,3$, taking into account only the terms specified in the above operator product expansion relations, are satisfied now. So this parafermionic algebra is selfconsistent, there are two free parameters: c and g. However, we have not specified all the singular terms in the operator product expansions, so the information contained in the generalized commutation relations extracted from the above operator product expansions is not sufficient to build the representation theory of the algebra. We have to specify all the singular terms in the operator product expansions of generating fields in terms of the generating fields, their derivatives and composite fields. By singular terms we understand all the n-products $\big[A,B\big]_n$, $n>0$, and by composite field we mean $\big[A,B\big]_n$, $n\le 0$, where A and B are two generating fields.

We have to make the additional assumptions about the missing singular terms in the operator product expansions. The first assumption is that there are no other dimension-2 fields in the algebra. It means that the field in $\big[T^{(\alpha)},T^{(\beta)}\big]_2$ is a linear combination of $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$. The second consequence of this assumption is that the total energy—momentum field $T(z)$ is proportional to the sum of $T^{(1)}(z)$, $T^{(2)}(z)$ and $T^{(3)}(z)$. The factor is easily calculated from the requirement that the weight of the fields $G^{(\alpha)}$ under the action of $T(z)$ is equal to their conformal dimension 3/2. So we get that the total energy–momentum field is

$$T=\frac{1}{1+\dfrac{c}{18}\left(1+16g^2\right)}\Big(T^{(1)}+T^{(2)}+T^{(3)}\Big). \tag{3.13}$$

The requirement $\big[T,T^{(\alpha)}\big]_2=2T^{(\alpha)}$ leads to fixing the coefficients in the second-order term in the operator product expansion of $T^{(\alpha)}$ and $T^{(\beta)}$:

$$\big[T^{(1)},T^{(2)}\big]_2=\frac{c}{18}\left(1+16g^2\right)\Big(T^{(1)}+T^{(2)}-T^{(3)}\Big), \tag{3.14}$$

and the same for cyclic permutations of the indices.

Now it is easy to verify that the energy–momentum field $T(z)$ indeed satisfies the Virasoro algebra, the central charge of which is

$$\mathbf{C}=\frac{3c}{1+\dfrac{c}{18}\left(1+16g^2\right)}. \tag{3.15}$$

Now we look at the next to leading term $\big[G^{(\alpha)},G^{(\beta)}\big]_{1/2}$ in the expansion of two superconformal generators. This is a dimension-5/2 field. The simplest assumption that it is just proportional to $\partial G^{(\gamma)}$ does not work. We have to introduce three new basic fields $U^{(1)},U^{(2)},U^{(3)}$ of dimension 5/2:

$$\begin{aligned}G^{(\alpha)}(z)G^{(\beta)}(w)={}&\mathrm{e}^{\mathrm{i}\pi/4}g\left(\frac{G^{(\gamma)}(w)}{(z-w)^{3/2}}+\frac{\frac{1}{2}\partial G^{(\gamma)}(w)+U^{(\gamma)}(w)}{(z-w)^{1/2}}\right)\\&+O\big((z-w)^{1/2}\big),\end{aligned} \tag{3.16}$$

where α, β, γ are cyclically ordered. The coefficient $\frac{1}{2}$ before $\partial G^{(\gamma)}(w)$ is chosen so to make the field $U^{(\gamma)}(w)$ primary with respect to the total energy–momentum Virasoro field. The field $U^{(\gamma)}$ has the same $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading as $G^{(\gamma)}$. Consequently $U^{(\gamma)}$ has the same commutation factors with other fields as $G^{(\gamma)}$, if defined appropriately:

$$\mu_{A,U^{(\gamma)}} = \mu_{A,G^{(\gamma)}}, \tag{3.17}$$

if $\alpha_{A,U^{(\gamma)}} = \alpha_{A,G^{(\gamma)}} + 2\,\mathbb{Z}$ in (2.2). Here A stands for any field.

The first-order singular term in the expansion of $T^{(\alpha)}$ and $G^{(\beta)}$ is also of dimension $5/2$ and a priori is not expressed in terms of $U^{(\gamma)}$ and $\partial G^{(\gamma)}$ only. But we would like to make the life easy assuming that no new dimension-$5/2$ basic fields have to be introduced. This assumption will cost us one free parameter in the algebra: c becomes a function of g. First note that with respect to the $N = 1$ superconformal algebra generated by $T^{(1)}$ and $G^{(1)}$, the two other dimension-$3/2$ fields $G^{(2)}$ and $G^{(3)}$ are the highest weight primary fields of Ramond type of weight $(1 + 16g^2)\, c/24$:

$$\begin{aligned} &G^{(1)}_0 |G^{(2)}\rangle = g\, e^{i\pi/4} |G^{(3)}\rangle, && G^{(1)}_n |G^{(2,3)}\rangle = 0, \;\; n > 0, \\ &T^{(1)}_0 |G^{(2,3)}\rangle = \tfrac{c}{24}(1 + 16g^2) |G^{(2,3)}\rangle, && T^{(1)}_n |G^{(2,3)}\rangle = 0, \;\; n > 0. \end{aligned} \tag{3.18}$$

Then the field $U^{(2)}$ is expressed in terms of $G^{(1)}_{-1} G^{(1)}_0 |G^{(2)}\rangle$ and $\big[T^{(1)}, G^{(2)}\big]_1$ is expressed in terms of $T^{(1)}_{-1} |G^{(2)}\rangle$. $G^{(1)}_{-1} G^{(1)}_0 |G^{(2)}\rangle$ and $T^{(1)}_{-1} |G^{(2)}\rangle$ are in general two independent states in the highest weight representation. However, if there is a null state on level 1 in the highest weight representation, then $G^{(1)}_{-1} G^{(1)}_0 |G^{(2)}\rangle \sim T^{(1)}_{-1} |G^{(2)}\rangle$ and $\big[T^{(1)}, G^{(2)}\big]_1$ is expressed through $U^{(2)}$. The null state appears on level 1 when the highest weight h and the central charge c are connected by the following equation: $3c - 72h + 16ch + 128h^2 = 0$. Upon substitution $h = (1 + 16g^2)c/24$ the relation is translated to

$$c = \frac{54g^2}{(1 + 4g^2)(1 + 16g^2)}. \tag{3.19}$$

So at last we can fix the following operator product expansions:

$$T^{(\alpha)}(z) G^{(\beta)}(w) = \frac{c(1 + 16g^2)}{24} \left(\frac{G^{(\beta)}(w)}{(z - w)^2} + \frac{\frac{2}{3} \partial G^{(\beta)}(w) - \sigma_{\alpha\beta} \frac{4}{3} U^{(\beta)}(w)}{z - w} \right) + O\big((z - w)^0\big), \tag{3.20}$$

where $\alpha \neq \beta$ and we introduced the following two-index symbol:

$$\sigma_{\alpha\beta} = \begin{cases} 0, & \alpha = \beta, \\ \epsilon_{\alpha\beta\gamma}, & \alpha \neq \beta \neq \gamma, \end{cases} \qquad \alpha, \beta, \gamma = 1, 2, 3, \tag{3.21}$$

i.e., $\sigma_{12} = \sigma_{23} = \sigma_{31} = 1$ and $\sigma_{21} = \sigma_{32} = \sigma_{13} = -1$.

Another new basic generator field W of dimension 3 appears in the first-order pole $\left[T^{(\alpha)}, T^{(\beta)}\right]_1$, $\alpha \neq \beta$:

$$\left[T^{(\alpha)}, T^{(\beta)}\right]_1 = \frac{c\left(1 + 16g^2\right)}{36} \left(\partial T^{(\alpha)} + \partial T^{(\beta)} - \partial T^{(\gamma)}\right) + \sigma_{\alpha\beta} W. \tag{3.22}$$

(A priori there are three such fields but since $\left[T, T^{(\beta)}\right]_1 = \partial T^{(\beta)}$ for $\beta = 1, 2, 3$ all the three dimension-3 primary fields are proportional to W.) The field W has the $\mathbb{Z}_2 \times \mathbb{Z}_2$ charge equal to $(0, 0)$. So its operator product expansions with all the other fields contain only integer powers of $(z - w)$, and it has the following exchange properties:

$$A(z)W(w) = W(w)A(z) \tag{3.23}$$

for any generator field $A(z)$.

It comes out that no other new fields are needed to close the algebra. The operator product expansions of the fields $U^{(\alpha)}$ and W with all the other basic fields are constructed using the dimensional and $\mathbb{Z}_2 \times \mathbb{Z}_2$ charge analysis, taking into account the basic fields ($G^{(\alpha)}, T^{(\alpha)}, U^{(\alpha)}, \alpha = 1, 2, 3,$ and W), their derivatives and the composite fields (e.g., $\left[G^{(1)}, G^{(2)}\right]_{-1/2}$, $\left[T^{(2)}, G^{(3)}\right]_0$, …). The structure constants are fixed by a routine check of the Jacobi identities. In the end, we obtain the operator product expansions listed in Appendix A. All the Jacobi identities are satisfied modulo a null field condition:

$$\partial W + \frac{(1 + 16g^2)c^2}{27} \left(\left[G^{(1)}, U^{(1)}\right]_0 + \left[G^{(2)}, U^{(2)}\right]_0 + \left[G^{(3)}, U^{(3)}\right]_0\right) = 0. \tag{3.24}$$

We want also to discuss here the subalgebras of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra. Obviously it has the three $N = 1$ superconformal subalgebras generated by $G^{(\alpha)}, T^{(\alpha)}$. Their bosonic parts generated by $T^{(\alpha)}$ only are also subalgebras. Another Virasoro subalgebra is generated by $T - T^{(\alpha)}$. It is commutative with both $T^{(\alpha)}$ and $G^{(\alpha)}$, so its central charge is equal to $\mathbf{C} - c$. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra has no other proper subalgebras.

4 Generalized commutation relations

The mode expansions of the fields are introduced as

$$A(z) = \sum_n A_n z^{-n-\Delta(A)}. \tag{4.1}$$

The generalized commutation relations between the field modes are obtained using the formula

$$\begin{aligned}\sum_{j=0}^{\infty} \binom{\alpha-k}{j} (-1)^j \Big(A_{m+\alpha-k-j} B_{n-\alpha+k+j} - \mu_{AB} (-1)^k B_{n-j} A_{m+j} \Big) \\ = \sum_{l=0}^{k-1} \binom{m+\Delta_A-1}{k-1-l} C^{(l)}_{m+n}.\end{aligned} \tag{4.2}$$

For the derivation see Section 4 in [9]. A_m and B_n are the modes of the fields $A(z)$ and $B(z)$, and $C^{(l)}(z)$ are the terms in the operator product expansion of $A(z)B(w)$:

$$A(z)B(w) = \frac{1}{(z-w)^\alpha} \Big(C^{(0)}(w) + C^{(1)}(w)(z-w) + C^{(2)}(w)(z-w)^2 + \cdots \Big). \tag{4.3}$$

The commutation factor μ_{AB} in (4.2) is chosen with respect to the α in the same formula. The integer number k in (4.2) is equal to the number of terms in the operator product expansion which are taken into account. The generalized commutation relation with smaller k can be obtained from that with larger k. Taking into account all the singular terms in the operator product expansion is sufficient to build the representation theory. However, in some calculations one can use the generalized commutation relation for the smaller number of terms, which is usually a more simple formula.

The singularities α in the operator product expansions of the fields in the $(0,0)$ charge $\mathbb{Z}_2 \times \mathbb{Z}_2$ sector ($T^{(\beta)}, \beta = 1, 2, 3$, and W) with all the fields in the algebra are integer. Therefore the corresponding generalized commutation relations are just usual commutators. The operator product expansions inside the same $\mathbb{Z}_2 \times \mathbb{Z}_2$ sector are also of standard type, so the relations between $G^{(\beta)}_n, U^{(\beta)}_m$ (for the same β) are anticommutation relations. The only relations which are of parafermionic type are those between the G, U fields from different $\mathbb{Z}_2 \times \mathbb{Z}_2$ sectors. For example:

$$\begin{aligned}\sum_{j=0}^{\infty} \binom{j-1/2}{j} \Big(\mathrm{e}^{-\mathrm{i}\pi/4} G^{(1)}_{m-1/2-j} G^{(2)}_{n+1/2+j} - \mathrm{e}^{\mathrm{i}\pi/4} G^{(2)}_{n-j} G^{(1)}_{m+j} \Big) \\ = g \left(\frac{m-n-1/2}{2} G^{(3)}_{n+m} + U^{(3)}_{n+m} \right),\end{aligned} \tag{4.4}$$

and the same for the cyclic permutation of the indices. Because of the space limitations we will not list all the generalized commutation relations here. The one which is important for the discussion below is the following commutation relation:

$$\begin{aligned}[T^{(1)}_m, T^{(2)}_n] = &\frac{(1+16g^2)c^2 m(m^2-1)}{432}\delta_{0,m+n} \\ &+ \frac{3g^2(m-n)}{2\,(1+4g^2)}\left(T^{(1)}_{m+n} + T^{(2)}_{m+n} - T^{(3)}_{m+n}\right) + W_{m+n},\end{aligned} \tag{4.5}$$

and the same for the cyclic permutation of the indices.

We should also draw your attention that some operator product expansions include composite operators, therefore the generalized commutation relations will include infinite sums of terms quadratic in modes also on the right hand side of (4.2). This happens for example in the case of the relation obtained from the operator product expansion $G^{(\alpha)}(z)U^{(\beta)}(w)$, $\alpha \neq \beta$ (A.13). The mode expansions of composite operators are derived in Appendix E of [10]. But here we can just extract the mode expansions for the composite operators from the same formula (4.2). One should choose k in such a way that the last term in the operator product expansion taken into account is the composite operator we are interested in. Then the "reversed" formula is

$$\begin{aligned}\Big([A,B]_\beta\Big)_{m+n} = &-\sum_{\gamma=\beta+1}^{\alpha}\binom{m+\Delta_A-1}{\gamma-\beta}\Big([A,B]_\gamma\Big)_{m+n} + \sum_{j=0}^{\infty}\binom{\beta-1}{j}(-1)^j \\ &\times\Big(A_{m+\beta-1-j}B_{n-\beta+1+j} + \mu_{AB}(-1)^{\alpha-\beta}B_{n-j}A_{m+j}\Big).\end{aligned} \tag{4.6}$$

There is some freedom in the choice of n and m (as long as $n+m$ is not affected). If $m \in -\Delta_A + \mathbb{Z}$, then the freedom can be used to simplify the formula: choose $m = -\Delta_A + 1$, then the first sum in (4.6) vanishes.

We should stress that all the infinite sums in the formulae above are nicely ordered in the sense that large positive modes are always from the right, so when applied to a state in a highest weight module the sum is truncated and becomes finite. Due to this fact we can use the generalized commutation relations in the computations on highest weight modules. The relation obtained from the operator product expansion $A(z)B(w)$ (even if includes formally infinite sums from both sides of the relation) should be used for exchanging the modes A_n and B_m. Although the calculations could be very complicated, they are very formal and can be held by a computer using the software for symbolic computations, like *Mathematica* (the one we have used). By exchanging the modes it should be possible to order them, i.e., a kind of Poincare–Birkhoff–Witt theorem should hold, but we do not know even how to choose the Poincare–Birkhoff–Witt basis.

5 Representation theory

Highest weight states are the states which are annihilated by positive modes of all the basic fields. Highest weight representations are obtained from the highest weight state by application of nonpositive field modes to it. The modes of the fields which belong to the $(0,0)$ charge $\mathbb{Z}_2 \times \mathbb{Z}_2$ sector ($T^{(\beta)}, \beta = 1,2,3$, and W) are always integer. The modes of the $G^{(\beta)}, U^{(\beta)}$ generators can be integer or half-integer depending on which state they are applied to. One can deduce from the generalized commutation relation (4.4) that the states in the highest weight module can be of four types. One is of the "NS-NS-NS" type, which means that all the $G_n^{(\beta)}, U_m^{(\beta)}$ $(\beta = 1,2,3)$ modes applied to it are half-integer: $n, m \in \mathbb{Z} + 1/2$. And three other are of "NS-Ramond-Ramond" type, which means that the modes of G, U fields from one sector should be half-integer, when applied to this state, and the modes of G, U fields from two other sectors should be integer. These four types of states correspond to the four elements of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group, and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading can be extended from the algebra to its representations in the following way. There will be highest weight states of four types: $|(0,0)\rangle, |(1,0)\rangle, |(0,1)\rangle, |(1,1)\rangle$, according to their $\mathbb{Z}_2 \times \mathbb{Z}_2$ charge. Then the charge of the state in the highest weight module is the sum (modulo 2) of charges of the highest weight state and the field modes applied to it. The states of $(0,0)$ charge are of course of "NS-NS-NS" type. And the states of $(1,0), (0,1), (1,1)$ charge are of "NS-Ramond-Ramond" type, if the G, U fields are in the same sector as the state then their modes are half-integer, if they are from the different sector then their modes are integer. To illustrate the above rule, we give an example of a valid state: $G_{-3}^{(3)} G_{-5/2}^{(1)} T_{-5}^{(3)} G_{-3/2}^{(3)} T_{-2}^{(2)} T_{-3}^{(1)} G_{-1}^{(1)} W_{-2} G_{-1/2}^{(2)} G_{-1/2}^{(1)} |(1,0)\rangle$, this state has the $\mathbb{Z}_2 \times \mathbb{Z}_2$ charge $(0,1)$.

Next we should discuss the zero modes. First we have to choose Cartan generators, a commuting set of zero modes. The eigenvalues of these operators on a highest weight state will be taken as weights labelling the highest weight state. It would be desirable to have the zero modes of the three Virasoro fields as Cartan generators, but unfortunately they do not commute, since according to (4.5)

$$[T_0^{(1)}, T_0^{(2)}] = W_0. \tag{5.1}$$

The maximal commuting set consists of two operators only: e.g., $T_0^{(3)}$ and $T_0^{(1)} + T_0^{(2)}$. We will label the highest weight representations by the total conformal weight, the eigenvalue of the total energy–momentum field (3.13), and the eigenvalue of one of the three Virasoro fields, say $T_0^{(3)}$:

$$\begin{aligned} A_n|h,a,q\rangle &= 0, \quad n > 0, \\ T_0|h,a,q\rangle &= h|h,a,q\rangle, \\ T_0^{(3)}|h,a,q\rangle &= a|h,a,q\rangle, \end{aligned} \tag{5.2}$$

where q denotes the $\mathbb{Z}_2 \times \mathbb{Z}_2$ charge of the highest weight state and A represents any basic field. There are two more zero modes coming from the $(0,0)$ sector generators. In the case $q = (0,0)$ there are no other zero modes, in the case $q = (1,0), (0,1)$, or $(1,1)$ there are four more zero modes coming from the G, U generators. Some linear combinations of zero modes (with h, a, q dependent coefficients) will also annihilate the highest weight state in (5.2).

6 Unitary models

All the generalized commutation relations are invariant under the following conjugation:

$$\begin{aligned} (G_n^{(\alpha)})^\dagger &= G_{-n}^{(\alpha)}, \quad (U_n^{(\alpha)})^\dagger = -U_{-n}^{(\alpha)}, \\ (T_n^{(\alpha)})^\dagger &= T_{-n}^{(\alpha)}, \quad (W_n)^\dagger = -W_{-n}, \end{aligned} \tag{6.1}$$

if the algebra parameter g is real. This conjugation is compatible with the standard conjugation on the three $N = 1$ superconformal subalgebras. We know that the $N = 1$ superconformal algebra has unitary representations either when the central charge $c \geq 3/2$, or when $c < 3/2$ at the following discrete set of values of the central charge:

$$c_p = \frac{3}{2} - \frac{12}{(p-1)(p+1)}, \quad p = 3, 4, 5, \ldots, \tag{6.2}$$

which correspond to the unitary minimal models of the $N = 1$ superconformal algebra. From this we can immediately deduce the restrictions on possible unitary models of the whole algebra. In our case, the $N = 1$ subalgebra central charge is connected to the coupling g by the formula (3.19). For real g we have $c \leq 3/2$. If $c = 3/2$, then $g^2 = 1/8$ and the total central charge (calculated from (3.15)) is $\mathbf{C} = 18/5$. If $c = c_p$, then there are two solutions for g^2 (and consequently for $\mathbf{C}$). Both solutions can be parametrized by the same formula but with different ranges for the parameter p:

$$g_p^2 = \frac{p+3}{8\,(p-3)},$$
$$\mathbf{C}_p = \frac{18}{5}\left(1 - \frac{4\,(p+11)}{(p+1)\,(5p-1)}\right), \tag{6.3}$$

where $p = 3, 4, 5, \ldots$ or $p = -3, -4, -5, \ldots$. In the case $p = 3$, the coupling g becomes formally infinite, but the algebra still makes sense, one has just to redefine the generators $G^{(\alpha)}$. The $N = 1$ subalgebra central charge c and the total central charge $\mathbf{C}$ both vanish in this case.

In fact the $p = 4$ model ($g^2 = 7/8, \mathbf{C} = 126/95$) is also excluded from the candidates for the unitary models, since the central charge of the Virasoro algebra generated by the field $T - T^{(3)}$ is equal to $126/95 - 7/10 = 119/190$, it is less than 1, but does not belong to the series of values of the central charge for the Virasoro algebra minimal models.

We should stress that we have no proof that the algebra indeed has unitary representations except two models for which we know explicit realization in terms of unitary fields. These realizations are described in the next section.

7 Explicit realizations

7.1 sl(3) fermions × affine so(3) on level 4

Here we present the construction of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ superconformal algebra at the central charge $\mathbf{C} = 18/5$ in terms of sl(3) fermions and the so(3) affine Kac–Moody algebra on level 4. This construction is in a sense a $\mathbb{Z}_2 \times \mathbb{Z}_2$ analogue of the realization of the standard $N = 1$ superconformal algebra at the central charge $c = 3/2$ in terms of one free boson and one free fermion.

The sl(3) fermion system is described in detail in [9]. We will briefly recall its definition here. It is also a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded algebra of parafermionic type, but the conformal dimensions of the main generating fields are equal to 1/2 and not to 3/2 like in the case of $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra. The algebra is generated by three fermion fields $\psi^{(\alpha)}$, $\alpha = 1, 2, 3$. The operator product expansion of each field with itself is the standard free fermion relation:

$$\psi^{(\alpha)}(z)\psi^{(\alpha)}(w) = \frac{1}{z-w} + O(z-w). \tag{7.1}$$

The operator product expansion of two different fields gives the third one:

$$\psi^{(\alpha)}(z)\psi^{(\beta)}(w) = \frac{c_{\alpha,\beta}\psi^{(\gamma)}(w)}{(z-w)^{1/2}} + O((z-w)^{1/2}), \quad \alpha \neq \beta \neq \gamma. \tag{7.2}$$

The fields in the operator product expansion are exchanged using our general prescription:

$$\begin{gathered} \psi^{(\alpha)}(z)\psi^{(\alpha)}(w) = -\psi^{(\alpha)}(w)\psi^{(\alpha)}(z), \\ \psi^{(\alpha)}(z)\psi^{(\beta)}(w)(z-w)^{1/2} = \mu_{\alpha,\beta}\psi^{(\beta)}(w)\psi^{(\alpha)}(z)(w-z)^{1/2}, \quad \alpha \neq \beta. \end{gathered} \tag{7.3}$$

The commutation factors can be obtained exactly in the same way as the commutation factors of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra (see section 3), the result is:

$$\mu_{1,2} = \mu_{2,3} = \mu_{3,1} = -\mathrm{i} = -\mu_{2,1} = -\mu_{3,2} = -\mu_{1,3}. \tag{7.4}$$

The structure constants are determined in [9] using Jacobi identities:

$$c_{1,2} = c_{2,3} = c_{3,1} = \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2}}, \quad c_{2,1} = c_{3,2} = c_{1,3} = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{2}}. \tag{7.5}$$

The sl(3) fermion model is given by the following coset construction [6]:

$$\frac{\mathrm{sl}(3)_2}{u(1)^2}. \tag{7.6}$$

The second part of our construction is the so(3) affine vertex algebra. It is also generated by three fields, and the algebra is also $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded. The fields $J^{(\alpha)}(z)$ are of conformal dimension 1, the defining operator product expansion is:

$$J^{(\alpha)}(z)J^{(\beta)}(w) = \frac{k\,\delta_{\alpha,\beta}}{(z-w)^2} + \frac{\mathrm{i}\epsilon_{\alpha\beta\gamma}J^{(\gamma)}(w)}{z-w} + O\big((z-w)^0\big). \tag{7.7}$$

The sl(3) fermions and the affine currents commute:

$$\psi^{(\alpha)}(z)J^{(\beta)}(w) = J^{(\beta)}(w)\psi^{(\alpha)}(z) = O\big((z-w)^0\big). \tag{7.8}$$

The superconformal generators $G^{(\alpha)}$ of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra are expressed as products of corresponding sl(3) fermions and affine currents:

$$G^{(\alpha)}(z) = \frac{1}{\sqrt{k}}\,\psi^{(\alpha)}(z)J^{(\alpha)}(z). \tag{7.9}$$

Then the g-coupling of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra is equal $g = 1/\sqrt{2k}$.

The Virasoro field associated with $G^{(\alpha)}(z)$ is

$$T^{(\alpha)} = \frac{1}{2k}\left[J^{(\alpha)}, J^{(\alpha)}\right]_0 + \frac{1}{2}\left[\psi^{(\alpha)}, \psi^{(\alpha)}\right]_{-1} = \frac{1}{2k}:J^{(\alpha)}J^{(\alpha)}: - \frac{1}{2}:\psi^{(\alpha)}\partial\psi^{(\alpha)}:. \tag{7.10}$$

So we see that it is the (free boson) $\times$ (free fermion) realization of the $N = 1$ superconformal algebra, the central charge of which is $c = 3/2$. From the relation (3.19) we obtain the coupling $g^2 = 1/8$, which means that we have to fix the level of the so(3) affine algebra to $k = 4$.

The total energy–momentum field is the sum of energy–momentum fields of the sl(3) fermion system and the so(3) affine algebra on level 4:

$$T = \frac{1}{10}\sum_{\alpha=1}^{3} :J^{(\alpha)}J^{(\alpha)}: - \frac{2}{5}\sum_{\alpha=1}^{3} :\psi^{(\alpha)}\partial\psi^{(\alpha)}:. \tag{7.11}$$

The central charge is the sum of the central charge of the sl(3) fermion system (6/5) and the central charge of the so(3) affine vertex algebra on level 4 (12/5):

$$\mathbf{C} = \frac{6}{5} + \frac{12}{5} = \frac{18}{5}, \tag{7.12}$$

as one would expect.

The $U^{(\alpha)}$and the W fields of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra can be expressed in terms of sl(n) fermions and affine currents using the operator product expansion relations (3.16) and (3.22), respectively.

7.2 Two free bosons

This is a realization of the $p = 5$ unitary model in the series (6.3). The coupling is $g^2 = 1/2$, the central charge of the $N = 1$ superconformal subalgebras is $c = 1$ and the total central charge is $\mathbf{C} = 2$. We take two free bosons ϕ_1 and ϕ_2:

$$\phi_i(z)\phi_j(w) = -\delta_{i,j}\log(z - w), \tag{7.13}$$

and built from them the vertex operators

$$\Gamma_\alpha(z) = c_\alpha :\mathrm{e}^{\mathrm{i}(\alpha,\phi)(z)}:. \tag{7.14}$$

α is a vector in 2-dimensional Euclidean space, and $(\cdot\,,\cdot)$ is the standard scalar product in this space. The factors c_α are the so-called cocycles, satisfying a 2-cocycle algebra, the exact definition of which is not important here.

The three superconformal generators are given by

$$G^{(\alpha)}(z) = \frac{\Gamma_\alpha(z) + \Gamma_{-\alpha}(z)}{\sqrt{2}}, \tag{7.15}$$

where α is the root of the sl(3) algebra normalized to $(\alpha, \alpha) = 3$.

The fields $T^{(\alpha)}(z)$ are obtained from the operator product expansion (A.6) and coincide with the well known energy–momentum field for the free boson system:

$$T^{(\alpha)}(z) = -\frac{:(\alpha, \partial\phi)(\alpha, \partial\phi):(z)}{2(\alpha, \alpha)}. \tag{7.16}$$

The total energy–momentum field is also the standard energy–momentum field of the system of two free bosons:

$$T(z) = -\frac{1}{2}\Big(:\partial\phi_1\partial\phi_1:(z) + :\partial\phi_2\partial\phi_2:(z)\Big). \tag{7.17}$$

The fields $U^{(\alpha)}(z)$ are obtained as

$$U^{(\alpha)} \sim :(\gamma, \partial\phi)\Big(\Gamma_\alpha - \Gamma_{-\alpha}\Big):, \tag{7.18}$$

where γ is a vector, which is orthogonal to the root α.

The field W is given by

$$W = \frac{\sqrt{3}}{8}\Big(\partial\phi_1\partial^2\phi_2 - \partial\phi_2\partial^2\phi_1\Big). \tag{7.19}$$

8 Discussion and speculations

We constructed a new chiral algebra of parafermionic type: the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra. The full set of operator product expansions is presented in Appendix A. The algebra has one continuous parameter: the coupling g, and contains three $N = 1$ superconformal subalgebras of the same central charge. We also discussed briefly the representation theory of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra. However the full description of the representation theory remains an open problem, in particular it would be important to understand what is the Poincare–Birkhoff–Witt basis for the highest weight modules of the algebra. We also obtained restrictions on the possible unitary models of the algebra, and provided two examples of explicit unitary realizations of the algebra.

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra is a generalization of the $\mathbb{Z}_2$ graded $N = 1$ superconformal algebra. Higher generalizations to the case of ${\mathbb{Z}_2}^n$ grading are possible. However in the case $n > 2$, there are less dimension-3/2 generating fields than $2^n - 1$, i.e., not every element (different from identity) of the ${\mathbb{Z}_2}^n$ group has an $N = 1$ superconformal generator associated with it. The natural structure in this case is the A_n-type root system. One should associate with every pair of opposite roots (the root direction) the standard $\mathbb{Z}_2$ graded $N = 1$ superconformal algebra, generated by $G^{(\alpha)}(z)$ and $T^{(\alpha)}(z)$, where α is the root direction. There are $n(n+1)/2$ such root directions, which is much less than $2^n - 1$ for greater n. Then the standard $\mathbb{Z}_2$ graded $N = 1$ superconformal algebra corresponds to the A_1 root system and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra, described in this paper, corresponds to the A_2 root system. Moreover this approach can be extended to any root system of A-D-E type. In fact the structure of relations between the $G^{(\alpha)}$ fields is the same as that of the operator product expansions of the so-called simply laced fermions defined in our previous work (Section 7 of [9]). The most singular term in the operator product expansions will be

$$\begin{aligned} &G^{(\alpha)}(z)G^{(\beta)}(w) \\ &\quad = \begin{cases} O((z-w)^0), & \alpha \text{ and } \beta \text{ are orthogonal,} \\ \dfrac{c_{\alpha,\beta}G^{(\alpha+\beta)}(w)}{(z-w)^{3/2}} + O((z-w)^{-1/2}), & \alpha \text{ and } \beta \text{ are not orthogonal.} \end{cases} \end{aligned} \tag{8.1}$$

Again we need many more fields to close the algebra: of conformal dimensions $5/2, 3$ and maybe of higher dimensions. But this is a subject for a separate publication. We want just to make a few predictions here. Since the root system has many A_2 root subsystems, the algebra has many subalgebras, which are the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebras. So we expect that there will be only one free parameter, the coupling g, which is connected to the central charge of the $N = 1$ superconformal subalgebras by the same relation (3.19). The total energy–momentum field is

$$T(z) = \frac{1 + 4g^2}{1 + (3h^\vee - 2)g^2} \sum_\alpha T^{(\alpha)}(z), \tag{8.2}$$

where $h^\vee$ is the dual Coxeter number of the simply laced algebra $\mathfrak{g}$, the root system of which is used in the construction of our parafermionic algebra. The total central charge is

$$\mathbf{C} = \frac{54g^2 R_{\mathfrak{g}}}{(1 + (3h^\vee - 2)g^2)(1 + 16g^2)}, \tag{8.3}$$

where $R_{\mathfrak{g}}$ is the number of root directions of the $\mathfrak{g}$ root system. Substituting the values of g corresponding to the unitary models from (6.3) (like in

Section 6), we get two series of central charge:

$$\mathbf{C}_p = \frac{6R_{\mathfrak{g}}(p^2-9)}{(p+1)\left((h^\vee+2)p+3h^\vee-10\right)}, \qquad \begin{array}{c} p = 3,4,5,6,\dots \\ \text{or} \\ p=-3,-4,-5,-6,\dots \end{array} \tag{8.4}$$

Unitary representations can appear only at these values of central charge or at the limit $p \to \pm\infty$ of these two series:

$$\mathbf{C} = \frac{6R_{\mathfrak{g}}}{h^\vee+2}. \tag{8.5}$$

In the case $\mathfrak{g} = \mathrm{sl}(n)$ this "limit" model is realized by the (sl(n) fermions) $\times$ (so(n) affine vertex algebra on level 4), exactly in the same way as described in Section 7.1.

We would like also to announce here the $N = 2$ superconformal algebras of parafermionic type. These are also associated with the root systems of a simple Lie algebra $\mathfrak{g}$ of the A-D-E type. But now there is a dimension-3/2 superconformal generator $G^{(\alpha)}(z)$ for every root α. The fields $G^{(\alpha)}(z)$ and $G^{(-\alpha)}(z)$ together with dimension-1 and dimension-2 fields $J^{(\alpha)}(z)$, $T^{(\alpha)}(z)$ form the standard $N = 2$ superconformal algebra. If $\alpha+\beta$ is a root, then the operator product expansion of $G^{(\alpha)}(z)$ and $G^{(\beta)}(w)$ is

$$G^{(\alpha)}(z)G^{(\beta)}(w) = \frac{c_{\alpha,\beta}G^{(\alpha+\beta)}(w)}{(z-w)^{3/2}} + O\big((z-w)^{-1/2}\big), \tag{8.6}$$

and it is not singular if $\alpha+\beta$ is not a root. The full field content and the operator product expansions defining the algebras are not known yet even in the sl(3) case, they are under investigation and will be reported in [11]. However we already know the minimal models of these simply laced $N = 2$ superconformal algebras. They are constructed using the idea from [12], where the minimal models of the sl(2) $N = 2$ superconformal algebra are constructed from $\mathbb{Z}_N$ parafermions and one free boson. Our minimal models are given by

$$\frac{\mathfrak{g}_k}{u(1)^r} \times u(1)^r, \tag{8.7}$$

there k is the level of the affine vertex algebra $\mathfrak{g}$ of A-D-E type, and r is its rank. The first part is generated by the Gepner parafermions [6], and the $u(1)^r$ part is just r free bosons. The main generators are obtained as

$$G^{(\alpha)}(z) = \psi_\alpha(z)\Gamma_{\alpha\sqrt{2+k}/\sqrt{2k}}(z), \tag{8.8}$$

ψ_α is the parafermion corresponding to the root α, the root is normalized to $(\alpha,\alpha) = 2$, the vertex operators Γ are defined in Section 7.2.

The formula for the total central charge of these unitary minimal models coincide with the formula for the central charge of the affine vertex algebra $\mathfrak{g}$:

$$\mathbf{C}_k(\mathfrak{g}) = \frac{k\,\mathrm{dim}\mathfrak{g}}{k + h^\vee}. \tag{8.9}$$

The algebras described in this paper may have interesting applications to string theory.

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A List of operator product expansions

We list here all the algebraic relations between the basic fields of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra. There are 10 basic fields: $G^{(\alpha)}, T^{(\alpha)}, U^{(\alpha)}, W,\ \alpha = 1, 2, 3$, of conformal dimensions 3/2, 2, 5/2 and 3, respectively. They are primary fields with respect to the total energy–momentum field

$$T = \frac{1 + 4g^2}{1 + 7g^2}\left(T^{(1)} + T^{(2)} + T^{(3)}\right). \tag{A.1}$$

This field $T(z)$ satisfies the Virasoro algebra with central charge

$$\mathbf{C} = \frac{162g^2}{(1 + 7g^2)\,(1 + 16g^2)}. \tag{A.2}$$

The central charge of the three $N = 1$ superconformal subalgebras is expressed in terms of the coupling g as

$$c = \frac{54g^2}{(1 + 4g^2)(1 + 16g^2)}. \tag{A.3}$$

In the formulae below we use some convenient notation, the following two-index symbol:

$$\sigma_{\alpha\beta} = \begin{cases} 0, & \alpha = \beta, \\ \epsilon_{\alpha\beta\gamma}, & \alpha \neq \beta \neq \gamma, \end{cases} \qquad \alpha, \beta, \gamma = 1, 2, 3, \tag{A.4}$$

(i.e., $\sigma_{12} = \sigma_{23} = \sigma_{31} = 1$ and $\sigma_{21} = \sigma_{32} = \sigma_{13} = -1$) and the following combination of the Virasoro fields:

$$\Theta^{(\alpha)}(w) = \sum_{\gamma=1}^{3} \sigma_{\alpha\gamma} T^{(\gamma)}(w). \tag{A.5}$$

In all the operator product expansions below the fields on the right hand side of the equations are taken at point w. The indices α, β, γ inside one equation are all different. There is no summation on repeated indices, unless the sum is explicitly written.

The operator product expansions defining the $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded $N = 1$ superconformal algebra read

$$G^{(\alpha)}(z)G^{(\alpha)}(w) = \frac{1}{(z-w)^3} + \frac{\frac{3}{c}T^{(\alpha)}}{z-w} + O\big((z-w)^0\big), \tag{A.6}$$

$$T^{(\alpha)}(z)G^{(\alpha)}(w) = \frac{\frac{3}{2}G^{(\alpha)}}{(z-w)^2} + \frac{\partial G^{(\alpha)}}{z-w} + O\big((z-w)^0\big), \tag{A.7}$$

$$T^{(\alpha)}(z)T^{(\alpha)}(w) = \frac{c/2}{(z-w)^4} + \frac{2T^{(\alpha)}}{(z-w)^2} + \frac{\partial T^{(\alpha)}}{z-w} + O\big((z-w)^0\big), \tag{A.8}$$

$$\begin{aligned} G^{(\alpha)}(z)G^{(\beta)}(w) = {} & \mathrm{e}^{\frac{\mathrm{i}\pi}{4}\sigma_{\alpha\beta}} g \left(\frac{G^{(\gamma)}}{(z-w)^{3/2}} + \frac{\frac{1}{2}\partial G^{(\gamma)} + \epsilon_{\alpha\beta\gamma}U^{(\gamma)}}{(z-w)^{1/2}} \right) \\ & + O\big((z-w)^{1/2}\big), \end{aligned} \tag{A.9}$$

$$\begin{aligned} T^{(\alpha)}(z)G^{(\beta)}(w) = {} & \frac{9g^2}{4\,(1+4g^2)} \left(\frac{G^{(\beta)}}{(z-w)^2} + \frac{\frac{2}{3}\partial G^{(\beta)} - \sigma_{\alpha\beta}\frac{4}{3}U^{(\beta)}}{z-w} \right) \\ & + O\big((z-w)^0\big), \end{aligned} \tag{A.10}$$

$$\begin{aligned} T^{(\alpha)}(z)T^{(\beta)}(w) = {} & \frac{\frac{c^2(1+16g^2)}{72}}{(z-w)^4} + \frac{\frac{3g^2}{2(1+4g^2)}\left(2T^{(\alpha)} + 2T^{(\beta)} - 2T^{(\gamma)}\right)}{(z-w)^2} + \\ & + \frac{\frac{3g^2}{2(1+4g^2)}\left(\partial T^{(\alpha)} + \partial T^{(\beta)} - \partial T^{(\gamma)}\right) + \sigma_{\alpha\beta}W}{z-w} \\ & + O\big((z-w)^0\big), \end{aligned} \tag{A.11}$$

$$\begin{aligned} G^{(\alpha)}(z)U^{(\alpha)}(w) = {} & \frac{-\frac{3}{c}\left(\sum_{\gamma=1}^{3}\sigma_{\alpha\gamma}T^{(\gamma)}\right)}{(z-w)^2} \\ & + \frac{-\frac{3}{4c}\left(\sum_{\gamma=1}^{3}\sigma_{\alpha\gamma}\partial T^{(\gamma)}\right) - \frac{27}{c^2(1+16g^2)}W}{z-w} + O\big((z-w)^0\big), \end{aligned} \tag{A.12}$$

$$\begin{aligned} G^{(\alpha)}(z)U^{(\beta)}(w) = {} & \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\sigma_{\alpha\beta}}}{4g} \Bigg(\frac{-(2+5g^2)\sigma_{\alpha\beta}G^{(\gamma)}}{(z-w)^{5/2}} \\ & + \frac{-\frac{(2+5g^2)}{6}\sigma_{\alpha\beta}\partial G^{(\gamma)} - \frac{2+17g^2}{3}U^{(\gamma)}}{(z-w)^{3/2}} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{g^2\sigma_{\alpha\beta}\partial^2 G^{(\gamma)} + 2g^2\partial U^{(\gamma)} - \frac{6}{c}\sigma_{\alpha\beta}\big[T^{(\alpha)}, G^{(\gamma)}\big]_0}{(z-w)^{1/2}}\Bigg) \\
&+ \frac{-\frac{1}{4}\sigma_{\alpha\beta}\big[G^{(\alpha)}, G^{(\beta)}\big]_{-1/2}}{(z-w)^{1/2}} + O\big((z-w)^{1/2}\big), \qquad \text{(A.13)}
\end{aligned}
$$

$$
\begin{aligned}
T^{(\alpha)}(z)U^{(\alpha)}(w) = {}& \frac{\frac{1+16g^2}{2(1+4g^2)}U^{(\alpha)}}{(z-w)^2} + \\
&+ \frac{\frac{1+16g^2}{2(1+4g^2)}\partial U^{(\alpha)} - \frac{1+16g^2}{12g^2}\sum_{\gamma=1}^{3}\sigma_{\alpha\gamma}\big[T^{(\gamma)}, G^{(\alpha)}\big]_0}{z-w} \\
&+ O\big((z-w)^0\big), \qquad \text{(A.14)}
\end{aligned}
$$

$$
\begin{aligned}
T^{(\alpha)}(z)U^{(\beta)}(w) = {}& \frac{1}{4(1+4g^2)}\Bigg(\frac{-3(2+5g^2)\sigma_{\alpha\beta}G^{(\beta)}}{(z-w)^3} \\
&+ \frac{-(2+5g^2)\sigma_{\alpha\beta}\partial G^{(\beta)} + (4+19g^2)U^{(\beta)}}{(z-w)^2} \\
&+ \frac{3g^2\sigma_{\alpha\beta}\partial^2 G^{(\beta)} - 6g^2\partial U^{(\beta)} - \frac{18}{c}\sigma_{\alpha\beta}\big[T^{(\alpha)}, G^{(\beta)}\big]_0}{z-w}\Bigg) \\
&+ \frac{-\frac{e^{\frac{i\pi}{4}\sigma_{\alpha\beta}}}{2g}\sigma_{\alpha\beta}\big[G^{(\alpha)}, G^{(\gamma)}\big]_{-1/2}}{z-w} + O\big((z-w)^0\big), \qquad \text{(A.15)}
\end{aligned}
$$

$$
\begin{aligned}
U^{(\alpha)}(z)U^{(\alpha)}(w) = {}& \frac{-\frac{2+5g^2}{4g^2}}{(z-w)^5} + \frac{-\frac{(1+g^2)(1+7g^2)}{c\,g^2(1+4g^2)}T + \frac{2-13g^2}{4c\,g^2}T^{(\alpha)}}{(z-w)^3} \\
&+ \frac{-\frac{(1+g^2)(1+7g^2)}{2c\,g^2(1+4g^2)}\partial T + \frac{2-13g^2}{8c\,g^2}\partial T^{(\alpha)}}{(z-w)^2} \\
&+ \frac{\frac{(1+7g^2)(8g^2-1)}{16c\,g^2(1+4g^2)}\partial^2 T - \frac{9}{8c}\partial^2 T^{(\alpha)}}{z-w} \\
&+ \frac{-\frac{9}{4c^2g^2}\left(\big[T^{(\beta)}, T^{(\gamma)}\big]_0 + \big[T^{(\gamma)}, T^{(\beta)}\big]_0\right)}{z-w} \\
&+ \frac{-\frac{1+4g^2}{6g^2}\left(\big[G^{(\beta)}, G^{(\beta)}\big]_{-1} + \big[G^{(\gamma)}, G^{(\gamma)}\big]_{-1}\right) - \frac{1}{4}\big[G^{(\alpha)}, G^{(\alpha)}\big]_{-1}}{z-w} \\
&+ \frac{-\frac{(1+10g^2)}{6g^2}\sum_{\delta=1}^{3}\sigma_{\alpha\delta}\big[G^{(\delta)}, U^{(\delta)}\big]_0}{z-w} + O\big((z-w)^0\big), \\
&\qquad \text{(A.16)}
\end{aligned}
$$

$$
\begin{aligned}
U^{(\alpha)}(z)U^{(\beta)}(w) = {} & \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{4}\sigma_{\alpha\beta}}}{12g}\left(\frac{\frac{\left(2+5g^2\right)\left(2+17g^2\right)}{4g^2}G^{(\gamma)}}{(z-w)^{7/2}}\right.\\
& + \frac{\frac{\left(2+5g^2\right)\left(2+17g^2\right)}{8g^2}\partial G^{(\gamma)} + \frac{251g^4+4g^2-4}{12g^2}\sigma_{\alpha\beta}U^{(\gamma)}}{(z-w)^{5/2}}\\
& + \frac{-\frac{4+19g^2}{4}\partial^2 G^{(\gamma)} + \frac{71g^4+7g^2-1}{6g^2}\sigma_{\alpha\beta}\partial U^{(\gamma)}}{(z-w)^{3/2}}\\
& + \frac{\frac{9\left(1+g^2\right)}{2\,\mathbf{C}\,g^2}\left[T, G^{(\gamma)}\right]_0 + \frac{3\left(11g^2-1\right)}{2\,c\,g^2}\left[T^{(\gamma)}, G^{(\gamma)}\right]_0}{(z-w)^{3/2}}\\
& + \frac{\left(\begin{gathered}-\frac{1+10g^2}{2}\partial^3 G^{(\gamma)} + \frac{3\left(1+4g^2\right)}{2}\sigma_{\alpha\beta}\partial^2 U^{(\gamma)}\\ +\frac{324}{c^2(1+16g^2)}\sigma_{\alpha\beta}\left[G^{(\gamma)}, W\right]_0\end{gathered}\right)}{(z-w)^{1/2}}\\
& + \frac{\left(\begin{gathered}\frac{81}{c^2(1+16g^2)}\partial\left[(T^{(\alpha)}+T^{(\beta)}), G^{(\gamma)}\right]_0\\ +\frac{9}{c}\partial\left[(T^{(\beta)}+T^{(\gamma)}), G^{(\gamma)}\right]_0\end{gathered}\right)}{(z-w)^{1/2}}\\
& \left. + \frac{\frac{9}{c}\left[(T^{(\alpha)}-T^{(\beta)}), G^{(\gamma)}\right]_{-1} + \frac{18}{c}\sigma_{\alpha\beta}\left[T^{(\gamma)}, U^{(\gamma)}\right]_0}{(z-w)^{1/2}}\right)\\
& + \frac{-\frac{8+11g^2}{48g^2}\left[G^{(\alpha)}, G^{(\beta)}\right]_{-1/2}}{(z-w)^{3/2}}\\
& + \frac{\frac{2+5g^2}{16g^2}\left[G^{(\alpha)}, G^{(\beta)}\right]_{-3/2} - \frac{1}{8g^2}\partial\left[G^{(\alpha)}, G^{(\beta)}\right]_{-1/2}}{(z-w)^{1/2}}\\
& + \frac{\frac{1+13g^2}{12g^2}\sigma_{\alpha\beta}\left(\left[G^{(\alpha)}, U^{(\beta)}\right]_{-1/2} + \left[U^{(\alpha)}, G^{(\beta)}\right]_{-1/2}\right)}{(z-w)^{1/2}}\\
& + O\left((z-w)^{1/2}\right),
\end{aligned} \tag{A.17}
$$

$$
\begin{aligned}
G^{(\alpha)}(z)W(w) = {} & \frac{2\left(1+4g^2\right)^2}{3g^2}\left(\frac{-2\left(1+g^2\right)U^{(\alpha)}}{(z-w)^2}\right.\\
& \left. + \frac{-\left(1+10g^2\right)\partial U^{(\alpha)} + \frac{9}{c}\sum_{\gamma=1}^{3}\sigma_{\alpha\gamma}\left[T^{(\gamma)}, G^{(\alpha)}\right]_0}{z-w}\right)\\
& + O\left((z-w)^0\right),
\end{aligned} \tag{A.18}
$$

$$
T^{(\alpha)}(z)W(w) = \frac{\frac{6g^2\left(1+g^2\right)}{\left(1+4g^2\right)^2}\Theta^{(\alpha)}}{(z-w)^3} + \frac{\frac{3g^2\left(1+g^2\right)}{2\left(1+4g^2\right)^2}\partial\Theta^{(\alpha)} + \frac{3c}{\mathbf{C}}W}{(z-w)^2}
$$

$$
+ \frac{\frac{1+16g^2}{2(1+4g^2)}\big[T^{(\alpha)}, \Theta^{(\alpha)}\big]_0 + \frac{54g^4}{(1+4g^2)^3}\big[G^{(\alpha)}, U^{(\alpha)}\big]_0}{z-w}
$$

$$
+ \frac{-\frac{9g^4}{2(1+4g^2)^2}\partial^2\Theta^{(\alpha)} - \frac{3g^2}{1+4g^2}\partial W}{z-w} + O\big((z-w)^0\big), \quad \text{(A.19)}
$$

$U^{(\alpha)}(z)W(w)$

$$
\begin{aligned}
&= \frac{1}{(1+4g^2)^2}\Bigg(\frac{\frac{3(1+g^2)(2+5g^2)}{4}G^{(\alpha)}}{(z-w)^4} + \frac{\frac{(1+g^2)(2+5g^2)}{4}\partial G^{(\alpha)}}{(z-w)^3} \\
&+ \frac{-\frac{15g^2(1+g^2)}{8}\partial^2 G^{(\alpha)} + \frac{81(1-2g^2)}{8\mathbf{C}}\big[T, G^{(\alpha)}\big]_0 + \frac{27(6g^2-1)}{8c}\big[T^{(\alpha)}, G^{(\alpha)}\big]_0}{(z-w)^2} \\
&+ \frac{-\frac{(1+4g^2)(1+10g^2)}{8g}\Big(\mathrm{e}^{-(\mathrm{i}\pi/4)\sigma_{\beta\gamma}}\big[G^{(\beta)}, G^{(\gamma)}\big]_{-1/2} + \mathrm{e}^{(\mathrm{i}\pi/4)\sigma_{\beta\gamma}}\big[G^{(\gamma)}, G^{(\beta)}\big]_{-1/2}\Big)}{(z-w)^2} \\
&+ \frac{-\frac{2+49g^2+128g^4}{32}\partial^3 G^{(\alpha)} + \frac{27(1-2g^2)}{8\mathbf{C}}\partial\big[T, G^{(\alpha)}\big]_0 + \frac{81g^2}{4\mathbf{C}}\big[T, G^{(\alpha)}\big]_{-1}}{z-w} \\
&+ \frac{\frac{9(14g^2-1)}{8c}\partial\big[T^{(\alpha)}, G^{(\alpha)}\big]_0 - \frac{81g^2}{4c}\big[T^{(\alpha)}, G^{(\alpha)}\big]_{-1} - \frac{27g^2}{2c}\big[\Theta^{(\alpha)}, U^{(\alpha)}\big]_0}{z-w} \\
&+ \frac{-\sigma_{\beta\gamma}\frac{3g(1+4g^2)}{2}\Big(\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}\sigma_{\beta\gamma}}\big[G^{(\beta)}, U^{(\gamma)}\big]_{-1/2} - \mathrm{e}^{(\mathrm{i}\pi/4)\sigma_{\beta\gamma}}\big[G^{(\gamma)}, U^{(\beta)}\big]_{-1/2}\Big)}{z-w} \\
&+ \frac{\frac{3g(1+4g^2)}{16}\Big(\mathrm{e}^{-(\mathrm{i}\pi/4)\sigma_{\beta\gamma}}\partial\big[G^{(\beta)}, G^{(\gamma)}\big]_{-1/2} + \mathrm{e}^{(\mathrm{i}\pi/4)\sigma_{\beta\gamma}}\partial\big[G^{(\gamma)}, G^{(\beta)}\big]_{-1/2}\Big)}{z-w}\Bigg) \\
&+ O\big((z-w)^0\big), \qquad \text{(A.20)}
\end{aligned}
$$

$W(z)W(w)$

$$
\begin{aligned}
&= \frac{-3g^2}{(1+4g^2)^3}\Bigg(\frac{\frac{c(1+g^2)(2+5g^2)}{2}}{(z-w)^6} + \frac{\frac{3c(1+g^2)(2+5g^2)}{\mathbf{C}}T}{(z-w)^4} + \frac{\frac{3c(1+g^2)(2+5g^2)}{2\mathbf{C}}\partial T}{(z-w)^3} \\
&+ \frac{\frac{9c(2g^2-1)^2}{16\mathbf{C}}\partial^2 T + \frac{81c(1-8g^2)}{4\mathbf{C}^2}\big[T, T\big]_0 + \frac{9(20g^2-1)}{4c}\sum_{\alpha=1}^3\big[T^{(\alpha)}, T^{(\alpha)}\big]_0}{(z-w)^2} \\
&+ \frac{-9g^2\sum_{\alpha=1}^3\big[G^{(\alpha)}, G^{(\alpha)}\big]_{-1}}{(z-w)^2}\Bigg) + \frac{\big[W, W\big]_1}{z-w} + O\big((z-w)^0\big), \qquad \text{(A.21)}
\end{aligned}
$$

where $\big[W, W\big]_1 = \frac{1}{2}\partial\big[W, W\big]_2 - \frac{1}{24}\partial^3\big[W, W\big]_4$.

The order of fields in the operator product expansions above is exchanged using the following rule:

$$\begin{aligned} B(z)A(w) &= A(w)B(z), \\ R^{(\alpha)}(z)S^{(\alpha)}(w) &= -S^{(\alpha)}(w)R^{(\alpha)}(z), \\ R^{(\alpha)}(z)S^{(\beta)}(w)(z-w)^{3/2} &= \mathrm{i}\sigma_{\alpha,\beta}S^{(\beta)}(w)R^{(\alpha)}(z)(w-z)^{3/2}, \quad \alpha \neq \beta, \end{aligned} \tag{A.22}$$

where B denotes any field from the set $\{T^{(1)}, T^{(2)}, T^{(3)}, W\}$, R and S stand for any field from the set $\{G, U\}$ and A is any of the 10 basic fields.

The generalized Jacobi identities (2.7) are satisfied modulo the following null field condition:

$$27\partial W + (1+16g^2)c^2 \sum_{\alpha=1}^{3} \left[G^{(\alpha)}, U^{(\alpha)}\right]_0 = 0. \tag{A.23}$$

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