Asian J. Math.
Vol. 13, No. 1, pp. 007–014, March 2009

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# THE POINCARÉ RECURRENCE PROBLEM OF INVISCID INCOMPRESSIBLE FLUIDS*

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**Abstract.** Nadirashvili presented a beautiful example showing that the Poincaré recurrence does not occur near a particular solution to the 2D Euler equation of inviscid incompressible fluids. Unfortunately, Nadirashvili's setup of the phase space is not appropriate, and details of the proof are missing. This note fixes that.

**Key words.** Poincaré recurrence, Euler equation of fluids, vorticity, Biot-Savart law, single layer potential.

**AMS subject classifications.** Primary 37, 76; Secondary 35, 34

## 1. Introduction.

2D Euler equation of inviscid incompressible fluids is the key in understanding the chaotic (turbulent) solutions of 2D Navier-Stokes equation in the infinite Reynolds number limit [8] [7] [12] [3] [2] [14]. Moreover, it is the simplest fluid equation. There are two distinct phenotypes of chaos: sensitive dependence on initial data, and recurrence. Sensitive dependence on initial data can often be proved by a shadowing technique or Smale horseshoe construction [8]. To accomplish this, one often needs detailed information on the dynamics (e.g. existence of a homoclinic orbit). This poses tremendous analytical challenge for the 2D Euler equation [7]. On the other hand, the well-known Poincaré recurrence theorem was proved primarily from the first-principle of measure theory under extremely general conditions of finite measure space and measure-preserving map. Therefore, it seems to have a good chance of success even for 2D Euler equation. The serious complication comes from the fact that natural finite dimensional measures (e.g. Gibbs measure) do not have good counterparts in infinite dimensions. It is well-known that the kinetic energy and enstrophy are invariant under the 2D Euler flow. But it is difficult to use them to define finite measures in infinite dimensions. It seems possible to study the Poincaré recurrence problem directly from Banach norms rather than measures. Of course, the most exciting and challenging problem shall be the general description of Poincaré recurrence or non-recurrence for 2D Euler equation directly from Banach norms. This is our main research project [9]. For a special multi-connected fluid domain, Nadirashvili gave a counter-example of Poincaré recurrence [11]. In this note, we will fix the problems in the Nadirashvili's proof of Poincaré non-recurrence near a particular solution to the 2D Euler equation. Because Nadirashvili's is the first and a beautiful counter-example in this content, we feel that such a note is worth-while. We also feel that this counter-example depends heavily upon the nature of a multi-connected domain. Nevertheless, from studies on periodic domain [7], we notice that there may be a lot of heteroclinic solutions to the 2D Euler equation. Thus, finding particular initial conditions like in the Nadirashvili's example, which has neighborhood never returning, is not surprising. On the other hand, in [9], we prove that in the periodic domain, all the solutions to the 2D Euler equation returns repeatedly to some neighborhood which may not be the neighborhood of the initial condition. This recurrence is due to the fact that the possible heteroclinic solutions here are not like that of (2.1),

*Received October 3, 2007; accepted for publication July 11, 2008.
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they are constrained by the kinetic energy and enstrophy.

## 2. The Poincaré Recurrence Theorem.

THEOREM 2.1. *Let $(X, \Sigma, \mu)$ be a finite measure space and $f : X \mapsto X$ be a measure-preserving transformation. For any $E \in \Sigma$ ($\sigma$-algebra of subsets of $X$), the measure*

$$\mu(\{x \in E \mid \exists N,\ f^n(x) \notin E\ \forall n > N\}) = 0\ .$$

*That is, almost every point returns infinitely often.*

*Proof.* This proof is a little further clarification of that in:
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Let

$$A_n = \bigcup_{k=n}^{+\infty} f^{-k}(E)\ ,$$

then

$$E \subset A_0\ , \quad A_j \subset A_i \quad \forall i \leq j\ ,$$

and

$$A_j = f^{i-j}(A_i)\ .$$

Thus

$$\mu(A_i) = \mu(A_j) \quad \forall i, j \geq 0\ ,$$

and

$$\mu(A_0 - A_n) = \mu(A_0) - \mu(A_n) = 0\ , \quad \forall n\ .$$

We have

$$\mu(E - \bigcap_{n=1}^{+\infty} A_n) \leq \mu(A_0 - \bigcap_{n=1}^{+\infty} A_n) = \mu(\bigcup_{n=1}^{+\infty}(A_0 - A_n)) = 0\ .$$

Notice that

$$E - \bigcap_{n=1}^{+\infty} A_n = \{x \in E \mid \exists N,\ f^n(x) \notin E\ \forall n > N\}\ .$$

The theorem is proved. ☐

REMARK 2.2. The proof in [15] is incorrect. The geometric intuition of the Poincaré recurrence theorem is that in a finite measure space (or invariant subset), the images of a positive measure set under a measure-preserving map will have no room left but intersect the original set repeatedly.

The measure of the space $X$ being finite is crucial. For example, consider the two-dimensional Hamiltonian system of the pendulum

$$\dot{x} = y\ , \quad \dot{y} = -\sin x\ . \tag{2.1}$$

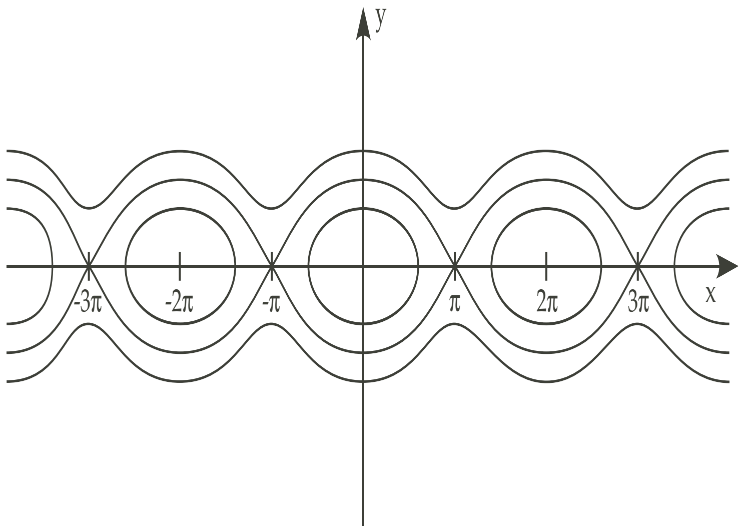


Fig. 1. *The phase plane diagram of the pendulum equation.*

Its phase plane diagram is shown in Figure 1. If the invariant region includes orbits outside the cat's eyes, then the measure of the region will not be finite, and the Poincaré recurrence theorem will not hold. One can see clearly that the orbits outside the cat's eyes will drift to infinity. The boundaries of the cat's eyes are called separatrices (heteroclinic orbits). For 2D Euler equation of inviscid incompressible fluids, it has been conjectured that unstable fixed points are connected by heteroclinic orbits [7]. So the Poincaré recurrence shall not be sought in the whole phase space.

## 3. The Poincaré Recurrence Problem of Inviscid Incompressible Fluids.

**3.1. Nadirashvili's Example.** In the original article [11] of Nadirashvili, the setup was incorrect, where a $C^1$ velocity space was taken as the phase space. The reason is that one needs at least $C^{1+\alpha}$ $(0 < \alpha < 1)$ initial data in velocity to get $C^1$ (in space) velocity solution of the 2D Euler equation [5] [6]. In general, $C^{1+\alpha}$ $(0 < \alpha < 1)$ initial data can lead to $C^{1+\beta}$ $(0 < \beta < \alpha < 1)$ solutions [6]. So $C^{1+\alpha}$ $(0 < \alpha < 1)$ is not a good space for a dynamical system study either. On the other hand, Nadirashvili's is a beautiful example; therefore, it should be set right. In this note, we will select an appropriate phase space.

Let $M$ be the annulus

$$M = \left\{ x \in \mathbb{R}^2 \mid 1 \leq |x| \leq 2 \right\} ,$$

$\Gamma_1$ and $\Gamma_2$ be the inner and outer boundaries, then $\partial M = \Gamma_1 \cup \Gamma_2$ (see Figure 2). Consider the 2D Euler equation in its vorticity form

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \ , \\ \omega = \ \text{curl } u \ , \quad \nabla \cdot u = 0 \ , \\ u \cdot n = 0 \quad \text{on } \partial M = \Gamma_1 \cup \Gamma_2 \ , \quad \int_{\Gamma_1} u \cdot dl = \sigma_1 \ , \\ u(0,x) = u^0(x) \ . \end{cases} \tag{3.1}$$

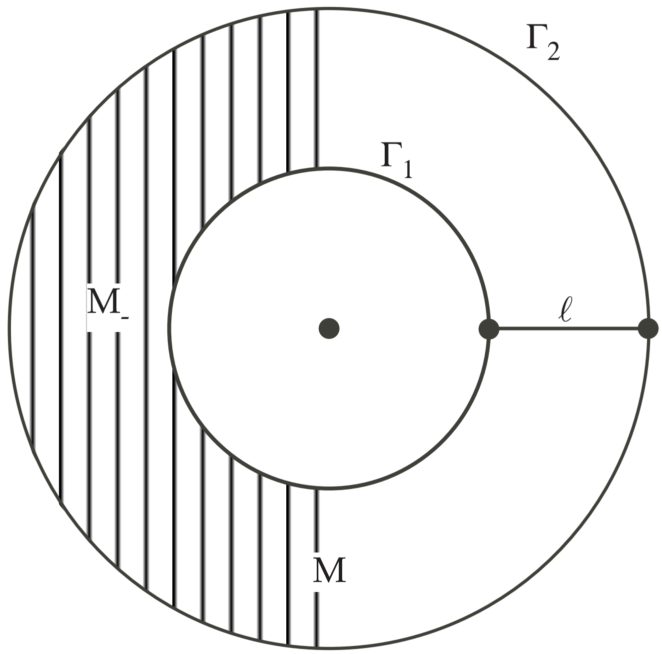


FIG. 2. *The fluid domain for the example of Nadirashvili.*

Let $S$ be the space

$$S = \{\omega \mid \omega \in C^1(M)\} \ .$$

Then for any $\omega^0 \in S$, there is a unique solution to the 2D Euler equation (3.1) $\omega(t) \in S$ for all $t \in \mathbb{R}$, with the initial data $\omega(0) = \omega^0$. In fact, $\omega(t)$ is a classical solution in the sense that (3.1) is satisfied for all $t$ and $x$ ([10], pp.73). In terms of the new phase space, the Nadirashvili's theorem [11] can be restated as follows:

THEOREM 3.1. *There exists a $\xi \in S$, $\epsilon > 0$ and $T > 0$ such that for any initial condition $\omega(0) \in S$ satisfying $\|\omega(0) - \xi\|_{C^1} < \epsilon$, the corresponding solution $\omega(t)$ of the 2D Euler equation satisfies $\|\omega(t) - \xi\|_{C^1} > \epsilon$ for all $t > T$.*

We need the following lemma to prove the theorem.

LEMMA 3.2. *For $(\omega, v)$ satisfying $\omega =$ curl $v$, $\nabla \cdot v = 0$, $v \cdot n = 0$ on $\partial M$, $\int_{\Gamma_1} v \cdot dl = 0$; one has the inequality*

$$\|v\|_{C^0} \le C\|\omega\|_{C^0} \ .$$

A proof of this lemma can be found in ([10], Lemma 3.1, pp.67). Below we give a rather detailed proof too.

*Proof.* By the Biot-Savart formula for multi-connected domain ([10], pp.16), we have

$$v = \hat{v} + \tilde{v} + \nabla\varphi$$

where

$$\hat{v} = \frac{\int_{\Gamma_1} v \cdot dl}{2\pi} |x|^{-2}(-x_2,\ x_1) = 0 \ ,$$
$$\tilde{v} = \frac{1}{2\pi}\int_M |x-\tilde{x}|^{-2}\omega(\tilde{x})\,(-(x_2-\tilde{x}_2),\ x_1-\tilde{x}_1)\,d\tilde{x} \ ,$$
$$\Delta\varphi = 0 \ , \quad \frac{\partial\varphi}{\partial n} = -\tilde{v}\cdot n \ \text{ on } \Gamma_1\cup\Gamma_2 \ , \quad n = \frac{x}{|x|} \ .$$

It is easy to estimate $\|\tilde{v}\|_{C^0}$:

$$\|\tilde{v}\|_{C^0} \leq \frac{\|\omega\|_{C^0}}{2\pi}\int_M |x-\tilde{x}|^{-1}d\tilde{x} \leq C\|\omega\|_{C^0} \ .$$

Estimating $\|\nabla\varphi\|_{C^0}$ is more complicated. We know that $\varphi$ is given by the single layer potential ([4], pp.171)

$$\varphi(x) = \int_{\Gamma_1\cup\Gamma_2} N(x,\tilde{x})f(\tilde{x})d\tilde{x} \ , \quad N(x,\tilde{x}) = \frac{1}{2\pi}\ln|x-\tilde{x}| \ , \tag{3.2}$$

where the moment $f$ is given by ([4], pp.174)

$$-\frac{1}{2}f(x) + \int_{\Gamma_1\cup\Gamma_2} \partial_{n_x}N(x,\tilde{x})f(\tilde{x})d\tilde{x} = -\tilde{v}\cdot n \ . \tag{3.3}$$

Using an approximation (e.g. as in [4], pp.95), one can show that

$$\nabla\cdot\tilde{v} = 0 \quad \text{for all } x\in\mathbb{R}^2 \ .$$

Thus

$$\int_{\Gamma_j} n\cdot\tilde{v}\ dx = 0 \quad \text{for } j = 1,2,$$

which is an if and only if condition for the existence of a solution to (3.3) in $L^2(\Gamma_1\cup\Gamma_2)$, and

$$\|f\|_{L^2(\Gamma_1\cup\Gamma_2)} \leq C\|\tilde{v}\cdot n\|_{L^2(\Gamma_1\cup\Gamma_2)} \leq C\|\omega\|_{C^0} \ .$$

In fact $f \in C(\Gamma_1\cup\Gamma_2)$ ([4], pp.160 (3.14)). Notice that ([4], pp.163)

$$\partial_{n_x}N(x,\tilde{x}) = \frac{1}{2\pi}\frac{(x-\tilde{x})\cdot n_x}{|x-\tilde{x}|^2} = \frac{1}{2\pi}\frac{\cos(x-\tilde{x},n_x)}{|x-\tilde{x}|} \ ,$$

and

$$\cos(x-\tilde{x},n_x) = \mathcal{O}(|x-\tilde{x}|) \quad \text{as } \tilde{x}\to x \text{ on } \Gamma_1\cup\Gamma_2 \ .$$

Thus $\partial_{n_x}N(x,\tilde{x})$ is bounded on $(\Gamma_1\cup\Gamma_2)\times(\Gamma_1\cup\Gamma_2)$. We have from (3.3) that

$$\|f\|_{C^0(\Gamma_1\cup\Gamma_2)} \leq C\|f\|_{L^2(\Gamma_1\cup\Gamma_2)} + \|\tilde{v}\cdot n\|_{C^0(\Gamma_1\cup\Gamma_2)} \leq C\|\omega\|_{C^0} \ .$$

This is the estimate we need. From (3.2),

$$\nabla\varphi(x) = \frac{1}{2\pi}\int_{\Gamma_1\cup\Gamma_2} \frac{x-\tilde{x}}{|x-\tilde{x}|^2} f(\tilde{x}) d\tilde{x} \ .$$

Thus

$$\|\nabla\varphi\|_{C^0} \leq C\|f\|_{C^0(\Gamma_1\cup\Gamma_2)} \leq C\|\omega\|_{C^0} \ .$$

The proof is complete. □

Remark 3.3. This lemma was not proved in the original article [11]. In [1], the authors claimed that it follows from the maximal principle. Maximal principle does not seem to have anything to do with this lemma. This lemma has to be a result of the Biot-Savart formula in multi-connected domain.

*Proof of the Theorem.* Let

$$u^* = |x|^{-2}(-x_2,\ x_1) \ .$$

Then

$$\text{curl } u^* = \nabla\cdot u^* = 0 \ ,$$
$$u^*\cdot n = 0 \quad \text{on } \partial M = \Gamma_1\cup\Gamma_2 \ ,$$
$$\int_{\Gamma_1} u^*\cdot dl = 2\pi \ .$$

Let

$$M_- = \{x\in M \mid x_1 < 0\} \ , \quad \ell = \{x\in M \mid x_2 = 0, x_1 > 0\} \ .$$

Choose $\xi$ such that

$$\|\xi\|_{C^1} < 4\epsilon \ , \quad \xi|_{M_-} = 0 \ , \quad \xi|_\ell > 2\epsilon \ .$$

For any initial condition $\omega(0)\in S$ such that $\|\omega(0)-\xi\|_{C^1} < \epsilon$, let $\omega(t)$ be the corresponding solution of the 2D Euler equation (3.1) with $\int_{\Gamma_1} u\cdot dl = \int_{\Gamma_1} u^*\cdot dl$. Let $v$ be the corresponding velocity given by Lemma 3.2. When $\epsilon$ is small enough, $\|v\|_{C^0} < \frac{1}{4}$ for all $t\in\mathbb{R}$. Let $u = u^* + v$. Then $(\omega, u)$ solves the 2D Euler equation (3.1) with $u(0,x) = u^*(x) + v(0,x)$, $\int_{\Gamma_1} u\cdot dl = \int_{\Gamma_1} u^*\cdot dl$. The velocity fields $u(t)$ defines for each $t\in\mathbb{R}$ an area-preserving diffeomorphism $g^t$ of the annulus $M$ ([10], Theorem 3.1, pp.72). The boundaries $\Gamma_1$ and $\Gamma_2$ are mapped into themselves by $g^t$. We have

$$u(t)|_{\Gamma_1} > 3/4 \ , \quad u(t)|_{\Gamma_2} < 3/4 \quad \text{in polar coordinate.}$$

Thus the corresponding angular velocities are greater than $3/4$ on $\Gamma_1$ and smaller than $3/8$ on $\Gamma_2$. The image $\ell_t = g^t(\ell)$ of the segment $\ell$ under the diffeomorphism $g^t$ will wrap around in the annulus with angular coordinates of the two ends diverging faster than $\frac{3t}{8}$. So $\ell_t$ will wrap around the inner circle with more and more loops. Thus when $t > \frac{8\pi}{3}$, $\ell_t$ will always intersect $M_-$. Notice that the value of $\omega(t)$ is carried over by $\ell_t$ and $\omega(t) > \epsilon$ on $\ell_t$. Hence, for $t > \frac{8\pi}{3}$, we have $\|\omega(t)-\xi\|_{C^1} > \epsilon$. □

Remark 3.4. The setup of curl $u|_\ell > \delta/4$ v.s. $\epsilon = \delta/4$ in ([1], pp.98) is not compatible for the later argument. From the above proof, we see that the non-recurrence is mainly due to the multi-connected nature of the fluid domain. In [9], we show that for a periodic fluid domain, certain recurrence can be established.

**3.2. Shnirelman's Theorem.** In terms of Lagrangian coordinates, Shnirelman proved a theorem on the non-recurrent nature of the configuration map induced by the 2D inviscid incompressible fluid motion in $\mathbb{T}^2$. [13]. The form of the fluid (particle trajectory) equation considered by Shnirelman is

$$\frac{d}{dt}(g,\omega) = \left(\text{rot}^{-1}(\omega \circ g^{-1}) \circ g, 0\right) \ , \tag{3.4}$$

defined on $\mathbb{T}^2$, where $g \ : \ \mathbb{T}^2 \mapsto \mathbb{T}^2$ is the configuration map induced by the fluid motion in $\mathbb{T}^2$, and $\omega$ is the vorticity.

Definition 3.5. Let $w(x) \in L^2(\mathbb{T}^2)$ and $\tilde{w}(\xi)$ ($\xi \in \mathbb{Z}^2$) be its Fourier transform. Besov space $B_s$ is the space of functions $w(x)$ with finite norm

$$\|w\|_{B_s}^2 = \sup_{k\geq 0} \left\{ 2^{2ks} \sum_{2^k \leq |\xi| < 2^{k+1}} |\tilde{w}(\xi)|^2 \right\} \ .$$

By $(g,\omega) \in X_s = DB_s \times B_{s-1}$, it means that $g(x) - x \in B_s$ and $\omega(x) \in B_{s-1}$. Denote by $G_t$ the evolution operator of the 2D Euler equation, i.e. $G_t(g(0),\omega(0)) = (g(t),\omega(t))$.

Theorem 3.6. *[13] For $s > 0$, there exists an open and dense set $Y_s \subset X_s$, such that for each point $(g,\omega) \in Y_s$ there is a wandering neighborhood $U$ of $(g,\omega)$, i.e. there exists a $T > 0$ (depending on $(g,\omega)$ and $U$) such that $G_t(U) \cap U = \emptyset$ for all $t$ such that $|t| > T$.*

Remark 3.7. The Besov space $B_s$ differs only slightly from the Sobolev spaces: $H^s \subset B_s \subset H^{s-\epsilon}$ for any $s$ and $\epsilon > 0$. The smooth functions are not dense in the Besov space $B_s$ (unlike the Sobolev space $H^s$). For $s > 3$, the fluid equation (3.4) is globally wellposed in $X_s$. The configuration map $g$ can be viewed as the family of all fluid trajectories. The theorem says that most of the families of all fluid trajectories are non-recurrent. As shown in [9], in the Eulerian coordinates, certain recurrence can be established for the 2D Euler equation on $\mathbb{T}^2$.

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